ACTA ARITHMETICA
152.1 (2012)

Local heights on Galois covers of the projective line

by

Robin de Jong (Leiden)

1. Introduction. Let K be a number field. Let X be a smooth projective curve of positive genus g defined over K, endowed with a Galois covering map $x\colon X \to \mathbb{P}^1_K$ of degree N to the projective line over K. Assume that x is totally ramified at some point $o \in X(K)$. The basic example that we have in mind is that of a hyperelliptic curve X endowed with a hyperelliptic map. In this paper we introduce and study, for each place v of K, a local height function

$$\lambda_v\colon X(\overline{K}_v) \setminus \{o\} \to \mathbb{R}$$

associated to x, generalizing the well known Néron–Tate local height function on an elliptic curve over K. Here K_v denotes the completion of K at v, and $\overline{K}_v$ an algebraic closure of K_v.

Fix a v-adic absolute value $|\cdot|_v$ on K_v, and assume that a coordinate function has been chosen on $\mathbb{P}^1_K$ such that o maps to ∞ under x. Our local height function will be obtained as an integral with a logarithmic integrand:

$$\lambda_v(p) = \frac{1}{N} \int_{X_v} \log|x - x(p)|_v \, \mu_v,$$

where X_v is a canonical analytic space associated to X at v and μ_v a canonical probability measure on X_v. To be precise, X_v will be the Berkovich analytic space associated to $X \otimes \hat{\overline{K}}_v$ if v is non-archimedean, and the complex analytic space $X(\overline{K}_v)$ if v is archimedean. The measure μ_v will be the canonical Arakelov measure on X_v, to be defined in Section 2 below.

Theorem A. *The function $\lambda_v\colon X(\overline{K}_v) \setminus \{o\} \to \mathbb{R}$ is a local Weil height with respect to the divisor o on X. The difference $\lambda_v(p) - N^{-1} \log|x(p)|_v$ tends to zero as p tends to o on $X(\overline{K}_v)$.*

2010 *Mathematics Subject Classification*: Primary 11G30; Secondary 11G50, 11J95.
Key words and phrases: Berkovich space, Galois cover, local height, Mahler measure, Néron–Tate height, Weierstrass point.

DOI: 10.4064/aa152-1-5

For $p \in X(K) \setminus \{o\}$ we will see that $\lambda_v(p)$ vanishes for almost all v. Let $\overline{K}$ be an algebraic closure of K, and let M_K be the set of places of K. The function $h_x \colon X(K) \setminus \{o\} \to \mathbb{R}$ defined via

$$h_x(p) = \frac{1}{[K\colon \mathbb{Q}]} \sum_{v \in M_K} n_v \lambda_v(p) = \frac{1}{[K\colon \mathbb{Q}]} \frac{1}{N} \sum_{v \in M_K} n_v \int_{X_v} \log|x - x(p)|_v \, \mu_v,$$

where the n_v are suitable local factors, extends to a global Weil height with respect to o on $X(\overline{K})$. We note that $[K\colon \mathbb{Q}]$ times this height can be viewed as a "Mahler measure" associated to the map $x\colon X \to \mathbb{P}^1_K$.

It will be no surprise that h_x relates directly to the Néron–Tate height on the jacobian of X via the Abel–Jacobi map based at o (cf. Proposition 4.1). Because of this connection, and the explicit nature of our local heights, the results in this paper may have an application to the actual calculation of Néron–Tate heights of rational points on jacobians. One possible starting point is the result in Theorem C below. Another possible starting point is to try to develop a "formulaire", à la Tate's formulaire for elliptic curves (cf. [Si2, Section VI.4]), for $\lambda_v(p)$ based, if v is non-archimedean, on the type of the reduction graph of X at v, and the specialization of p onto that graph.

The function λ_v is integrable against μ_v. This leads to a local invariant

$$\int_{X_v} \lambda_v \, \mu_v = \frac{1}{N} \int_{X_v} \int_{X_v} \log|x(p) - x(q)|_v \, \mu_v(p) \, \mu_v(q) \tag{1.1}$$

associated to each v. This invariant vanishes for almost all v and hence the global invariant:

$$\sum_{v \in M_K} n_v \int_{X_v} \lambda_v \, \mu_v = \frac{1}{N} \sum_{v \in M_K} n_v \int_{X_v} \int_{X_v} \log|x(p) - x(q)|_v \, \mu_v(p) \, \mu_v(q) \tag{1.2}$$

is well-defined. One expects this invariant to be comparable with more "classical" invariants of X, such as the (admissible) self-intersection of the relative dualizing sheaf of X as studied in [Zh]. For x a hyperelliptic map we prove that this is indeed the case:

Theorem B. *Assume X is a hyperelliptic curve of genus $g \geq 2$ and assume that $x\colon X \to \mathbb{P}^1_K$ is a hyperelliptic map. Let $(\omega, \omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of X. Then*

$$\sum_{v \in M_K} n_v \int_{X_v} \lambda_v \, \mu_v = \frac{1}{2} \sum_{v \in M_K} n_v \int_{X_v} \int_{X_v} \log|x(p) - x(q)|_v \, \mu_v(p) \, \mu_v(q) = \frac{(\omega, \omega)_a}{4g(g-1)}.$$

In fact we prove that in general the invariant (1.2) equals $-(o,o)_a$, where $(o,o)_a$ is the admissible self-intersection of the point o. For hyperelliptic curves, or more generally curves such that $(2g-2)o$ is a canonical divisor, this specializes to the formula in Theorem B. For hyperelliptic curves we are able to analyze the local invariants (1.1) in more detail and connect them to a local invariant that we introduced in an earlier paper [dJ].

Our next result states that the local heights $\lambda_v(p)$ for $p \in X(K) \setminus \{o\}$ can be obtained by averaging, and taking a limit, over higher-order Weierstrass points on X. More precisely, let X_n for $n \geq g$ be the divisor of Weierstrass points of the line bundle $\mathcal{O}_X(o)^{\otimes n+g-1}$ on X (see Section 6 for definitions). This X_n is an effective divisor of degree gn^2, generalizing the divisor of n-torsion points on (X, o) when (X, o) is an elliptic curve. We usually view X_n as a $\mathrm{Gal}(\overline{K}/K)$-invariant multi-set in $X(\overline{K})$.

Theorem C. *Let p be a point in $X(K) \setminus \{o\}$. Assume that p is not in X_n for at least one $n \geq g$. Let v be a place of K. Then*

$$\frac{1}{gn^2} \sum_{\substack{q \in X_n \\ x(q) \neq x(p), \infty}} \log|x(p) - x(q)|_v \to \int_{X_v} \log|x - x(p)|_v \, \mu_v$$

for some sequence of natural numbers n tending to infinity. In the sum on the left hand side, the points in X_n are counted with multiplicity.

It is clear that there can be at most finitely many points p in $X(\overline{K})$ such that $p \in X_n$ for all $n \geq g$. In any case, apart from some "obvious" cases, such points appear to be extremely rare [SV]. If $p \notin X_n$ for some $n \geq g$ then $p \notin X_n$ for infinitely many $n \geq g$. The sequence of these n can be taken as in Theorem C. It seems likely that Theorem C holds without the assumption that p should not be in every X_n, and that the limit can be taken over all natural numbers.

A celebrated theorem of Mumford–Neeman [Ne] implies that, at least if v is archimedean, the multi-sets X_n are weakly equidistributed with respect to μ_v. As far as we know, the equidistribution of Weierstrass points at non-archimedean v has not yet been proven. In some sense the theorem of Mumford–Neeman "explains" Theorem C but it seems there is no direct implication. For example, the condition that the point p should be algebraic seems to be essential.

