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On the limit distribution of Frobenius numbers

by

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1. Introduction. We denote by $\widehat{\mathbb{N}}^d$ the set of integer vectors in $\mathbb{R}^d$ with positive coprime coefficients (viz. the greatest common divisor of all coefficients is one). Given $\boldsymbol{a} = (a_1, \ldots, a_d) \in \widehat{\mathbb{N}}^d$, the *Frobenius number* $g(\boldsymbol{a}) = g(a_1, \ldots, a_d)$ is defined as the largest integer which is not representable as a non-negative integer combination of $a_1, \ldots, a_d$. The problem of computing $g(\boldsymbol{a})$ is known as the *Frobenius problem* or the *coin exchange problem*, and it has been studied extensively (see, e.g., [23] and [16, Problem C7]).

In the majority of problems related to Frobenius numbers, it is more convenient to consider the function

$$f(\boldsymbol{a}) = f(a_1, \ldots, a_d) = g(a_1, \ldots, a_d) + a_1 + \cdots + a_d. \tag{1.1}$$

Clearly, $f(\boldsymbol{a})$ is the largest integer which is not a *positive* integer combination of $a_1, \ldots, a_d$.

In the case of two variables, $d = 2$, the Frobenius number is given by Sylvester's formula ([23, Theorem 2.1.1]),

$$g(a_1, a_2) = a_1 a_2 - a_1 - a_2 \quad (\text{viz., } f(a_1, a_2) = a_1 a_2). \tag{1.2}$$

For $d \geq 3$ no explicit formula is known. Arnold [4]–[6] asked about the behavior of $g(a_1, \ldots, a_d)$ for a "random" large vector $(a_1, \ldots, a_d) \in \mathbb{R}^d$. Davison had previously asked similar questions for $d = 3$, in [11, Sec. 5]. Recently Marklof [19] obtained a definitive result for arbitrary $d \geq 3$, generalizing previous results by Bourgain and Sinai [9] in the case $d = 3$ (cf. also Shchur, Sinai and Ustinov [32]):

THEOREM 1 (Marklof [19]). *Given $d \geq 3$, there exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ with non-empty interior and boundary of Lebesgue*

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measure zero, and any $R \geq 0$,

$$\lim_{T\to\infty} \frac{1}{\#(\widehat{\mathbb{N}}^d \cap T\mathcal{D})} \#\left\{ \boldsymbol{a} \in \widehat{\mathbb{N}}^d \cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} = \Psi_d(R). \tag{1.3}$$

For arbitrary $d \geq 3$, Li [18, Thm. 1.3] has recently obtained an effective version of Theorem 1, where (1.3) is proved to hold with a power convergence rate (with respect to T).

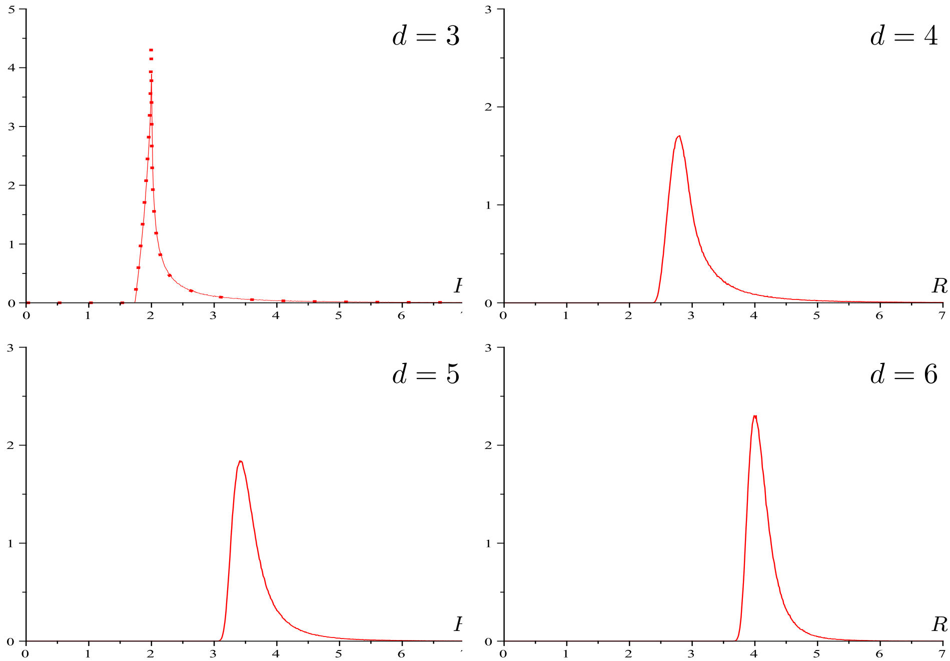

Fig. 1. Experimental graphs of the density functions $\psi_d(R) = -\frac{d}{dR}\Psi_d(R)$ of the limit distribution in Theorem 1, for $d = 3, 4, 5, 6$. The graphs were obtained by computing $(a_1 \cdots a_d)^{-1/(d-1)} f(\boldsymbol{a})$ for $1.2 \cdot 10^6$ integer vectors $\boldsymbol{a}$ picked at random in $\widehat{\mathbb{N}}^d \cap [0, T]^d$ with $T = 10^{15}$, and collecting the results into bins of width 0.01 along the R-axis. The computations of $f(\boldsymbol{a})$ were performed using the Frobby software package by Roune [28]; cf. also [29]. We repeated the computations using other random seeds and/or changing T to 10^{14}, as well as to $10^{13}, 10^{12}, 10^{11}$ in some cases, and the resulting graphs were consistently found to be practically indistinguishable, except for $d = 3$ and R very near 2. For $d = 3$ also the graph of the exact function in (1.7) is drawn (the dotted curve, which is distinguishable from the experimental graph only for R very near 2).

Marklof also proved an explicit formula for $\Psi_d(R)$, namely that $\Psi_d(R)$ equals the probability that the simplex

$$\Delta = \{\boldsymbol{x} \in \mathbb{R}^{d-1}_{\geq 0} : \boldsymbol{x} \cdot \boldsymbol{e} \leq 1\}, \qquad \boldsymbol{e} := (1, \ldots, 1), \tag{1.4}$$

has covering radius larger than R with respect to a random lattice $L \subset \mathbb{R}^{d-1}$ of covolume one. In other words ([19, Thm. 2]),

$$\Psi_d(R) = \mu_{d-1}(\{L \in X_{d-1} : \rho(L) > R\}), \tag{1.5}$$

where X_{d-1} is the set of all lattices $L \subset \mathbb{R}^{d-1}$ of covolume one, μ_{d-1} is Siegel's measure ([33]) on X_{d-1}, normalized to be a probability measure, and $\rho(L)$ is the covering radius of Δ with respect to L, viz.

$$\rho(L) = \inf\{\rho > 0 : L + \rho\Delta = \mathbb{R}^{d-1}\}. \tag{1.6}$$

In the special case $d = 3$, Ustinov [37] (cf. also [36]) proved a more precise version of (1.3), where the averaging is performed over only two of the three arguments a_1, a_2, a_3, and the limit is obtained with a power rate of convergence. Ustinov in fact gave a completely explicit formula for the limit density $\psi_3(R) = -\frac{d}{dR}\Psi_3(R)$ in terms of elementary functions:

$$\psi_3(R) = \begin{cases} 0 & (0 \le R \le \sqrt{3}), \\ \dfrac{12}{\pi}\left(\dfrac{R}{\sqrt{3}} - \sqrt{4-R^2}\right) & (\sqrt{3} \le R \le 2), \\ \dfrac{12}{\pi^2}\left(R\sqrt{3}\arccos\left(\dfrac{R+3\sqrt{R^2-4}}{4\sqrt{R^2-3}}\right) \right. & \\ \qquad \left. + \dfrac{3}{2}\sqrt{R^2-4}\log\left(\dfrac{R^2-4}{R^2-3}\right)\right) & (R > 2). \end{cases} \tag{1.7}$$

See also [22] for a derivation of (1.7) from (1.5).

Our purpose in the present note is to discuss the behavior of $\Psi_d(R)$ for d fixed and R large, as well as for d large. For fixed $d \ge 3$, it was proved by Li [18] that $\Psi_d(R) \ll_d R^{-(d-1)}$ for all $R > 0$, and Marklof in an unpublished note [20] pointed out that a corresponding lower bound also holds: $\Psi_d(R) \gg_d R^{-(d-1)}$ for all $R \ge 1$. Our first result, which we will prove in Section 2, is an asymptotic formula refining these bounds:

THEOREM 2. *Let $d \ge 3$. Then*

$$\Psi_d(R) = \frac{d}{2\zeta(d-1)}R^{-(d-1)} + O_d(R^{-d-1/(d-2)}) \quad \text{as } R \to \infty. \tag{1.8}$$

Here the error term is sharp; in fact there exists a constant $c > 0$ which only depends on d, such that for all sufficiently large R,

$$\Psi_d(R) > \frac{d}{2\zeta(d-1)}R^{-(d-1)} + cR^{-d-1/(d-2)}. \tag{1.9}$$

In particular we may note that (1.7) implies $\Psi_3(R) = \frac{9}{\pi^2}R^{-2} + \frac{33}{2\pi^2}R^{-4} + O(R^{-6})$ as $R \to \infty$, which is consistent with Theorem 2.

Combining Theorems 1 and 2 we conclude that if R is large, and if $\boldsymbol{a}$ is picked at random from a set of the type $\widehat{\mathbb{N}}^d \cap T\mathcal{D}$ with T sufficiently large—where the notion of "sufficiently large" may depend on R—then the probability that the normalized Frobenius number $\frac{f(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}}$ is greater than R is approximately $\frac{d}{2\zeta(d-1)}R^{-(d-1)}$. It is an interesting problem to try to get a more uniform control on the probability of $\frac{f(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}}$ being large, i.e. to give bounds from above and below, *uniformly with respect to large T and R,* on

$$P_d(T,R) := \frac{1}{\#(\widehat{\mathbb{N}}^d\cap T\mathcal{D})}\#\Big\{\boldsymbol{a}\in\widehat{\mathbb{N}}^d\cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}} > R\Big\}. \tag{1.10}$$

Results related to this question have recently been obtained by Aliev and Henk [2] and Aliev, Henk and Hinrichs [3], by making use of Schmidt's results on the distribution of similarity classes of sublattices of $\mathbb{Z}^m$ [31]. We will show that the application of [31] can be refined—using in particular the strong uniform error bounds which Schmidt provides for his asymptotic formulas—so as to give a uniform bound which significantly improves upon the bounds obtained in [2], [3], and which can be viewed as a T*-uniform* version of Li's upper bound $\Psi_d(R)\ll_d R^{-(d-1)}$.

For technical reasons we will consider the Frobenius number normalized not with the factor $(a_1\cdots a_d)^{-1/(d-1)}$, but with $s(\boldsymbol{a})^{-1}$, where

$$s(\boldsymbol{a}) := \frac{\sum_{j=1}^d a_j\sqrt{\|\boldsymbol{a}\|^2-a_j^2}}{\|\boldsymbol{a}\|^{1-1/(d-1)}}, \tag{1.11}$$

with $\|\boldsymbol{a}\|$ denoting the standard Euclidean norm of $\boldsymbol{a}$. Thus, we set

$$\widetilde{P}_d(T,R) := \frac{1}{\#(\widehat{\mathbb{N}}^d\cap T\mathcal{D})}\#\Big\{\boldsymbol{a}\in\widehat{\mathbb{N}}^d\cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} > R\Big\}. \tag{1.12}$$

Note that $P_d(T,R)$ and $\widetilde{P}_d(T,R)$ are defined for any $T>0$ such that $\widehat{\mathbb{N}}^d\cap T\mathcal{D}\neq\emptyset$; in particular, for any fixed $\mathcal{D}\subset\mathbb{R}^d_{\geq 0}$ with non-empty interior, $P_d(T,R)$ and $\widetilde{P}_d(T,R)$ are defined for all $T\gg_{\mathcal{D}} 1$.

The normalizing factor $s(\boldsymbol{a})$ was used also in Aliev and Henk [2]; cf. also Fukshansky and Robins [13]. Note that if we assume that the coefficients of $\boldsymbol{a}$ are ordered so that $a_1\leq\cdots\leq a_d$ then $s(\boldsymbol{a})\asymp_d a_{d-1}a_d^{1/(d-1)}$; in particular we have

$$(a_1\cdots a_d)^{1/(d-1)}\ll_d s(\boldsymbol{a})\ll_d \|\boldsymbol{a}\|^{d/(d-1)},\qquad \forall\boldsymbol{a}\in\mathbb{R}^d_{>0}. \tag{1.13}$$

Hence there exists a constant $c_1>0$ which only depends on d such that

$$\widetilde{P}_d(T,c_1R)\leq P_d(T,R) \tag{1.14}$$

for any $R>0$ and any $\mathcal{D}\subset\mathbb{R}^d_{\geq 0}$ and $T>0$ such that $\widehat{\mathbb{N}}^d\cap T\mathcal{D}\neq\emptyset$. On the other hand, if $\mathcal{D}$ is bounded and satisfies $\overline{\mathcal{D}}\subset\mathbb{R}^d_{>0}$, then $s(\boldsymbol{a})\asymp$

$(a_1 \cdots a_d)^{1/(d-1)}$ holds uniformly over all $\boldsymbol{a} \in \mathbb{R}_{>0}\mathcal{D}$, and thus we have $P_d(T,R) \leq \widetilde{P}_d(T, c_2 R)$ for all $T, R > 0$ with $\widehat{\mathbb{N}}^d \cap T\mathcal{D} \neq \emptyset$, where $c_2 > 0$ is a constant which only depends on $\mathcal{D}$. Hence for any such region $\mathcal{D}$, any of the two functions $P_d(T,R)$ and $\widetilde{P}_d(T,R)$ can essentially be bounded in terms of the other, as long as we allow an implied constant which may depend on $\mathcal{D}$.

Our main result on $\widetilde{P}_d(T,R)$ is the following bound, which we will prove in Section 3.

THEOREM 3. *Let $d \geq 3$, and let $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ be bounded with non-empty interior. Then*

$$\widetilde{P}_d(T,R) \ll_{d,\mathcal{D}} R^{-(d-1)} \tag{1.15}$$

uniformly over all $T > 0$ with $\widehat{\mathbb{N}}^d \cap T\mathcal{D} \neq \emptyset$, and all $R > 0$. Furthermore, for any such T,

$$\widetilde{P}_d(T,R) = 0 \quad \text{whenever} \quad R \geq (T \sup_{\boldsymbol{x}\in\mathcal{D}} \|\boldsymbol{x}\|)^{1-1/(d-1)}. \tag{1.16}$$

Theorem 3 strengthens the bound $\widetilde{P}_d(T,R) \ll R^{-2}$ which was given in [2, Thm. 1.1]. Note also that if the set $\mathcal{D}$ satisfies $\overline{\mathcal{D}} \subset \mathbb{R}^d_{>0}$, then by the previous discussion Theorem 3 implies $P_d(T,R) \ll_{d,\mathcal{D}} R^{-(d-1)}$.

From many points of view, the normalization factor $(a_1 \cdots a_d)^{-1/(d-1)}$ is the most natural one to use in the Frobenius problem. A clear indication of this is for example the fact that the limit distribution obtained in Theorem 1 is independent of the choice of $\mathcal{D}$. Hence it is interesting to ask whether the bound in Theorem 3 is valid also for $P_d(T,R)$, *without* the extra assumption $\overline{\mathcal{D}} \subset \mathbb{R}^d_{>0}$. We conjecture that this is so. However in the present paper we will content ourselves with pointing out a weaker bound, which follows fairly directly from Theorem 3 by an argument along the lines of [3], and which strengthens the bound (1) $P_d(T,R) \ll R^{-2\frac{d-1}{d+1}+\varepsilon}$ obtained in [3].