In the case that (X, o) is an elliptic curve, Theorem C occurs in an article by Everest and Fhlathúin [EF]. A result similar in spirit to Theorem C but in the context of dynamical systems on $\mathbb{P}^1_K$ has been proven by Szpiro and Tucker [ST]. They show that if $\varphi \colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a non-constant rational map of degree $d > 1$, and $p \in \mathbb{P}^1(K) \setminus \{\infty\}$ is a rational point, then

$$\frac{1}{d^n} \sum_{\substack{q \in \mathrm{Fix}_n \\ q \neq p, \infty}} \log|x(p) - x(q)|_v \to \int_{\mathbb{P}^1_v} \log|x - x(p)|_v \, \mu_{\varphi,v}$$

as $n \to \infty$. Here Fix_n is the multi-set (divisor) of fixed points of the iterate φ^n for all positive integers n, and $\mu_{\varphi,v}$ is the local canonical measure on $\mathbb{P}^1_v$

associated to φ by Call–Silverman [CS]. The results of Everest–ní Fhlathúin and Szpiro–Tucker were proved using diophantine approximation in some form. We will prove Theorem C as a corollary of Faltings's diophantine approximation results on abelian varieties [Fa].

A possible application of Theorem C is a method to actually calculate, or at least approximate, the canonical height of p by evaluating a sequence of polynomials—the "division polynomials" associated to the X_n—at $x(p)$. In the case of elliptic curves, this method is discussed in [EW]. The next interesting case would be the case of genus two (or more generally, hyperelliptic) curves. The division polynomials associated to the X_n are then known to satisfy some explicit recurrence relations that one could use [Ca]. The upshot of this method is that one does not need algebraic equations for the jacobian of X. We have stated our result as Theorem 6.8 below.

Let us summarize the contents of this paper. In Section 2 we recall some useful facts from potential theory on Berkovich spaces associated to curves, mainly based on Thuillier's thesis [Th]. This is the "right" context to make sense of the integrals involved in defining λ_v in the non-archimedean setting. We also connect the theory with Zhang's theory of local admissible pairing on a curve [Zh].

In Section 3 we extend the theory to the global setting and make the connection, following a classical result of Hriljac–Faltings, with the Néron–Tate height on the jacobian.

From Section 4 on we assume X is endowed with a Galois covering map $x\colon X \to \mathbb{P}^1_K$. We start by introducing λ_v and proving Theorem A. We verify that for (X, o) an elliptic curve, the function λ_v coincides with Tate's classical Néron function. Finally we prove Theorem B.

We analyze the case of hyperelliptic curves more closely in Section 5. Theorem C is proved in Section 6.

2. Potential theory on Berkovich analytic curves. Let X be a geometrically connected smooth projective curve over a local field $(k, |\cdot|)$. Let $\overline{k}$ be an algebraic closure of k, and let $\hat{\overline{k}}$ be the completion of $\overline{k}$. The absolute value $|\cdot|$ extends in a unique way to $\hat{\overline{k}}$. One has associated to X a locally ringed space $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}})$, where the underlying topological space $|\mathfrak{X}|$ has the following properties: $|\mathfrak{X}|$ is compact, metrizable, path-connected, and contains $X(\overline{k})$ with its topology induced from $|\cdot|$ naturally as a dense subspace. If k is archimedean, we just take $X(\overline{k})$ itself, with its structure of complex analytic space; if k is non-archimedean we let $\mathfrak{X}$ be the Berkovich analytic space associated to $X \otimes \hat{\overline{k}}$, as in [Be].

Our purpose in this section is to put a canonical probability measure $\mu_{\mathfrak{X}}$ on $|\mathfrak{X}|$, and to discuss a few results from potential theory on $\mathfrak{X}$. Everything is standard for archimedean k; however for non-archimedean k the results seem to be less known. We base our discussion on the thesis of Thuillier [Th], in particular Chapter 3. For more background on Berkovich spaces we refer to [Ba]. As an application of the formalism we present a formula (see Proposition 2.2 below) for the integral $\int_{\mathfrak{X}} \log|f|\,\mu_{\mathfrak{X}}$ where f is an arbitrary non-zero rational function on $X \otimes \overline{k}$. This formula establishes a link with Zhang's theory of local admissible pairing [Zh].

We start by considering the natural exact sequence

$$0 \to \mathcal{H} \to \mathcal{A}^0 \xrightarrow{dd^c} \mathcal{A}^1 \to 0$$

of sheaves of $\mathbb{R}$-vector spaces on $\mathfrak{X}$. Here $\mathcal{H}$ is the sheaf of harmonic functions on $\mathfrak{X}$, $\mathcal{A}^0$ is the sheaf of smooth functions on $\mathfrak{X}$, $\mathcal{A}^1$ is the sheaf of smooth forms on $\mathfrak{X}$, and dd^c is the Laplace operator. The sheaf $\mathcal{A}^0$ is in fact a sheaf of $\mathbb{R}$-algebras and the sheaf $\mathcal{A}^1$ is naturally a sheaf of modules over $\mathcal{A}^0$.

We let $\mathcal{A}^0(\mathfrak{X})$ and $\mathcal{A}^1(\mathfrak{X})$ be the spaces of global sections of $\mathcal{A}^0$ and $\mathcal{A}^1$. Further we put $\mathcal{D}^0(\mathfrak{X}) = \mathcal{A}^1(\mathfrak{X})^*$ and $\mathcal{D}^1(\mathfrak{X}) = \mathcal{A}^0(\mathfrak{X})^*$ for their $\mathbb{R}$-linear duals. We have a natural $\mathbb{R}$-linear integration map $\int_{\mathfrak{X}}\colon \mathcal{A}^1(\mathfrak{X}) \to \mathbb{R}$ and a natural $\mathbb{R}$-bilinear pairing $\mathcal{A}^0(\mathfrak{X}) \times \mathcal{A}^1(\mathfrak{X}) \to \mathbb{R}$ given by $(\varphi, \omega) \mapsto \int_{\mathfrak{X}} \varphi\omega$. This pairing yields a natural commutative diagram

$$\begin{tikzcd} \mathcal{D}^0(\mathfrak{X}) \arrow[r] & \mathcal{D}^1(\mathfrak{X}) \\ \mathcal{A}^0(\mathfrak{X}) \arrow[u] \arrow[r, "dd^c"] & \mathcal{A}^1(\mathfrak{X}) \arrow[u] \end{tikzcd}$$

where the upward arrows are injections, and the map $\mathcal{D}^0(\mathfrak{X}) \to \mathcal{D}^1(\mathfrak{X})$ is the dual of the map $\mathcal{A}^0(\mathfrak{X}) \xrightarrow{dd^c} \mathcal{A}^1(\mathfrak{X})$, also to be denoted by dd^c. The kernel of $dd^c : \mathcal{D}^0(\mathfrak{X}) \to \mathcal{D}^1(\mathfrak{X})$ is a one-dimensional $\mathbb{R}$-vector space, to be identified with the set of constant functions on $\mathfrak{X}$. The elements of $\mathcal{D}^\alpha(\mathfrak{X})$ are called (α, α)-*currents*; $(1,1)$-currents can be viewed as measures on $|\mathfrak{X}|$. The unit element of $\mathcal{A}^0(\mathfrak{X})$ gives, under the natural map $\mathcal{A}^0(\mathfrak{X}) \to \mathcal{D}^1(\mathfrak{X})^*$, a natural $\mathbb{R}$-linear integration map $\int_{\mathfrak{X}}\colon \mathcal{D}^1(\mathfrak{X}) \to \mathbb{R}$, extending $\int_{\mathfrak{X}}$ on $\mathcal{A}^1(\mathfrak{X})$. Associated to each non-zero rational function f on $X \otimes \overline{k}$ we have a natural $(0,0)$-current $\log|f| \in \mathcal{D}^0(\mathfrak{X})$. For each closed point p on $X \otimes \overline{k}$ we have a Dirac measure $\delta_p \in \mathcal{D}^1(\mathfrak{X})$, and by linear extension we obtain a measure $\delta_D \in \mathcal{D}^1(\mathfrak{X})$ for each divisor D on $X \otimes \overline{k}$.