COROLLARY 1. *Let $d \geq 3$, and let $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ be bounded with non-empty interior. Then*

$$P_d(T,R) \ll_{d,\mathcal{D}} R^{-(d-1)/2}(\log(R+2))^{(d-3)/2} \tag{1.17}$$

uniformly over all $T > 0$ with $\widehat{\mathbb{N}}^d \cap T\mathcal{D} \neq \emptyset$, and all $R > 0$. Furthermore,

$$P_d(T,R) = 0 \quad \text{whenever} \quad R \geq d(T \sup_{\boldsymbol{x}\in\mathcal{D}} \|\boldsymbol{x}\|)^{1-1/(d-1)}. \tag{1.18}$$

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(1) We here correct a mistake in [3, p. 530, lines 5–6] by adding ε in the exponent: In the notation of [3], the choice of $t = \frac{n-1}{n+1}$ yields the bound $\beta^{-2\frac{(n-1)^2}{n(n+1)}}$ and not $\beta^{-2\frac{n-1}{n+1}}$ as claimed; choosing t optimally yields the bound $\beta^{-2\frac{(n-1)^2}{n^2+1}}$, and using also [3, p. 529, Remark 1] brings the bound down to $\beta^{-2\frac{n-1}{n+1}+\varepsilon}$.

We remark that in the special case $d = 3$, it follows from Ustinov [37, pp. 1025, 1044] that the stronger bound $P_3(T,R) \ll_{\mathcal{D}} R^{-2}$ is valid at least so long as we keep $T \gg R^{22+\varepsilon}$.

It is also interesting to consider the *moments* of the (normalized) Frobenius number; in particular the *expected value* has been considered by many authors (cf., e.g., [3]–[6], [11, Sec. 5], [36]). Note that it follows from Theorem 2 (or just from the upper and lower bounds by Li [18] and Marklof [20]) that the limit distribution described by $\Psi_d(R)$ possesses qth moment for $0 < q < d-1$ (q not necessarily an integer), and for no larger q. Let us write $M_{d,q}$ for this moment:

$$M_{d,q} := -\int_0^\infty R^q \, d\Psi_d(R) = q \int_0^\infty R^{q-1}\Psi_d(R)\, dR, \qquad 0 < q < d-1. \tag{1.19}$$

Now the following is an easy consequence of Theorem 1 combined with Theorem 3 and Corollary 1.

Corollary 2. *Let $d \geq 3$, and let $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ be a bounded set with non-empty interior and boundary of Lebesgue measure zero. Then for any q in the interval $0 < q < \frac{1}{2}(d-1)$, we have convergence of moments:*

$$\lim_{T\to\infty} \frac{1}{\#(\widehat{\mathbb{N}}^d \cap T\mathcal{D})} \sum_{\boldsymbol{a}\in \widehat{\mathbb{N}}^d \cap T\mathcal{D}} \left(\frac{f(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}}\right)^q = M_{d,q}. \tag{1.20}$$

If furthermore $\overline{\mathcal{D}} \subset \mathbb{R}^d_{>0}$, then (1.20) *holds for all $0 < q < d-1$.*

We expect that (1.20) should hold for all $0 < q < d-1$ also without the extra assumption $\overline{\mathcal{D}} \subset \mathbb{R}^d_{>0}$ (indeed this would follow from our previous conjecture that $P_d(T,R) \ll_{d,\mathcal{D}} R^{-(d-1)}$ holds for general $\mathcal{D}$). We note that in the special case $d = 3$ and $q = 1$, (1.20) *does* hold for general $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ (bounded with non-empty interior); this follows from Ustinov [36, Thm. 1]. We also remark that for $d \geq 4$ and $q = 1$, (1.20) was proved in [3].

Finally let us turn to a slightly different question: What can be said about the limit distribution of Frobenius numbers for d *large*? Let ρ_{d-1} be the absolute inhomogeneous minimum of Δ, viz.

$$\rho_{d-1} = \inf\{\rho(L) : L \in X_{d-1}\}. \tag{1.21}$$

Using (1.5) and the fact that Ψ_d is continuous ([19, Lemma 7]), one easily shows that

$$\Psi_d(R) = 1 \quad \text{for } 0 \leq R \leq \rho_{d-1}, \quad \text{and} \quad \Psi_d(R) < 1 \quad \text{for } R > \rho_{d-1}, \tag{1.22}$$

i.e. the limit distribution described by $\Psi_d(R)$ has support exactly in the interval $[\rho_{d-1}, \infty)$. In fact ρ_{d-1} is not only a lower bound for the support of the limit distribution, but also a lower bound on the normalized Frobenius number for *any* input vector; we have

$$\frac{f(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} \geq \rho_{d-1}, \qquad \forall \boldsymbol{a} \in \widehat{\mathbb{N}}^d \tag{1.23}$$

(cf. Aliev and Gruber [1, Thm. 1.1(i)] as well as Rödseth [24]). It was noted in [1, (7)] that

$$\rho_{d-1} > (d-1)!^{1/(d-1)}. \tag{1.24}$$

On the other hand the number ρ_{d-1} is quite near $(d-1)!^{1/(d-1)}$ for d large: It follows from a bound by Rogers [27] on lattice coverings by general convex bodies, refined by Gritzmann [14] in the case of convex bodies satisfying a mild symmetry condition (cf. also [12, Sec. 9], and use the fact that Δ can be mapped to a regular $(d-1)$-simplex by a volume preserving linear map), that

$$\rho_{d-1} \leq (d-1)!^{1/(d-1)}\left(1 + O\left(\frac{\log d}{d}\right)\right) \qquad \text{as } d \to \infty. \tag{1.25}$$

When computing the Frobenius numbers for modest d and several random large vectors $\boldsymbol{a}$, one notes that the normalized values $f(\boldsymbol{a})/(a_1 \cdots a_d)^{1/(d-1)}$ most often do not exceed the experimental value for the lower bound ρ_{d-1} by more than a constant factor < 2. This is seen in Figure 1 above in the cases $d = 3, 4, 5, 6$; the same phenomenon was also noted in [7, Sec. 5 (esp. Fig. 17)] for $d = 4$ and $d = 8$. The following result shows that this behavior continues as $d \to \infty$; indeed, for d large, the distribution described by $\Psi_d(R)$ has almost all of its mass concentrated in the interval between $(d-1)!^{1/(d-1)}$ and $1.757 \cdot (d-1)!^{1/(d-1)}$.

THEOREM 4. *Let $\eta_0 = 0.756\ldots$ be the unique real root of $e \log \eta + \eta = 0$. Then for any $\alpha > 1 + \eta_0$ we have*

$$\Psi_d(\alpha(d-1)!^{1/(d-1)}) \to 0 \qquad \textit{as } d \to \infty. \tag{1.26}$$

We remark that the proof will show that (1.26) holds with an exponential rate, for any fixed $\alpha > 1 + \eta_0$.

Combining Theorem 4 with Theorem 1 and (1.24) implies that for large d, the normalized Frobenius number $f(\boldsymbol{a})/(a_1 \cdots a_d)^{1/(d-1)}$ is very likely to lie between $(d-1)!^{1/(d-1)}$ and $1.757 \cdot (d-1)!^{1/(d-1)}$. In precise terms, for any fixed $\alpha > \eta_0$ we have

$$\lim_{d\to\infty} \liminf_{T\to\infty} \frac{1}{\#(\widehat{\mathbb{N}}^d \cap [0,T]^d)} \#\Big\{\boldsymbol{a} \in \widehat{\mathbb{N}}^d \cap [0,T]^d : \\ (d-1)!^{1/(d-1)} < \frac{f(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} < \alpha(d-1)!^{1/(d-1)}\Big\} = 1. \tag{1.27}$$

Theorem 4 follows from a modification of a general bound by Rogers on lattice coverings of space with convex bodies [25], further improved by Schmidt [30]. We carry this out in Section 4 below.

Remark 1. It is an interesting question whether the bound on α in Theorem 4 can be further improved. Could it be that the limit distribution of Frobenius numbers in fact concentrates near $(d-1)!^{1/(d-1)}$ as $d \to \infty$, in the sense that (1.26) holds for *all* $\alpha > 1$?