PROPOSITION 2.1.

(i) (Poincaré–Lelong equation) *Let f be a non-zero rational function on $X \otimes \overline{k}$. Then*

$$dd^c \log|f| - \delta_{\operatorname{div} f} = 0$$

in $\mathcal{D}^1(\mathfrak{X})$.

(ii) (The Laplace operator is self-adjoint) *We have*

$$\int_{\mathfrak{X}} \varphi \, dd^c \, \psi = \int_{\mathfrak{X}} \psi \, dd^c \, \varphi$$

for all $\varphi, \psi \in \mathcal{D}^0(\mathfrak{X})$.

(iii) (Existence and uniqueness of Green's functions) *Let* $\mu \in \mathcal{A}^1(\mathfrak{X})$ *be a smooth measure with* $\int_{\mathfrak{X}} \mu = 1$, *and let* $p \in X(\overline{k})$. *Then there exists a unique current* $g_{\mu,p} \in \mathcal{D}^0(\mathfrak{X})$ *such that*

$$dd^c \, g_{\mu,p} = \mu - \delta_p, \qquad \int_{\mathfrak{X}} g_{\mu,p} \, \mu = 0.$$

The symmetry relation $g_{\mu,p}(q) = g_{\mu,q}(p)$ *holds for all* $p \neq q \in X(\overline{k})$.

Proof. This is well known for archimedean k. For non-archimedean k we find (i)–(iii) respectively in [Th, Propositions 3.3.15, 3.2.12 and 3.3.13]. ∎

Our next goal is to designate a canonical probability measure $\mu_{\mathfrak{X}} \in \mathcal{A}^1(\mathfrak{X})$. We assume from now on that the genus g of X is positive. If k is archimedean we let $\mu_{\mathfrak{X}}$ be the Arakelov probability measure on $X(\overline{k})$. One way of defining $\mu_{\mathfrak{X}}$ is as follows: let $\iota \colon X(\overline{k}) \to J(\overline{k})$ be an immersion of $X(\overline{k})$ into the complex torus $J(\overline{k})$, where $J = \mathrm{Pic}^0 X$ is the jacobian of X. Then $\mu_{\mathfrak{X}} = g^{-1} \iota^* \nu$, where ν is the unique translation-invariant $(1,1)$-form representing the first Chern class of the line bundle defining the canonical principal polarization on $J(\overline{k})$.

Now suppose that k is non-archimedean. Let $\mathcal{R}$ be the reduction graph in the sense of Chinburg–Rumely [CR] of X. This is a metrized graph, with a canonical surjective continuous specialization map $\mathrm{sp} \colon |\mathfrak{X}| \to \mathcal{R}$. In particular $\mathcal{R}$ is compact and path-connected. The map sp has a canonical section $i \colon \mathcal{R} \to |\mathfrak{X}|$, and via i the reduction graph $\mathcal{R}$ is identified with the minimal skeleton of $\mathfrak{X}$. The Laplace operator on $\mathfrak{X}$ extends in a natural way the Laplace operator on $\mathcal{R}$.

Now in [Zh], Section 3 a canonical probability measure $\mu_{\mathcal{R}}$ is constructed on $\mathcal{R}$, called the admissible measure. We will not recall its definition; let us just say that it has properties analogous to the Arakelov measure in the archimedean setting. For example, it gives rise to an adjunction formula. We define the canonical Arakelov measure $\mu_{\mathfrak{X}}$ on $|\mathfrak{X}|$ by putting $\mu_{\mathfrak{X}} = i_* \mu_{\mathcal{R}}$.

Let $g_{\mu_{\mathfrak{X}},p}$ be the Green's function based on $\mu_{\mathfrak{X}}$ from Proposition 2.1(iii). We then obtain a canonical symmetric pairing $(\,,)_a$ on $X(\overline{k})$ by putting $(p,q)_a = g_{\mu_{\mathfrak{X}},p}(q)$ for $p \neq q$. This pairing coincides with the admissible pairing constructed in [Zh, Section 4] using Green's functions on $\mathcal{R}$ with respect to $\mu_{\mathcal{R}}$ and the specialization map. We refer to [He] where this connection is made explicit.

We have the following proposition relating the integrals $\int_{\mathfrak{X}} \log |f| \, \mu_{\mathfrak{X}}$ to Zhang's admissible pairing.

PROPOSITION 2.2. *Let f be a non-zero rational function on $X \otimes \overline{k}$. Assume that a coordinate has been chosen on $\mathbb{P}^1_k$. Then*

$$\int_{\mathfrak{X}} \log|f|\,\mu_{\mathfrak{X}} = \log|f|(r) + (\operatorname{div} f, r)_a.$$

Here r is an arbitrary point in $X(\overline{k})$ outside the support of f.

Proof. By Proposition 2.1(i) we have

$$dd^c \log|f| = \delta_{\operatorname{div} f}.$$

By integrating against $g_{\mu_{\mathfrak{X}},r}$ we obtain

$$\int_{\mathfrak{X}} g_{\mu_{\mathfrak{X}},r}\, dd^c \log|f| = g_{\mu_{\mathfrak{X}},r}(\operatorname{div} f) = (\operatorname{div} f, r)_a.$$

On the other hand, by Proposition 2.1(ii), (iii) we have

$$\begin{aligned}\int_{\mathfrak{X}} g_{\mu_{\mathfrak{X}},r}\, dd^c \log|f| &= \int_{\mathfrak{X}} (\log|f|)\, dd^c g_{\mu_{\mathfrak{X}},r} = \int_{\mathfrak{X}} (\log|f|)(\mu_{\mathfrak{X}} - \delta_r)\\ &= -\log|f|(r) + \int_{\mathfrak{X}} \log|f|\,\mu_{\mathfrak{X}}.\end{aligned}$$

The proposition follows. ∎

In [Zh, Theorem 4.6(iii)] it is stated that with notation as in the above proposition $\log|f|(r) + (\operatorname{div} f, r)_a$ is constant outside the support of f. Using Thuillier's thesis we are thus able to interpret this constant as a suitable integral over $\mathfrak{X}$.

3. Connection with the Néron–Tate height on the jacobian. We now apply the global admissible intersection theory as developed in [Zh, Section 5]. Let K be a number field and let X be a smooth projective curve of positive genus g over K. For each place v of K we denote by $\overline{K}_v$ an algebraic closure of K_v. We endow each K_v with a (standard) absolute value $|\cdot|_v$ as follows. If v is archimedean, we take the euclidean norm on K_v. If v is non-archimedean, we choose $|\cdot|_v$ such that $|\pi|_v = \mathrm{e}$, where π is a uniformizer of K_v. Let X_v be the analytic space associated to $X \otimes \widehat{\overline{K}}_v$, and μ_v be the canonical measure on X_v, as in Section 2.

Let M_K be the set of places of K. For each place $v \in M_K$, let n_v be the (standard) local factor defined as follows: if v is real, then $n_v = 1$; if v is complex, then $n_v = 2$; if v is non-archimedean, then n_v is the log of the cardinality of the residue field at v. Note that we have a product formula $\sum_{v \in M_K} n_v \log|\alpha|_v = 0$ for all α in K^*.