It is also an interesting task to try to prove a good *uniform* bound on $\Psi_d(R)$ valid for all large d and R, uniting Theorem 4 and the fact that $\Psi_d(R) \ll_d R^{-(d-1)}$ as $R \to \infty$. Even more generally, we may ask for a good uniform bound on $P_d(T,R)$ valid for all large d, T, R.

2. The asymptotic behavior of $\Psi_d(R)$ as $R \to \infty$. In this section we will prove Theorem 2.

2.1. Preliminaries. Let us write $n = d-1$. Recall that Δ denotes the standard n-dimensional simplex defined in (1.4). Given $L \in X_n$ and $\rho > 0$, we have $L + \rho\Delta = \mathbb{R}^n$ if and only if $\boldsymbol{\zeta} - \rho\Delta$ has non-empty intersection with L for each $\boldsymbol{\zeta} \in \mathbb{R}^n$. Thus, since $L = -L$,

$$\rho(L) = \sup\{\rho > 0 : \text{there is } \boldsymbol{\zeta} \in \mathbb{R}^n \text{ such that } L \cap (\rho\Delta - \boldsymbol{\zeta}) = \emptyset\}. \tag{2.1}$$

It follows that the formula for $\Psi_d(R)$, (1.5), may be rewritten as

$$\Psi_d(R) = \mu_n(\{L \in X_n : \text{there is } \boldsymbol{\zeta} \in \mathbb{R}^n \text{ such that } L \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset\}). \tag{2.2}$$

Let us write $G = G^{(n)} = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \Gamma^{(n)} = \mathrm{SL}(n,\mathbb{Z})$. For any $M \in G$, $\mathbb{Z}^n M$ is an n-dimensional lattice of covolume one, and this gives an identification of the space X_n with the homogeneous space $\Gamma\backslash G$. Note that μ_n is the measure on X_n coming from Haar measure on G, normalized to be a probability measure; we write μ_n also for the corresponding Haar measure on G. Let $A = A^{(n)}$ be the subgroup of G consisting of the diagonal matrices with positive entries

$$\mathsf{a}(a) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in G, \qquad a_j > 0, \tag{2.3}$$

and let $N = N^{(n)}$ be the subgroup of upper triangular matrices

$$\mathsf{n}(u) = \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & u_{n-1,n} \\ & & & 1 \end{pmatrix} \in G. \tag{2.4}$$

Every element $M \in G$ has a unique Iwasawa decomposition

$$M = \mathsf{n}(u)\mathsf{a}(a)\mathsf{k} \tag{2.5}$$

with $\mathsf{k} \in \mathrm{SO}(n)$. We set

$$\mathcal{F}_N = \{u : u_{jk} \in (-1/2, 1/2],\, 1 \le j < k \le n\}; \tag{2.6}$$

then $\{\mathsf{n}(u) : u \in \mathcal{F}_N\}$ is a fundamental region for $(\Gamma \cap N)\backslash N$. We define the following Siegel set:

$$\begin{aligned}\mathcal{S}_n := \big\{&\mathsf{n}(u)\mathsf{a}(a)\mathsf{k} \in G : u \in \mathcal{F}_N,\\ &0 < a_{j+1} \le \tfrac{2}{\sqrt{3}} a_j \ (j = 1, \ldots, n-1),\ \mathsf{k} \in \mathrm{SO}(n)\big\}.\end{aligned} \tag{2.7}$$

It is known that $\mathcal{S}_n$ contains a fundamental region for $X_n = \Gamma\backslash G$, and on the other hand $\mathcal{S}_n$ is contained in a finite union of fundamental regions for X_n (cf. [8]).

LEMMA 1. *If $R > 0$ and $M = \mathsf{n}(u)\mathsf{a}(a)\mathsf{k} \in \mathcal{S}_n$ satisfy $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ for some $\boldsymbol{\zeta} \in \mathbb{R}^n$, then $a_1 \gg_d R$.*

Proof. Note that $R\Delta$ contains a ball of radius $\gg_d R$. Now the lemma follows from [35, Lemma 2.1]. ■

Alternatively, Lemma 1 follows from Jarník's inequalities (cf., e.g., [15, p. 99]) together with the fact that $a_1 \asymp_d \lambda_n$, where λ_n is the last successive mimimum of the lattice $\mathbb{Z}^n M$ (cf. (3.6) below).

Using the above lemma together with (2.2) and the bound

$$\mu_n(\{M \in \mathcal{S}_n : a_1 > A\}) \ll_d A^{-n}, \qquad \forall A > 0 \tag{2.8}$$

(cf. the proof of [35, Lemma 2.4]), we immediately deduce the upper bound

$$\Psi_d(R) \ll_d R^{-n}, \tag{2.9}$$

which was proved by Li [18, Thm. 1.2] in a different (but closely related) way.

We next recall the parametrization of $G = G^{(n)}$ by $\mathbb{R}_{>0} \times \mathrm{S}_1^{n-1} \times \mathbb{R}^{n-1} \times G^{(n-1)}$ introduced in [35, (2.9)–(2.11)]. Let us fix a function f (smooth except possibly at one point, say) $\mathrm{S}_1^{n-1} \to \mathrm{SO}(n)$ such that $\boldsymbol{e}_1 f(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathrm{S}_1^{n-1}$ (where $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$). Given $M = \mathsf{n}(u)\mathsf{a}(a)\mathsf{k} \in G$, the matrices $\mathsf{n}(u)$, $\mathsf{a}(a)$ and k can be split uniquely as

$$\mathsf{n}(u) = \begin{pmatrix} 1 & \boldsymbol{u} \\ {}^{\mathrm{t}}\mathbf{0} & \mathsf{n}(\underset{\sim}{u}) \end{pmatrix}, \quad \mathsf{a}(a) = \begin{pmatrix} a_1 & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & a_1^{-1/(n-1)} \mathsf{a}(\underset{\sim}{a}) \end{pmatrix}, \quad \mathsf{k} = \begin{pmatrix} 1 & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \underset{\sim}{\mathsf{k}} \end{pmatrix} f(\boldsymbol{v}), \tag{2.10}$$

where $\boldsymbol{u} \in \mathbb{R}^{n-1}$, $\mathsf{n}(\underset{\sim}{u}) \in N^{(n-1)}$, $a_1 > 0$, $\mathsf{a}(\underset{\sim}{a}) \in A^{(n-1)}$, $\underset{\sim}{\mathsf{k}} \in \mathrm{SO}(n-1)$ and

$\boldsymbol{v} \in \mathrm{S}_1^{n-1}$. We set

$$M_{\sim} = \mathsf{n}(\underset{\sim}{u})\mathsf{a}(\underset{\sim}{a})\underset{\sim}{\mathsf{k}} \in G^{(n-1)}. \tag{2.11}$$

In this way we get a bijection between G and $\mathbb{R}_{>0} \times \mathrm{S}_1^{n-1} \times \mathbb{R}^{n-1} \times G^{(n-1)}$; we write $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$ for the element in G corresponding to the 4-tuple $\langle a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}\rangle \in \mathbb{R}_{>0} \times \mathrm{S}_1^{n-1} \times \mathbb{R}^{n-1} \times G^{(n-1)}$. The Haar measure μ_n takes the following form in the parametrization $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$:

$$d\mu_n(M) = \zeta(n)^{-1}\, d\mu_{n-1}(\underset{\sim}{M})\, d\boldsymbol{u}\, d\boldsymbol{v} \frac{da_1}{a_1^{n+1}}, \tag{2.12}$$

where $d\boldsymbol{u}$ is standard Lebesgue measure on $\mathbb{R}^{n-1}$ and $d\boldsymbol{v}$ is the $(n-1)$-dimensional volume measure on S_1^{n-1} ([35, (2.12)]). Note that all of the above claims are valid also for $n = 2$, with the natural interpretation that $\mathcal{S}_1 = \mathrm{SL}(1, \mathbb{R}) = \{1\}$ with $\mu_1(\{1\}) = 1$.

2.2. On the intersection of Δ and a hyperplane orthogonal to $\boldsymbol{v}$. For $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$, the points in the lattice $\mathbb{Z}^n M$ are given by the formula

$$(k, \boldsymbol{m})M = k a_1 \boldsymbol{v} + a_1^{-1/(n-1)}(0, k\boldsymbol{u}\mathsf{a}(\underset{\sim}{a})\underset{\sim}{\mathsf{k}} + \boldsymbol{m}\underset{\sim}{M}) f(\boldsymbol{v}) \qquad (\forall k \in \mathbb{Z},\ \boldsymbol{m} \in \mathbb{Z}^{n-1}). \tag{2.13}$$

In particular $\mathbb{Z}^n M$ is contained in the union of the (parallel) hyperplanes $k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp}$:

$$\mathbb{Z}^n M \subset \bigcup_{k \in \mathbb{Z}} (k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp}). \tag{2.14}$$

Note that for each k, the $(n-1)$-dimensional affine lattice $\mathbb{Z}^n M \cap (k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp})$ has covolume a_1^{-1} inside $k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp}$. Hence if a_1 is large then this point set typically covers $k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp}$ well in the sense that the maximal distance from $\mathbb{Z}^n M \cap (k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp})$ to any point in $k a_1 \boldsymbol{v} + \boldsymbol{v}^{\perp}$ is small.

Given $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathrm{S}_1^{n-1}$ we let $P_{\boldsymbol{v}} : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the line $\mathbb{R}\boldsymbol{v}$, viz.

$$P_{\boldsymbol{v}}(\boldsymbol{x}) := (\boldsymbol{x} \cdot \boldsymbol{v})\boldsymbol{v}. \tag{2.15}$$

Note that $P_{\boldsymbol{v}}(\Delta)$ is a closed line segment; let $\ell(\boldsymbol{v})$ denote its length. In other words, $\ell(\boldsymbol{v})$ is the width of Δ in the direction $\boldsymbol{v}$. Since Δ is the convex hull of $\{\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, where $\boldsymbol{e}_j$ is the jth standard basis vector of $\mathbb{R}^n$, $P_{\boldsymbol{v}}(\Delta)$ is the convex hull of $\{P_{\boldsymbol{v}}(\boldsymbol{0}), P_{\boldsymbol{v}}(\boldsymbol{e}_1), \ldots, P_{\boldsymbol{v}}(\boldsymbol{e}_n)\}$, and here $P_{\boldsymbol{v}}(\boldsymbol{0}) = \boldsymbol{0}$ and $P_{\boldsymbol{v}}(\boldsymbol{e}_j) = v_j \boldsymbol{v}$. Hence

$$\ell(\boldsymbol{v}) = \ell_+(\boldsymbol{v}) - \ell_-(\boldsymbol{v}), \tag{2.16}$$

where

$$\ell_+(\boldsymbol{v}) := \max(0, v_1, \ldots, v_n), \qquad \ell_-(\boldsymbol{v}) := \min(0, v_1, \ldots, v_n). \tag{2.17}$$

In particular $1/\sqrt{n} \le \ell(\boldsymbol{v}) \le \sqrt{2}$.

Lemma 2. *If* $R > 0$, $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$ *and* $a_1 > \ell(\boldsymbol{v})R$, *then there exists* $\boldsymbol{\zeta} \in \mathbb{R}^n$ *such that* $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$.

Proof. Because of (2.14), $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ certainly holds whenever $R\Delta - \boldsymbol{\zeta}$ lies completely inside the open strip contained between the two parallel hyperplanes $\boldsymbol{v}^\perp$ and $a_1\boldsymbol{v} + \boldsymbol{v}^\perp$, and this holds if and only if $P_{\boldsymbol{v}}(R\Delta - \boldsymbol{\zeta}) \subset \{t\boldsymbol{v} : 0 < t < a_1\}$. There exist vectors $\boldsymbol{\zeta}$ satisfying the last inclusion if and only if $\ell(\boldsymbol{v})R < a_1$. ∎

We next seek to obtain restrictions on those lattices $\mathbb{Z}^n M$ with $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$ and $a_1 \le \ell(\boldsymbol{v})R$ which still satisfy $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ for some $\boldsymbol{\zeta} \in \mathbb{R}^n$. We first prove the following simple geometric fact.

Lemma 3. *For any* $\boldsymbol{v} \in \mathrm{S}_1^{n-1}$ *and* $x \in \mathbb{R}$, *the hyperplane* $x\boldsymbol{v} + \boldsymbol{v}^\perp$ *intersects* Δ *if and only if* $x \in [\ell_-(\boldsymbol{v}), \ell_+(\boldsymbol{v})]$, *and furthermore when this happens,* $(x\boldsymbol{v} + \boldsymbol{v}^\perp) \cap \Delta$ *contains an* $(n-1)$*-dimensional ball of radius*

$$r := (2\sqrt{n} + n)^{-1} \min(x - \ell_-(\boldsymbol{v}), \ell_+(\boldsymbol{v}) - x). \tag{2.18}$$

Proof. The first statement follows since $x\boldsymbol{v} + \boldsymbol{v}^\perp$ intersects Δ if and only if $x\boldsymbol{v} \in P_{\boldsymbol{v}}(\Delta)$, and $P_{\boldsymbol{v}}(\Delta) = \{t\boldsymbol{v} : \ell_-(\boldsymbol{v}) \le t \le \ell_+(\boldsymbol{v})\}$.

To prove the second statement we will prove the stronger fact that if $x \in [\ell_-(\boldsymbol{v}), \ell_+(\boldsymbol{v})]$ then there is some $\boldsymbol{y} \in x\boldsymbol{v} + \boldsymbol{v}^\perp$ such that $\boldsymbol{y} + \mathcal{B}_r^n \subset \Delta$, where $\mathcal{B}_r^n$ denotes the closed n-dimensional ball of radius r centered at $\mathbf{0}$ (thus $\boldsymbol{y} + \mathcal{B}_r^n$ is the ball of radius r centered at $\boldsymbol{y}$).

For an arbitrary point $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we note that $\boldsymbol{y} + \mathcal{B}_r^n \subset \Delta$ holds if and only if $y_1, \ldots, y_n \ge r$ and $y_1 + \cdots + y_n \le 1 - \sqrt{n}r$, which is equivalent to saying that $(\sqrt{n} + n)r \le 1$ and $\boldsymbol{y} - r\boldsymbol{e} \in (1 - (\sqrt{n} + n)r)\Delta$. The condition $(\sqrt{n} + n)r \le 1$ is clearly met for our r, since $\min(x - \ell_-(\boldsymbol{v}), \ell_+(\boldsymbol{v}) - x) \le \frac{1}{2}\ell(\boldsymbol{v}) \le 2^{-1/2}$.

Hence, since Δ is the convex hull of $\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, it follows that there exists a point $\boldsymbol{y} \in x\boldsymbol{v} + \boldsymbol{v}^\perp$ with $\boldsymbol{y} + \mathcal{B}_r^n \subset \Delta$ if and only if x lies in the (1-dimensional) convex hull of the $n+1$ numbers

$$r\boldsymbol{v} \cdot \boldsymbol{e} \quad \text{and} \quad r\boldsymbol{v} \cdot \boldsymbol{e} + (1 - (\sqrt{n} + n)r)v_j \quad \text{for } j = 1, \ldots, n. \tag{2.19}$$

Recalling (2.17) we see that this holds if and only if $x \in [\alpha_-, \alpha_+]$, where

$$\alpha_\pm := r\boldsymbol{v} \cdot \boldsymbol{e} + (1 - (\sqrt{n} + n)r)\ell_\pm(\boldsymbol{v}). \tag{2.20}$$

However

$$|\alpha_\pm - \ell_\pm(\boldsymbol{v})| \le r|\boldsymbol{v} \cdot \boldsymbol{e}| + (\sqrt{n} + n)r|\ell_\pm(\boldsymbol{v})| \le r(\sqrt{n} + \sqrt{n} + n). \tag{2.21}$$