Let $o \in X(K)$ be a point and $J = \operatorname{Pic}^0 X$ be the jacobian of X. We denote by $h\colon J(\overline{K}) \to \mathbb{R}$ the Néron–Tate height associated to the canonical principal polarization of J. Let f be a non-zero rational function on X and assume that a coordinate has been chosen on $\mathbb{P}^1_K$. It follows from Proposition 2.2 that the real number $\int_{X_v} \log|f|_v \, \mu_v$ vanishes for almost all v.

PROPOSITION 3.1. *Let* $\operatorname{div} f = \sum_q m_q \cdot q$ *be the divisor of the non-zero rational function* f *on* $X \otimes \overline{K}$. *Then*

$$[K\colon \mathbb{Q}] \sum_q m_q \, h([q-o]) = g \sum_{v \in M_K} n_v \int_{X_v} \log|f|_v \, \mu_v .$$

Proof. We follow the formalism and results of [Zh, Section 5]. First of all, for any point $q \in X(K)$ one has $-2[K\colon \mathbb{Q}]h([q-o]) = (q-o, q-o)_a$, where now $(\,,)_a$ denotes the global admissible pairing. This formula is essentially due to Hriljac and Faltings. By the adjunction formula (see *op. cit.*) we next have $(q-o, q-o)_a = -(\omega + 2\,o, q)_a - (\omega, o)_a$, where ω denotes the admissible relative dualizing sheaf of X. We obtain

$$\begin{aligned}
-2[K\colon \mathbb{Q}] \sum_q m_q h([q-o]) &= -(\omega + 2o, \operatorname{div} f)_a \\
&= -\Big(\omega + 2o, \sum_{v \in M_K} n_v \Big(\int_{X_v} \log|f|_v \, \mu_v \Big) \cdot X_v \Big)_a \\
&= -2g \sum_{v \in M_K} n_v \int_{X_v} \log|f|_v \, \mu_v .
\end{aligned}$$

The proposition follows. ∎

4. The local canonical height. Now assume that X is equipped with a Galois covering map $x\colon X \to \mathbb{P}^1_K$ of degree N, and that it has a point $o \in X(K)$ for which x is totally ramified. Also assume that $x(o) = \infty$. Let again v be a place of K. Take a point $p \in X(\overline{K}_v) \setminus \{o\}$ and put $f = x - x(p)$. This is then a well-defined rational function on $X \otimes \overline{K}_v$. We define $\lambda_v(p)$ to be the integral

$$\lambda_v(p) = \frac{1}{N} \int_{X_v} \log|x - x(p)|_v \, \mu_v .$$

This will be our basic object of study from now on. Let $p \in X(K) \setminus \{o\}$ and as in Section 3 let $h\colon J(\overline{K}) \to \mathbb{R}$ be the Néron–Tate height associated to the canonical principal polarization of the jacobian J of X.

PROPOSITION 4.1. *The real number* $\lambda_v(p)$ *vanishes for almost all* v, *and*

$$[K\colon \mathbb{Q}]h([p-o]) = g \sum_{v \in M_K} n_v \lambda_v(p) = \frac{g}{N} \sum_{v \in M_K} n_v \int_{X_v} \log|x - x(p)|_v \, \mu_v .$$

Proof. This is a straightforward consequence of Proposition 3.1. Let $-No + \sum_q m_q q$ be the divisor of $f = x - x(p)$. The formula in Proposition 3.1 specializes to the following:

$$[K\colon \mathbb{Q}]\sum_q m_q h([q-o]) = gN \sum_{v\in M_K} n_v\lambda_v(p).$$

As all the $h([q-o])$ where q runs through $x^{-1}(x(p))$ are equal (use the automorphism group of $x\colon X\to \mathbb{P}^1_K$) we obtain the desired formula. ∎

We find that the $\lambda_v(p)$ sum up to essentially the Néron–Tate height of the Abel–Jacobi image of p, with reference point o. It is instructive to compare this result with the main result of [PST], which writes the global canonical height associated to a morphism (dynamical system) $\varphi\colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ as a sum of local canonical heights given by logarithmic integrals à la λ_v.

For (X,o) an elliptic curve, the function λ_v coincides with Tate's classical Néron function (see for instance [Se, Section 6.5], the "first normalization").

PROPOSITION 4.2. *Assume that (X,o) is an elliptic curve and that $x\colon X\to\mathbb{P}^1_K$ is a hyperelliptic map. Let v be a place of K. Then λ_v is equal to the unique Néron function with respect to o on (X,o) normalized so that $\lambda_v(p)-\frac{1}{2}\log|x(p)|_v\to 0$ as $p\to o$.*

Proof. Let $\tilde\lambda_v\colon X(\overline{K}_v)\setminus\{o\}\to\mathbb{R}$ be the Néron function described in the proposition. It satisfies the "quasi-parallelogram law" (cf. *op. cit.*)

$$\tilde\lambda_v(p+q)+\tilde\lambda_v(p-q) = 2\,\tilde\lambda_v(p)+2\,\tilde\lambda_v(q)-\log|x(p)-x(q)|_v$$

for all $p,q\in X(\overline{K}_v)\setminus\{o\}$ such that $p\neq\pm q$. By fixing p and integrating against $\mu_v(q)$ one finds, using the translation-invariance of μ_v and cancelling three terms

$$\tilde\lambda_v(p) = \frac{1}{2}\int_{X_v}\log|x-x(p)|_v\,\mu_v.$$

This shows that $\tilde\lambda_v=\lambda_v$. ∎

The following theorem analyzes the properties of the local functions λ_v in more detail. Let again v be any place of K.

THEOREM 4.3. *The function $\lambda_v\colon X(\overline{K}_v)-\{o\}\to\mathbb{R}$ extends naturally and uniquely as a $(0,0)$-current on X_v. It satisfies the dd^c-equation*

$$dd^c\,\lambda_v = \mu_v - \delta_o.$$

As a consequence,

$$\lambda_v(p) = (p,o)_a + \int_{X_v}\lambda_v\,\mu_v,$$

where $(\,,)_a$ is the local admissible pairing on $X(\overline{K}_v)$. Furthermore,

$$\lambda_v(p)-\frac{1}{N}\log|x(p)|_v\to 0$$

as $p \to o$ on $X(\overline{K}_v)$. In particular λ_v defines a local Weil function with respect to the divisor o on X.

Proof. Let G be the automorphism group of $x\colon X \to \mathbb{P}^1_K$ over $\overline{K}$. Note that $\operatorname{div}(x - x(p)) = -No + \sum_{\sigma \in G} \sigma(p)$. From Proposition 2.2 we obtain

$$N\lambda_v(p) = \log|x(p) - x(r)|_v + \sum_{\sigma\in G} (\sigma(p) - o, r)_a, \tag{4.1}$$

where $r \in X(\overline{K}_v)$ is an arbitrary point not in the support of $x - x(p)$. Now consider equation (4.1) with p as a variable and r fixed. Both $(\sigma(p) - o, r)_a$ and $\log|x(p) - x(r)|_v$ extend as $(0, 0)$-currents over X_v. Hence so does λ_v. The extension is unique, as $X(\overline{K}_v)$ is dense in X_v. To prove the first formula of the theorem, note that as $(\, ,)_a$ is canonical, it is invariant under G. We can thus rewrite (4.1) as

$$N\lambda_v(p) = \log|x(p) - x(r)|_v + \sum_{\sigma\in G} (p - o, \sigma(r))_a.$$

Taking dd^c we have, by Proposition 2.1,

$$N\, dd^c \lambda_v = \sum_{\sigma\in G} (\delta_{\sigma(r)} - \delta_o) + \sum_{\sigma\in G} (\mu_v - \delta_{\sigma(r)}).$$