Hence $x \in [\alpha_-, \alpha_+]$ certainly holds whenever

$$\ell_-(\boldsymbol{v}) + (2\sqrt{n} + n)r \le x \le \ell_+(\boldsymbol{v}) - (2\sqrt{n} + n)r, \tag{2.22}$$

and this condition is clearly fulfilled for our r in (2.18). ∎

LEMMA 4. *If $R > 0$, $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}] \in \mathcal{S}_n$ and $a_1 \leq \ell(\boldsymbol{v})R$, and if $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ for some $\boldsymbol{\zeta} \in \mathbb{R}^n$, then $\underset{\sim}{a}_1 \gg_d (\ell(\boldsymbol{v})R - a_1)a_1^{1/(n-1)}$ in $\underset{\sim}{M} = \mathsf{n}(\underset{\sim}{u})\mathsf{a}(\underset{\sim}{a})\underset{\sim}{\mathsf{k}} \in G^{(n-1)}$.*

Proof. Set $X = \ell(\boldsymbol{v})R - a_1 \geq 0$. Since $P_{\boldsymbol{v}}(R\Delta - \boldsymbol{\zeta})$ is a closed line segment in $\mathbb{R}\boldsymbol{v}$ of length $\ell(\boldsymbol{v})R$, there exists some $k \in \mathbb{Z}$ such that $ka_1\boldsymbol{v} \in P_{\boldsymbol{v}}(R\Delta - \boldsymbol{\zeta})$ and furthermore such that $ka_1\boldsymbol{v}$ has distance $\geq \frac{1}{2}X$ to both the endpoints of $P_{\boldsymbol{v}}(R\Delta - \boldsymbol{\zeta})$. Hence by Lemma 3, $(ka_1\boldsymbol{v} + \boldsymbol{v}^\perp) \cap (R\Delta - \boldsymbol{\zeta})$ contains an $(n-1)$-dimensional ball B of radius $\gg_d X$. Now $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ implies that the $(n-1)$-dimensional affine lattice $(ka_1\boldsymbol{v} + \boldsymbol{v}^\perp) \cap \mathbb{Z}^n M$ must be disjoint from B. In view of (2.13) it follows that the $(n-1)$-dimensional lattice $a_1^{-1/(n-1)}(0, \mathbb{Z}^{n-1}\underset{\sim}{M})f(\boldsymbol{v}) \subset \boldsymbol{v}^\perp$ is disjoint from a certain translate of B inside $\boldsymbol{v}^\perp$. Hence $\mathbb{Z}^{n-1}\underset{\sim}{M}$ is disjoint from a ball of radius $\gg_d a_1^{1/(n-1)}X$ in $\mathbb{R}^{n-1}$, and so $\underset{\sim}{a}_1 \gg_d a_1^{1/(n-1)}X$ by [35, Lemma 2.1]. ∎

2.3. The main computation. Recall that by Lemma 1, if $M = \mathsf{n}(u)\mathsf{a}(a)\mathsf{k} \in \mathcal{S}_n$ satisfies $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ for some $\boldsymbol{\zeta} \in \mathbb{R}^n$, then $a_1 \geq \kappa R$, where $\kappa > 0$ is a constant which only depends on d. We set

$$A := \kappa R, \tag{2.23}$$

and from now on we keep $R > \kappa^{-1}$, so that $A > 1$.

We next recall some definitions and facts from [21, Sec. 3.2]. We fix a subset $\mathrm{S}_\pm^{n-1} \subset \mathrm{S}_1^{n-1} \cap \{v_1 \geq 0\}$ which contains exactly one of the vectors $\boldsymbol{v}$ and $-\boldsymbol{v}$ for every $\boldsymbol{v} \in \mathrm{S}_1^{n-1}$. Let us also fix a (set-theoretical, measurable) fundamental region $\mathcal{F}_{n-1} \subset \mathcal{S}_{n-1}$ for $\Gamma^{(n-1)} \backslash G^{(n-1)}$. We set (cf. [21, (3.15), (3.18)])

$$\mathcal{G}_A := \{[a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}] \in G : a_1 > A,\ \boldsymbol{v} \in \mathrm{S}_\pm^{n-1},\ \boldsymbol{u} \in (-1/2, 1/2]^{n-1},\ \underset{\sim}{M} \in \mathcal{F}_{n-1}\} \tag{2.24}$$

and

$$\mathcal{S}_n' := \{[a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}] \in \mathcal{S}_n : \boldsymbol{v} \in \mathrm{S}_\pm^{n-1}\}. \tag{2.25}$$

LEMMA 5. *There exists a (set-theoretical, measurable) fundamental region $\mathcal{F}_n \subset \mathcal{S}_n'$ for $X_n = \Gamma \backslash G$ and a (measurable) subset $\mathcal{C} \subset \mathcal{S}_n' \cup \mathcal{G}_A$ such that*

$$\mathcal{G}_A \setminus \mathcal{C} \subset \{M \in \mathcal{F}_n : a_1 > A\} \subset \mathcal{G}_A \cup \mathcal{C} \tag{2.26}$$

and $\mu_n(\mathcal{C}) \ll_d A^{-2n}$ if $n \geq 3$, while $\mathcal{C} = \emptyset$ if $n = 2$.

Proof. For $n \geq 3$ this follows from [21, Lemma 3.4], together with the computation in [21, (3.23), (3.24)]. In the remaining case $n = 2$ we use the well-known fact that a fundamental region for $X_2 = \Gamma^{(2)} \backslash G^{(2)}$ is provided by

$$\mathcal{F}_2 := \{\mathsf{n}(u)\mathsf{a}(a)f(\boldsymbol{v}) \in G^{(2)} : u + a_1^2 i \in \mathcal{F}_{\mathbb{H}},\ \boldsymbol{v} \in \mathrm{S}^1_{\pm}\}, \tag{2.27}$$

where $\mathcal{F}_{\mathbb{H}}$ is the usual fundamental region for the action of $\Gamma^{(2)}$ on the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$, viz.

$$\mathcal{F}_{\mathbb{H}} := \{z = x+iy \in \mathbb{H} : -1/2 < x \leq 1/2,\ |z| \geq 1,\ (x < 0 \Rightarrow |z| > 1)\}. \tag{2.28}$$

In particular for this choice of $\mathcal{F}_2$ we have $\mathcal{F}_2 \subset \mathcal{S}_2'$ and $\{M \in \mathcal{F}_2 : a_1 > A\} = \mathcal{G}_A$, since $A > 1$. ∎

It follows from Lemma 5 and (2.2) that

$$\Psi_d(R) = \int_{\mathcal{G}_A} I(\exists \boldsymbol{\zeta} \in \mathbb{R}^n : \mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset)\, d\mu_n(M) + O(\mu_n(\mathcal{C})), \tag{2.29}$$

where the error term is $\ll_d A^{-2n} \ll_d R^{-2n}$ if $n \geq 3$, while if $n = 2$ then the error term vanishes. Hence, using (2.24) and (2.12), we obtain

$$\Psi_d(R) = \frac{1}{\zeta(n)} \int_A^{\infty} \int_{\mathrm{S}^{n-1}_{\pm}} \int_{(-1/2,1/2)^{n-1}} \int_{\mathcal{F}_{n-1}} I(\exists \boldsymbol{\zeta} \in \mathbb{R}^n : \mathbb{Z}^n[a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}] \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset)\, d\mu_{n-1}(\underset{\sim}{M})\, d\boldsymbol{u}\, d\boldsymbol{v}\, \frac{da_1}{a_1^{n+1}} + O_d(I(n \geq 3) \cdot R^{-2n}). \tag{2.30}$$

Here it follows from Lemma 2 that the right hand side is

$$\geq \frac{1}{\zeta(n)} \int_{\mathrm{S}^{n-1}_{\pm}} \int_{\ell(\boldsymbol{v})R}^{\infty} \frac{da_1}{a_1^{n+1}}\, d\boldsymbol{v} = \frac{R^{-n}}{n\zeta(n)} \int_{\mathrm{S}^{n-1}_{\pm}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v}. \tag{2.31}$$

(Note here that by Lemma 2 and our definition of A we have $A \leq \ell(\boldsymbol{v})R$ for all $\boldsymbol{v} \in \mathrm{S}^{n-1}_1$.) On the other hand it follows from Lemma 4 that there is a constant $\kappa' > 0$ which only depends on d such that the difference between the integral in (2.30) and the right hand side of (2.31) is

$$\leq \frac{1}{\zeta(n)} \int_{\mathrm{S}^{n-1}_{\pm}} \int_A^{\ell(\boldsymbol{v})R} \mu_{n-1}(\{\underset{\sim}{M} \in \mathcal{F}_{n-1} : \underset{\sim}{a}_1 \geq \kappa'(\ell(\boldsymbol{v})R - a_1)a_1^{1/(n-1)}\})\, \frac{da_1}{a_1^{n+1}}\, d\boldsymbol{v}. \tag{2.32}$$

Here $A = \kappa R$; hence $R \ll_d a_1 \ll_d R$ in the integral, and we get, with a new constant $\kappa'' > 0$ which only depends on d,

$$\ll_d R^{-(n+1)} \int_{\mathrm{S}_\pm^{n-1}} \int_{\kappa R}^{\ell(\boldsymbol{v})R} \mu_{n-1}(\{\underset{\sim}{M} \in \mathcal{F}_{n-1} : \underset{\sim}{a}_1 \geq \kappa''(\ell(\boldsymbol{v})R - a_1)R^{1/(n-1)}\})\, da_1\, d\boldsymbol{v} \tag{2.33}$$

$$\leq R^{-(n+1)} \int_{\mathrm{S}_\pm^{n-1}} \int_{0}^{\ell(\boldsymbol{v})R} \mu_{n-1}(\{\underset{\sim}{M} \in \mathcal{F}_{n-1} : \underset{\sim}{a}_1 \geq \kappa'' t R^{1/(n-1)}\})\, dt\, d\boldsymbol{v}$$

$$\ll_d R^{-(n+1)} \int_{0}^{\sqrt{2}R} \mu_{n-1}(\{\underset{\sim}{M} \in \mathcal{F}_{n-1} : \underset{\sim}{a}_1 \geq \kappa'' t R^{1/(n-1)}\})\, dt.$$

Now if $n \geq 3$ then by a computation as in the proof of [35, Lemma 2.4] we get

$$\ll_d R^{-(n+1)} \int_0^{\sqrt{2}R} (1 + tR^{1/(n-1)})^{-(n-1)}\, dt \ll_d R^{-n-1-1/(n-1)}. \tag{2.34}$$

On the other hand if $n = 2$ then $\mathcal{F}_{n-1} = \{1\}$ and hence the last line of (2.33) equals $R^{-3} \cdot \min(\sqrt{2}R, \kappa''^{-1} R^{-1})$, which is $\ll R^{-4}$. Hence we conclude that

$$\Psi_d(R) = \frac{R^{-n}}{n\zeta(n)} \int_{\mathrm{S}_\pm^{n-1}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v} + O_d(R^{-n-1-1/(n-1)}). \tag{2.35}$$

Now to prove the asymptotic formula for $\Psi_d(R)$ stated in Theorem 2, it only remains to compute the integral $\int_{\mathrm{S}_\pm^{n-1}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v}$.

2.4. Computing the constant in the main term

Lemma 6. *For every $n \geq 2$ we have*

$$\int_{\mathrm{S}_\pm^{n-1}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v} = \frac{n(n+1)}{2}. \tag{2.36}$$

Proof. Set

$$K = \{r\boldsymbol{v} : \boldsymbol{v} \in \mathrm{S}_1^{n-1},\ 0 \leq r \leq \ell(\boldsymbol{v})^{-1}\} \subset \mathbb{R}^n; \tag{2.37}$$

then clearly

$$\int_{\mathrm{S}_\pm^{n-1}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v} = \frac{1}{2} \int_{\mathrm{S}_1^{n-1}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v} = \frac{n}{2} \operatorname{vol}(K). \tag{2.38}$$

But for any $\boldsymbol{x} = r\boldsymbol{v}$ with $r > 0$ and $\boldsymbol{v} \in \mathrm{S}_1^{n-1}$ we have

$$\ell(\boldsymbol{v}) = \|\boldsymbol{x}\|^{-1}(\max(0, x_1, \ldots, x_n) - \min(0, x_1, \ldots, x_n)), \tag{2.39}$$

so that $r \leq \ell(\boldsymbol{v})^{-1}$ if and only if $\max(0, x_1, \ldots, x_n) - \min(0, x_1, \ldots, x_n) \leq 1$. In other words,

$$K = \{\boldsymbol{x} \in [-1,1]^n : |x_j - x_k| \leq 1, \forall j, k\}. \tag{2.40}$$

Hence by easy symmetry considerations we have

$$\begin{aligned} \operatorname{vol}(K) = {} & \operatorname{vol}(K \cap [0,1]^n) + \operatorname{vol}(K \cap [-1,0]^n) \\ & + n(n-1) \operatorname{vol}(\{\boldsymbol{x} \in K : x_1 < 0 < x_2 \text{ and } \\ & \qquad x_1 < x_j < x_2 \text{ for } j = 3, \ldots, n\}) \\ = {} & 2 + n(n-1) \int_{-1}^{0} \int_{0}^{1+x_1} (x_2 - x_1)^{n-2}\, dx_2\, dx_1 = n+1. \end{aligned} \tag{2.41}$$

The lemma follows from (2.38) and (2.41). ■

2.5. Bound from below. Finally we will prove the lower bound (1.9) in Theorem 2.

The key step is the following lemma, which says that for "good" directions $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathrm{S}_1^{n-1}$, we may weaken the restriction $a_1 > \ell(\boldsymbol{v})R$ in Lemma 2 by a small but uniform amount, and still be sure to have $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$ for some $\boldsymbol{\zeta} \in \mathbb{Z}^n$.

LEMMA 7. *Let c be a fixed number in the interval $(0, n^{-1/2})$, and set*

$$c' = (n-1)!^{1/(n-1)} c^{n/(n-1)}. \tag{2.42}$$

Then for any $R \geq (2c'\sqrt{n})^{1-1/n}$ and any $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$ with $a_1 > \ell(\boldsymbol{v})R - c'R^{-1/(n-1)}$ and $v_j > c$ (for all j), there exists $\boldsymbol{\zeta} \in \mathbb{R}^n$ such that $\mathbb{Z}^n M \cap (R\Delta - \boldsymbol{\zeta}) = \emptyset$.

Proof. Let R and $M = [a_1, \boldsymbol{v}, \boldsymbol{u}, \underset{\sim}{M}]$ satisfy the given assumptions. If $a_1 > \ell(\boldsymbol{v})R$ then the desired statement is in Lemma 2; hence from now on we may assume $a_1 \leq \ell(\boldsymbol{v})R$. We will choose

$$\boldsymbol{\zeta} = c'R^{-1/(n-1)}\boldsymbol{v} + \boldsymbol{w} \tag{2.43}$$

for some $\boldsymbol{w} \in \boldsymbol{v}^{\perp}$ which will be fixed at the end of the proof. Then for every $\boldsymbol{x} \in R\Delta - \boldsymbol{\zeta}$ we have

$$\boldsymbol{x} \cdot \boldsymbol{v} \leq \ell_+(\boldsymbol{v})R - \boldsymbol{\zeta} \cdot \boldsymbol{v} = \ell(\boldsymbol{v})R - c'R^{-1/(n-1)} \tag{2.44}$$

and

$$\boldsymbol{x} \cdot \boldsymbol{v} \geq -\boldsymbol{\zeta} \cdot \boldsymbol{v} = -c'R^{-1/(n-1)} \geq -(\ell(\boldsymbol{v})R - c'R^{-1/(n-1)}), \tag{2.45}$$

where we used the assumption $R \geq (2c'\sqrt{n})^{1-1/n}$ in the last step. Using (2.44), (2.45) and $a_1 > \ell(\boldsymbol{v})R - c'R^{-1/(n-1)}$ we conclude that

$$(R\Delta - \boldsymbol{\zeta}) \cap (ka_1\boldsymbol{v} + \boldsymbol{v}^{\perp}) = \emptyset, \qquad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{2.46}$$