It follows that $dd^c \lambda_v = \mu_v - \delta_o$ as required. As $(p, o)_a = g_{\mu_v,o}(p)$ satisfies the same dd^c-equation, we obtain the second formula. To prove the last formula, let $p \to o$ in (4.1). Then the sum $\sum_{\sigma\in G}(\sigma(p) - o, r)_a$ converges to zero. ∎

THEOREM 4.4. *Let $(o, o)_a$ be the admissible self-intersection of the point o on X. Then*

$$\sum_{v\in M_K} n_v \int_{X_v} \lambda_v\, \mu_v = \frac{1}{N} \sum_{v\in M_K} n_v \int_{X_v}\int_{X_v} \log|x(p) - x(q)|_v\, \mu_v(p)\, \mu_v(q) = -(o, o)_a.$$

Proof. Choose a $p \in X(K) \setminus \{o\}$ arbitrarily. Let again G be the automorphism group of x over $\overline{K}$. Note that $\mathcal{O}_X(\operatorname{div}(x - x(p))) = \mathcal{O}_X(-No + \sum_{\sigma\in G} \sigma(p))$ is a trivial admissible line bundle at all places v of K. It follows that the global pairing $(-No + \sum_{\sigma\in G} \sigma(p), r)_a$ is independent of the choice of r. We can choose $r = o$ and we derive from equation (4.1) in global admissible theory the relation

$$N \sum_{v\in M_K} n_v \lambda_v(p) = \Big(-No + \sum_{\sigma\in G} \sigma(p), o\Big)_a = N(p - o, o)_a$$

and hence

$$\sum_{v\in M_K} n_v \lambda_v(p) = (p - o, o)_a = (p, o)_a - (o, o)_a.$$

By Theorem 4.3 we have, on the other hand,

$$(p,o)_a = \sum_{v\in M_K} n_v\Big(\lambda_v(p) - \int_{X_v} \lambda_v\,\mu_v\Big).$$

The theorem follows. ∎

Note that if $g \geq 1$ and $(2g-2)o$ is a canonical divisor on X, then

$$((2g-2)o-\omega,(2g-2)o-\omega)_a = 0, \quad \text{i.e.,} \quad -(o,o)_a = \frac{(\omega,\omega)_a}{4g(g-1)}.$$

In that case the formula in the Theorem becomes

$$\sum_{v\in M_K} n_v \int_{X_v} \lambda_v\,\mu_v = \frac{1}{N}\sum_{v\in M_K} n_v \int_{X_v}\int_{X_v} \log|x(p)-x(q)|_v\,\mu_v(p)\,\mu_v(q) = \frac{(\omega,\omega)_a}{4g(g-1)}.$$

The condition that $(2g-2)o$ is a canonical divisor is fulfilled when X is a hyperelliptic curve, and more generally, when the coordinate ring of $X\setminus\{o\}$ is generated by two elements. Such curves go by different names in the literature: plane model curves, C_{ab}-curves, Burchnall–Chaundy curves, ...

5. Hyperelliptic curves. The purpose of this section is to study the local invariants $\int_{X_v}\lambda_v\,\mu_v$ in more detail for hyperelliptic maps. In particular we connect them with the local invariants $\chi(X_v)$ introduced in [dJ]. We start with a rather remarkable formula that computes the special value of λ_v at a hyperelliptic ramification point.

Let (X,o) be a hyperelliptic curve of genus $g\geq 2$ over K given by an equation $y^2=f(x)$, where $f(x)\in K[x]$ is monic and separable of degree $m=2g+1$. Fix a place v of K, as well as an algebraic closure $\overline{K}_v$ of K_v. Keep the v-adic absolute values on K_v and $\overline{K}_v$ as defined in Section 3.

Proposition 5.1. *Let $w\in X(\overline{K}_v)\setminus\{o\}$ be a hyperelliptic ramification point of X and let $\alpha = x(w)$ in $\overline{K}_v$. Then*

$$2\lambda_v(w) = \int_{X_v} \log|x-\alpha|_v\,\mu_v = \frac{1}{2g}\log|f'(\alpha)|_v.$$

Proof. We use a result on the arithmetic of symmetric roots from [dJ]. Let $\alpha_1,\dots,\alpha_{2g+2}=\infty$ on $\mathbb{P}^1(\overline{K}_v)$ be the branch points of x. The *symmetric root* of a triple $(\alpha_i,\alpha_j,\alpha_k)$ of distinct branch points is then defined to be an element

$$\ell_{ijk} = \frac{\alpha_i-\alpha_k}{\alpha_j-\alpha_k}\sqrt[2g]{-\frac{f'(\alpha_j)}{f'(\alpha_i)}}$$

of $\overline{K}_v^*$. The actual choice of $2g$th root will be immaterial in the discussion below. If α_j equals infinity, the formula is to be read as follows:

$$\ell_{i\infty k} = (\alpha_i-\alpha_k)\,\sqrt[2g]{-f'(\alpha_i)}^{\,-1} \tag{5.1}$$

(recall that f is monic). Now let $w_1, \ldots, w_{2g+2} \in X(\overline{K}_v)$ be the hyperelliptic ramification points corresponding to $\alpha_1, \ldots, \alpha_{2g+2}$. Theorem C of [dJ] then states that if (w_i, w_j, w_k) is a triple of distinct ramification points, we have

$$(5.2) \qquad (w_i - w_j, w_k)_a = -\frac{1}{2} \log |\ell_{ijk}|_v .$$

Here, as before $(\,,)_a$ denotes Zhang's local admissible pairing on $X(\overline{K}_v)$. Applying Proposition 2.2 to the rational function $x - \alpha_i$, with α_i a finite branch point, we find

$$\int_{X_v} \log |x - \alpha_i|_v \, \mu_v = \log |x(p) - \alpha_i|_v + 2(w_i - o, p)_a$$

for any $p \neq o, w_i$. Taking $p = w_k$ and applying (5.2) we find

$$\begin{aligned} \int_{X_v} \log |x - \alpha_i|_v \, \mu_v &= \log |\alpha_i - \alpha_k|_v + 2(w_i - o, w_k)_a \\ &= \log |\alpha_i - \alpha_k|_v - \log |\ell_{i\infty k}|_v . \end{aligned}$$

Hence, by (5.1),

$$2\,\lambda_v(w_i) = \int_{X_v} \log |x - \alpha_i|_v \, \mu_v = \frac{1}{2g} \log |f'(\alpha_i)|_v .$$

The proposition is proven. ■

The function λ_v depends on the choice of the monic equation f for the pointed curve (X, o). Let $\Delta = 2^{4g} \Delta(f)$ where $\Delta(f)$ is the discriminant of f. We refer to [Lo] for properties of Δ. The discriminant Δ generalizes the usual definition $\Delta = 2^4 \Delta(f)$ in the case where (X, o) is an elliptic curve. We renormalize λ_v by putting

$$\hat{\lambda}_v(p) = \lambda_v(p) - \frac{1}{4g(2g+1)} \log |\Delta|_v .$$

Then $\hat{\lambda}_v$ is independent of the choice of the monic equation f for (X, o), as one checks by replacing x by $u^2 x + t$ for $u \in K^*$, $t \in K$. We obtain the familiar renormalization

$$\hat{\lambda}_v = \lambda_v - \frac{1}{12} \log |\Delta|_v$$

in the case where (X, o) is an elliptic curve (cf. [Se, Section 6.5], the "second normalization").

Let i be an index with $1 \leq i \leq 2g+2$. Then put

$$\chi(X_v) = -2g \Big(\log |2|_v + \sum_{k \neq i} (w_i, w_k)_a \Big).$$

It is proved in [dJ, Theorem B] that $\chi(X_v)$ is independent of the choice of i, hence is an invariant of X_v. One can prove that $\chi(X_v) \geq 0$ for all v, and in fact $\chi(X_v) > 0$ if v is archimedean. We have $\chi(X_v) = 0$ if v is non-archimedean and X has potentially good reduction at v. Let $(\omega,\omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of X. We have

$$(\omega,\omega)_a = \frac{2g-2}{2g+1} \sum_{v\in M_K} n_v\, \chi(X_v), \tag{5.3}$$

where v runs over the places of K. The following result says that $\chi(X_v)$ is essentially equal to $\int_{X_v} \hat{\lambda}_v\, \mu_v$.

Corollary 5.2. *Let v be a place of K. Then*

$$\chi(X_v) = 2g(2g+1) \int\limits_{X_v} \hat{\lambda}_v\, \mu_v.$$

Proof. Let $w_1, \ldots, w_{2g+1}$ be the ramification points of X distinct from o. From Proposition 5.1 and the identity $\Delta(f) = \prod_{i=1}^{2g+1} f'(\alpha_i)$ we obtain the formula

$$\sum_{i=1}^{2g+1} \hat{\lambda}_v(w_i) = -\log|2|_v.$$

From Proposition 4.3 we deduce that

$$\hat{\lambda}_v(p) = (p,o)_a + \int\limits_{X_v} \hat{\lambda}_v\, \mu_v$$

for each $p \in X(\overline{K}_v) \setminus \{o\}$. The corollary follows by combining these two relations. ∎

Note that the above result yields an alternative approach to Theorem B. Namely we derive

$$\sum_{v\in M_K} n_v \int\limits_{X_v} \hat{\lambda}_v\, \mu_v = \frac{(\omega,\omega)_a}{4g(g-1)}$$

from (5.3) and Corollary 5.2, with the additional information that in this local decomposition, all contributions on the left hand side are non-negative, and are canonically associated to X. The equivalence of the above local decomposition with the formula from Theorem B is clear by the product formula.

It would be interesting to have explicit formulas for $\hat{\lambda}_v(p)$, for genus two curves say, à la the ones of Tate (see [Si2, Chapter VI]) in the context of elliptic curves, given the type of the reduction graph $\mathcal{R}_v$ of X at v, and the specialization of p on $\mathcal{R}_v$, if v is non-archimedean. A natural case to start would be the case where X is a Mumford curve at v. This occurs if the branch points of X come in pairs of points closer to one another than to the other branch points, where the distance is measured by rational affinoid subsets of the projective line [Br].

6. Proof of Theorem C. In this section we prove Theorem C. Let K be a number field. We will make use of the following general diophantine approximation result due to Faltings (see [Fa, Theorem II]):

THEOREM 6.1. *Let A be an abelian variety over K and let D be an ample divisor on A. Let v be a place of K and let $\lambda_{D,v}$ be a Néron function on $A(\overline{K}_v)$ with respect to D. Let h be a height on $A(\overline{K})$ associated to an ample line bundle on A, and let $\kappa \in \mathbb{R}_{>0}$. Then there exist only finitely many K-rational points z in $A - D$ such that $\lambda_{D,v}(z) > \kappa \cdot h(z)$.*

Again let X be a smooth projective curve of positive genus g defined over K, let $x\colon X \to \mathbb{P}^1_K$ be a Galois covering map and assume that o is a totally ramified point for x such that $x(o) = \infty$. Let $J = \operatorname{Pic}^0 X$ be the jacobian of X, and let $\iota\colon X \to J$ be the Abel–Jacobi embedding given by $p \mapsto [p - o]$. Then we have a natural theta divisor Θ on J represented by the classes $[q_1 + \cdots + q_{g-1} - (g-1)o]$ for $q_1, \ldots, q_{g-1}$ running through X. We note that Θ is invariant under the automorphism group of $x\colon X \to \mathbb{P}^1_K$ acting on J in its natural fashion.

For $n \geq g$ any integer we define X_n to be the divisor $\iota^*[n]^*\Theta$ of X. This is an effective K-divisor on X of degree gn^2, as can be seen for example by noting that X_n coincides with the divisor of Weierstrass points of the line bundle $\mathcal{O}_X(o)^{\otimes n+g-1}$ as considered in [Ne]. The points in the support of X_n are called the *nth order Weierstrass points* of (X, o). By our remark above, X_n is invariant under the automorphism group of $x\colon X \to \mathbb{P}^1_K$. We note that $X_g = W + go$, where $W = \sum_{p \in X} w(p) \cdot p$ is the "classical" divisor of Weierstrass points of X determined by the Weierstrass weights $w(p)$ based on the gap sequence at p.

For $p \in X(\overline{K})$ we define

$$T(p) = \{n \in \mathbb{Z}_{\geq g} \mid p \notin X_n\} = \{n \in \mathbb{Z}_{\geq g} \mid n[p - o] \notin \Theta\}.$$

For example, if $g = 1$ then $\Theta = \{o\}$ and $T(p)$ is the set of positive integers in the complement of a subgroup of $\mathbb{Z}$.

LEMMA 6.2. *Let $p \in X(\overline{K})$. If $T(p)$ is not empty, then $T(p)$ contains infinitely many elements.*

Proof. For p such that $[p-o]$ is torsion in J the statement is immediate: assume $n_0[p-o] \notin \Theta$, then if k is the order of $[p-o]$ we can take those $n \geq g$ such that $n \equiv n_0 \bmod k$. Assume therefore that $[p-o]$ is not torsion in J. We prove that infinitely many points of the form $n[p-o]$ where $n \in \mathbb{Z}_{\geq 0}$ are not in Θ. Let Z^+ be the Zariski closure of the set $\mathbb{Z}_{\geq 0} \cdot [p-o]$ in J, and let Z be the Zariski closure in J of the subgroup $\mathbb{Z} \cdot [p-o]$ of J. Then Z is a closed algebraic subgroup of J, by Lemma 6.3 below, and using the involution $x \mapsto -x$ on J one sees that actually $Z^+ = Z$. Suppose that only finitely many of the $n[p-o]$ with $n \in \mathbb{Z}_{\geq 0}$ are outside Θ. Then $Z = Z^+$ is the union of a finite positive number of isolated points and a closed subset of Θ. It follows that Z has dimension zero, contradicting the assumption that $[p-o]$ is not torsion. ∎

LEMMA 6.3. *Let G be an algebraic group variety over a field k and let H be a subgroup of G. Then the Zariski closure of H in G is an algebraic subgroup of G.*

Proof. Let Z be the Zariski closure of H in G and for every h in H denote by $t_h Z$ the left translate of Z under h in G. As $t_h Z$ is closed in G and contains H we find that $t_h Z$ contains Z and in fact $t_h Z = Z$. This implies that H is contained in the stabiliser $\mathrm{Stab}(Z)$ of Z, which is a closed algebraic subgroup of G. We conclude that Z is contained in $\mathrm{Stab}(Z)$ and hence Z is itself an algebraic subgroup of G. ∎

Note that $T(p)$ can be empty for $p \neq o$: for example if $x\colon X \to \mathbb{P}^1_K$ is totally ramified at p as well and $N \leq g$. We refer to [SV] for a study of the sets $T(p)$ in a more general setting.

We have the following theorem.

THEOREM 6.4. *Let $p \in X(K) \setminus \{o\}$ be a rational point. Assume that p is not in X_n for at least one $n \geq g$. Let v be a place of K. Then*

$$\frac{1}{gn^2} \sum_{\substack{q \in X_n \\ x(q) \neq \infty}} \log |x(p) - x(q)|_v \to \int_{X_v} \log |x - x(p)|_v \, \mu_v$$

as $n \to \infty$ over the infinite set $T(p)$. In the sum on the left hand side, points are counted with multiplicity.

We note that Theorem 6.4 implies Theorem C. Indeed, if $n \in T(p)$ then $x(q) \neq x(p)$ for all $q \in X_n$ since X_n is invariant under the automorphism group of $x\colon X \to \mathbb{P}^1_K$ over $\overline{K}$, and this automorphism group acts transitively on each fiber of x.

The proof of Theorem 6.4 is based on the existence of an identity (6.1) below, between (generalized) functions on $X(\overline{K}_v)$. We obtain Theorem 6.4 by dividing by gn^2 and letting n tend to infinity over $T(p)$, using Faltings's result to see that $\lim_{n\to\infty} \lambda_{\Theta,v}(n[p-o])/gn^2 = 0$.

Let us make these ideas precise now.

PROPOSITION 6.5. *Let v be a place of K, and let $\lambda_{\Theta,v}$ be a Néron function with respect to Θ on $J(\overline{K}_v)$. There exists a polynomial $a(u) \in \overline{K}_v[u]$ such that for all integers n with $n \geq g$ and for all $p \in X(\overline{K}_v)$ with $p \notin X_n$,*

$$(6.1)\quad \log|a(n)|_v + \frac{1}{N}\sum_{\substack{q\in X_n\\ x(q)\neq\infty}} \log|x(p)-x(q)|_v = -\lambda_{\Theta,v}(n[p-o]) + gn^2\lambda_v(p).$$

In the sum, points are counted with their multiplicity.

Proof. Fix an integer $n \geq g$. First of all neglect the term $\log|a(n)|_v$ on the left hand side. Write $\ell_{n,v}(p)$ as shorthand for $\lambda_{\Theta,v}(n[p-o])$. One can view $L = \mathcal{O}_J(\Theta)$ as an adelic line bundle on J by putting $\|1\|_{L,v}(z) = \exp(-\lambda_{\Theta,v}(z))$, where 1 is the canonical global section of $\mathcal{O}_J(\Theta)$. By pullback one obtains a structure of adelic line bundle on each $L_n = \mathcal{O}_X(X_n) = \iota^*[n]^*\mathcal{O}_J(\Theta)$ given by $\|1\|_{L_n,v}(p) = \exp(-\ell_{n,v}(p))$ where now 1 is the canonical global section of $\mathcal{O}_X(X_n)$. By [Zh, Section 4.7] (see also [He, Section 4]) the resulting adelic metric is admissible; in particular $\ell_{n,v}(p)$ is equal to the admissible pairing $(p, X_n)_a$ up to an additive constant. As a result $\ell_{n,v}$ extends to $\mathcal{D}^0(X_v)$, the space of $(0,0)$-currents on X_v. As the other terms in the equality to be proven do so as well, we try to prove the equality as an identity in $\mathcal{D}^0(X_v)$. By Proposition 2.1(iii) we are done once we prove that both sides of the claimed equality have the same image under dd^c, and the difference of both sides tends to zero as p tends to o over $X(\overline{K}_v)$ avoiding X_n.

From the observation that $\|1\|_{L_n,v}(p) = \exp(-\ell_{n,v}(p))$ defines an admissible metric on L_n we obtain first of all

$$dd^c\,\ell_{n,v} = (\deg X_n)\mu_v - \delta_{X_n} = gn^2\mu_v - \delta_{X_n}.$$

We deduce from Proposition 4.3 that

$$dd^c\,\lambda_v = \mu_v - \delta_o.$$

Finally by the Poincaré–Lelong equation of Proposition 2.1(i) we have

$$dd^c\,\frac{1}{N}\sum_{\substack{q\in X_n\\ x(q)\neq\infty}} \log|x(p)-x(q)|_v = \delta_{X_n} - gn^2\delta_o,$$

and hence the difference of both sides of (6.1) vanishes under dd^c. Now we consider the behavior of both left and right hand sides as $p \to o$ avoiding X_n. Let $z_1,\ldots,z_g$ be local coordinates around the origin on $J(\overline{K}_v)$, and let $s \in \overline{K}_v[[z_1,\ldots,z_g]]$ be a local equation for Θ such that

$$\lambda_{\Theta,v}(z) + \log|s(z_1,\ldots,z_g)|_v \to 0$$

as $z \to 0$. Let t be a local equation for o on $X(\overline{K}_v)$ and write $\iota^* z_j = a_j t^{m_j}(1+O(t))$ with $a_j \in \overline{K}_v$, $m_j \in \mathbb{Z}_{>0}$. We have $[n]^* z_j \equiv n z_j \bmod (z_1,\ldots,z_g)^2$ and it follows that there exists a polynomial $a(u) \in \overline{K}_v[u]$ and $m \in \mathbb{Z}_{>0}$ such that

$$\iota^*[n]^* s(z_1,\ldots,z_g) = a(n)t^m(1+O(t))$$

for all integers $n \geq g$. Note that m is the multiplicity of o in X_n; it is independent of n. We can assume without loss of generality that $t^{-N} = x$. We find:

$$\begin{aligned} \lambda_v(p) &\to -\log|t(p)|_v, \\ \lambda_{\Theta,v}(n[p-o]) + \log|a(n)|_v &\to -m\log|t(p)|_v, \\ \frac{1}{N}\sum_{\substack{q\in X_n \\ x(q)\neq\infty}} \log|x(p)-x(q)|_v &\to -(gn^2-m)\log|t(p)|_v \end{aligned}$$

as $p \to o$ avoiding X_n. The proposition follows by combining these asymptotics. ∎

REMARK 6.6. Let (X, o) be an elliptic curve. We can choose $\lambda_{\Theta,v}$ to be λ_v itself; then $a(n) = n$ and the left hand side of (6.1) is equal to $\log|\psi_n(x(p))|_v$ with $\psi_n \in K[x]$ the nth division polynomial of (X, o) with respect to x (cf. [Si1, Exercise 3.7]). One finds the identity

$$\log|\psi_n(x(p))|_v = -\lambda_v(np) + n^2\lambda_v(p),$$

which seems to be well known.

REMARK 6.7. If (X, o) is a hyperelliptic curve and v is archimedean, one finds in [Ôn, Theorem 8.3] a holomorphic analogue of (6.1), based on Klein's hyperelliptic sigma-function. It would be interesting to generalize the result from [Ôn] to the case of more general (X, o).

Proof of Theorem 6.4. Let $p \in X(K)\backslash\{o\}$ be a point such that $T(p)$ is infinite, and let v be a place of K. By Proposition 6.5 we are done once we prove that $\log|a(n)|_v/n^2 \to 0$ as $n \to \infty$ and $\lambda_{\Theta,v}(n[p-o])/n^2 \to 0$ as $n \to \infty$ over $T(p)$. The first statement is immediate since $a(n)$ is a polynomial in n. As to the second statement, note that it follows immediately if $[p-o]$ is torsion since then the set of values $\lambda_{\Theta,v}(n[p-o])$ as n ranges over $T(p)$ is bounded. Assume therefore that $[p-o]$ is not torsion. Then the $n[p-o]$ with n running through $T(p)$ form an infinite set of K-rational points of $J - \Theta$. Since

$$\frac{\lambda_{\Theta,v}(n[p-o])}{n^2} = h([p-o]) \cdot \frac{\lambda_{\Theta,v}(n[p-o])}{h(n[p-o])}$$

with $h([p-o]) > 0$, Faltings's Theorem 6.1 can be applied to give

$$\limsup_{\substack{n\to\infty \\ n\in T(p)}} \frac{\lambda_{\Theta,v}(n[p-o])}{n^2} \leq 0.$$

On the other hand $\lambda_{\Theta,v}$ is bounded from below so that

$$\liminf_{\substack{n\to\infty \\ n\in T(p)}} \frac{\lambda_{\Theta,v}(n[p-o])}{n^2} \geq 0.$$

The result follows. ∎

We finish with an application of Theorem 6.4. Let (X,o) be a hyperelliptic curve over K of genus $g \geq 2$ and let $y^2 = f(x)$ with $f \in O_K[x]$ monic of degree $2g+1$ be an equation for (X,o) putting o at infinity. Here O_K denotes the ring of integers of K. In [Ca] polynomials $\psi_n \in O_K[x]$ are constructed with leading coefficient polynomially growing in n and with zero divisor on X equal to $X_n - m \cdot o$, where m is the multiplicity of o in X_n (actually $m = g(g+1)/2$). The sequence $(\psi_n)_n$ is determined by a non-linear recurrence relation.

THEOREM 6.8. *Assume that $p = (x(p), y(p)) \in O_K^2$ is an o-integral point of X, with $y(p) \neq 0$. Let S be a finite set of places of K containing the places of bad reduction for X as well as the infinite ones. Then*

$$[K:\mathbb{Q}]h([p-o]) = \lim_{\substack{n\to\infty \\ n\in T(p)}} \frac{1}{gn^2} \log \prod_{v\in S} |\psi_n(x(p))|_v^{n_v}.$$

Proof. Let $\hat{\lambda}_v$ be the renormalized version of λ_v introduced in Section 5. By the product formula and Proposition 4.1 we can write

$$[K:\mathbb{Q}]h([p-o]) = \sum_{v\in M_K} n_v\, \hat{\lambda}_v(p).$$

By Theorem 4.3, for the local admissible pairing $(p,o)_a$ we have

$$\hat{\lambda}_v(p) = (p,o)_a + \int_{X_v} \hat{\lambda}_v\, \mu_v.$$

By the results in Section 5 we have $\int_{X_v} \hat{\lambda}_v\, \mu_v = 0$ if v is a place of good reduction. Further we have $(p,o)_a = 0$ if v is a place of good reduction since p is o-integral by assumption. We obtain

$$[K:\mathbb{Q}]h([p-o]) = \sum_{v\in S} n_v\, \hat{\lambda}_v(p).$$

By assumption p is not a Weierstrass point of X. Hence according to Theorem 6.4 we have

$$\hat{\lambda}_v(p) = -\frac{1}{4g(2g+1)} \log|\Delta|_v + \lim_{\substack{n\to\infty \\ n\in T(p)}} \frac{1}{gn^2} \log|\psi_n(x(p))|_v$$

for any place v of K. By combining these two formulas and interchanging the limit and the (finite) sum we obtain the theorem, upon noting that

$$\sum_{v\in S} n_v \log|\Delta|_v = 0$$

by the product formula and the fact that $\log|\Delta|_v = 0$ if $v \notin S$. ∎

Theorem 6.8 generalizes the main result of [EW], which is the analogue in the case of elliptic curves. For points p that are not necessarily o-integral one has to make sure that the set S contains the primes of K where p reduces to o; then the same formula works.

The limit formula in Theorem 6.8 gives a method, in principle, of approximating to high precision Néron–Tate heights of points on X without exhibiting a model for the jacobian of X. Here one uses the results of [Ca] to compute recursively the sequence of division polynomials ψ_n. Note that in order to have the formula in Theorem 6.8 work in practice one has to know in advance that the gaps in the sequence $T(p)$ are bounded independent of p. Here is an argument that indicates that a gap between two consecutive integers in $T(p)$ should not be larger than g: a gap of length $g+1$ would give rise to an element in an intersection $\Theta \cap \Theta_{[p-o]} \cap \cdots \cap \Theta_{g[p-o]}$ of $g+1$ translates of Θ. These translates are distinct if p is not a Weierstrass point. The intersection should be empty for dimension reasons.

Acknowledgements. The research done for this paper was supported by VENI grant 639.033.402 from the Netherlands Organisation for Scientific Research (NWO). Part of the research was done at the Max Planck Institute in Bonn, whose hospitality is gratefully acknowledged.

References

[Ba] M. Baker, *An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves*, in: p-Adic Geometry. Lectures from the 2007 Arizona Winter School, Univ. Lecture Ser. 45, Amer. Math. Soc., Providence, RI, 2008, 123–174.

[Be] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Math. Surveys Monogr. 33, Amer. Math. Soc., Providence, RI, 1990.

[Br] P. Bradley, *Cyclic coverings of the p-adic projective line by Mumford curves*, Manuscripta Math. 124 (2007), 77–95.

[CS] G. Call and J. Silverman, *Canonical heights on varieties with morphisms*, Compos. Math. 89 (1993), 163–205.

[Ca] D. Cantor, *On the analogue of the division polynomials for hyperelliptic curves*, J. Reine Angew. Math. 447 (1994), 91–145.

[CR] T. Chinburg and R. Rumely, *The capacity pairing*, ibid. 434 (1993), 1–44.

[EF] G. Everest and B. ní Fhlathúin, *The elliptic Mahler measure*, Math. Proc. Cambridge Philos. Soc. 120 (1996), 13–25.

[EW] G. Everest and T. Ward, *The canonical height of an algebraic point on an elliptic curve*, New York J. Math. 6 (2000), 331–342.

[Fa] G. Faltings, *Diophantine approximation on abelian varieties*, Ann. of Math. 133 (1991), 549–576.

[He] N. Heinz, *Admissible metrics for line bundles on curves and abelian varieties over non-archimedean local fields*, Arch. Math. (Basel) 82 (2004), 128–139.

[dJ] R. de Jong, *Symmetric roots and admissible pairing*, Trans. Amer. Math. Soc. 363 (2011), 4263–4283.

[Lo] P. Lockhart, *On the discriminant of a hyperelliptic curve*, ibid. 342 (1994), 729–752.

[Ne] A. Neeman, *The distribution of Weierstrass points on a compact Riemann surface*, Ann. of Math. 120 (1984), 317–328.

[Ôn] Y. Ônishi, *Determinant expressions for hyperelliptic functions* (with an appendix by S. Matsutani), Proc. Edinburgh Math. Soc. 48 (2005), 705–742.

[PST] J. Pineiro, L. Szpiro and T. Tucker, *Mahler measure for dynamical systems on* $\mathbb{P}^1$ *and intersection theory on a singular arithmetic surface*, in: F. Bogomolov and Y. Tschinkel (eds.), Geometric Methods in Algebra and Number Theory, Progr. Math. 235, Birkhäuser, Boston, 2005, 219–250.

[Se] J.-P. Serre, *Lectures on the Mordell–Weil Theorem*, Aspects Math. E15, Vieweg, Braunschweig, 1989.

[Si1] J. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer, 1986.

[Si2] —, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, New York, 1994.

[SV] J. Silverman and J. Voloch, *Multiple Weierstrass points*, Compos. Math. 79 (1991), 123–134.

[ST] L. Szpiro and T. Tucker, *Equidistribution and generalized Mahler measures*, Serge Lang Memorial Volume, to appear.

[Th] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*, Ph.D. thesis, Univ. de Rennes, 2005.

[Zh] S. Zhang, *Admissible pairing on a curve*, Invent. Math. 112 (1993), 171–193.

Robin de Jong
Mathematical Institute
University of Leiden
P.O. Box 9512
2300 RA Leiden, The Netherlands
E-mail: rdejong@math.leidenuniv.nl

Received on 1.2.2011
and in revised form on 20.6.2011 (6611)