Hence, using also (2.14), we get

$$(R\Delta - \boldsymbol{\zeta}) \cap \mathbb{Z}^n M = (R\Delta - \boldsymbol{\zeta}) \cap L_{M,\boldsymbol{v}}, \tag{2.47}$$

where $L_{M,\boldsymbol{v}}$ is the $(n-1)$-dimensional lattice $L_{M,\boldsymbol{v}} = \mathbb{Z}^n M \cap \boldsymbol{v}^{\perp}$. Recall that $L_{M,\boldsymbol{v}}$ has covolume a_1^{-1} in $\boldsymbol{v}^{\perp}$. Using also $R\Delta \subset \mathbb{R}^n_{\geq 0}$ and $\boldsymbol{\zeta} = c' R^{-1/(n-1)} \boldsymbol{v} + \boldsymbol{w}$, $\boldsymbol{w} \in \boldsymbol{v}^{\perp}$, we obtain

$$\begin{aligned}(R\Delta - \boldsymbol{\zeta}) \cap \mathbb{Z}^n M &\subset (\mathbb{R}^n_{\geq 0} - c' R^{-1/(n-1)} \boldsymbol{v} - \boldsymbol{w}) \cap L_{M,\boldsymbol{v}} \\ &= (((\mathbb{R}^n_{\geq 0} - c' R^{-1/(n-1)} \boldsymbol{v}) \cap \boldsymbol{v}^{\perp}) - \boldsymbol{w}) \cap L_{M,\boldsymbol{v}}.\end{aligned} \tag{2.48}$$

Here $(\mathbb{R}^n_{\geq 0} - c' R^{-1/(n-1)} \boldsymbol{v}) \cap \boldsymbol{v}^{\perp}$ is a closed $(n-1)$-dimensional simplex, and a simple computation yields for its volume (cf. [13, (17)], or the simpler computation in [5, Lemma 1])

$$\begin{aligned}\operatorname{vol}_{n-1}&((\mathbb{R}^n_{\geq 0} - c' R^{-1/(n-1)} \boldsymbol{v}) \cap \boldsymbol{v}^{\perp}) \\ &= \frac{\prod_{j=1}^n v_j^{-1}}{(n-1)!} (c' R^{-1/(n-1)})^{n-1} < R^{-1}.\end{aligned} \tag{2.49}$$

In the last step we used $v_j > c$ (for all j) and (2.42). However the covolume of $L_{M,\boldsymbol{v}}$ in $\boldsymbol{v}^{\perp}$ is, since we assumed $a_1 \leq \ell(\boldsymbol{v}) R$ from the start,

$$\operatorname{vol}_{n-1}(\boldsymbol{v}^{\perp}/L_{M,\boldsymbol{v}}) = a_1^{-1} \geq (\ell(\boldsymbol{v}) R)^{-1} > R^{-1}. \tag{2.50}$$

(Indeed $\ell(\boldsymbol{v}) = \ell_+(\boldsymbol{v}) < 1$ since all v_j are positive.) The above shows that the volume of $(\mathbb{R}^n_{\geq 0} - c' R^{-1/(n-1)} \boldsymbol{v}) \cap \boldsymbol{v}^{\perp}$ is smaller than the covolume of $L_{M,\boldsymbol{v}}$, and hence there is some $\boldsymbol{w} \in \boldsymbol{v}^{\perp}$ such that the intersection in (2.48) is empty. ∎

We now return to the computation in Section 2.3. We will bound the difference between the integral in (2.30) and the right hand side of (2.31) from *below.* Fix a constant $c \in (0, n^{-1/2})$ as in Lemma 7, let $c' > 0$ be as in (2.42), and let Ω be the non-empty, relatively open subset of $\mathrm{S}^{n-1}_{\pm}$ consisting of all $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathrm{S}^{n-1}_1$ with $v_j > c$ (for all j). It now follows from Lemma 7 that, for any $R \geq (2c'\sqrt{n})^{1-1/n}$, the difference between the integral in (2.30) and the right hand side of (2.31) is

$$\geq \frac{1}{\zeta(n)} \int_{\ell(\boldsymbol{v})R - c' R^{-1/(n-1)}}^{\ell(\boldsymbol{v})R} \int_{\Omega} d\boldsymbol{v}\, \frac{da_1}{a_1^{n+1}} \gg_d R^{-n-1-1/(n-1)}. \tag{2.51}$$

In particular note that this contribution is asymptotically larger than the error term in (2.30). Hence we conclude that there exist constants $c, c' > 0$ which only depend on n such that for all $R > c'$,

$$\Psi_d(R) > \frac{R^{-n}}{n\zeta(n)} \int_{\mathrm{S}^{n-1}_{\pm}} \ell(\boldsymbol{v})^{-n}\, d\boldsymbol{v} + c R^{-n-1-1/(n-1)}. \tag{2.52}$$

In view of Lemma 6 we have thus proved (1.9) in Theorem 2. Since the asymptotic relation (1.8) follows from (2.35) and Lemma 6, this concludes the proof of Theorem 2. ∎

3. Uniform bounds on $\widetilde{P}_d(T,R)$ and $P_d(T,R)$. In this section we will prove Theorem 3 and Corollary 1.

3.1. Proof of Theorem 3. Let us first note that the claim (1.16), i.e.

$$\widetilde{P}_d(T,R) = 0 \quad \text{whenever } R \geq \kappa_{\mathcal{D}}^{1-1/(d-1)} T^{1-1/(d-1)} \tag{3.1}$$

where

$$\kappa_{\mathcal{D}} := \sup_{\boldsymbol{x}\in\mathcal{D}} \|\boldsymbol{x}\|, \tag{3.2}$$

is a direct consequence of any among several known bounds on the Frobenius number (cf., e.g., [23]). For example, the classical bound by Schur (cf. [10]) asserts that for any $\boldsymbol{a} \in \widehat{\mathbb{N}}^d$ satisfying $a_1 \leq \cdots \leq a_d$,

$$g(\boldsymbol{a}) \leq a_1 a_d - a_1 - a_d \quad (\text{thus } f(\boldsymbol{a}) \leq a_1 a_d + a_2 + \cdots + a_{d-1} < d a_1 a_d). \tag{3.3}$$

Using this together with the fact that $s(\boldsymbol{a}) \geq d a_1 a_d \|\boldsymbol{a}\|^{-1+1/(d-1)}$ for any such $\boldsymbol{a}$, we deduce

$$\frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} < \|\boldsymbol{a}\|^{1-1/(d-1)}. \tag{3.4}$$

Here both the left and the right hand sides are invariant under permutations of the coefficients of $\boldsymbol{a}$; hence (3.4) in fact holds for *all* $\boldsymbol{a} \in \widehat{\mathbb{N}}^d$. Finally, (3.1) follows from (3.4).

We next turn to the proof of (1.15) in Theorem 3. As in the previous section we write $n = d-1$. Given $\boldsymbol{a} \in \widehat{\mathbb{N}}^d$ we set

$$\Lambda_{\boldsymbol{a}} = \mathbb{Z}^d \cap \boldsymbol{a}^{\perp} = \{\boldsymbol{x} \in \mathbb{Z}^d : \boldsymbol{a}\cdot\boldsymbol{x} = 0\}. \tag{3.5}$$

This is an n-dimensional sublattice of $\mathbb{Z}^d$ of determinant $\det(\Lambda_{\boldsymbol{a}}) = \|\boldsymbol{a}\|$. (By the determinant, $\det(\Lambda)$, of a lattice Λ of not necessarily full rank in $\mathbb{R}^d$, we mean the covolume of Λ in $\operatorname{span}_{\mathbb{R}} \Lambda$.) Given any n-dimensional lattice $\Lambda \subset \mathbb{R}^d$ we write $0 < \lambda_1(\Lambda) \leq \cdots \leq \lambda_n(\Lambda)$ for the Minkowski successive minima of Λ, i.e.

$$\lambda_j(\Lambda) = \inf\{r > 0 : \dim \operatorname{span}_{\mathbb{R}}(\mathcal{B}_r^d \cap \Lambda) \geq j\}. \tag{3.6}$$

(Recall that $\mathcal{B}_r^d$ is the closed d-dimensional ball of radius r centered at $\mathbf{0}$.) Then by Aliev and Henk [2, (14)] (2) (cf. also Kannan [17, Thm. 2.5]) we have

(2) Note that λ_j in [2] equals $\|\boldsymbol{a}\|^{-1/n}\lambda_j(\Lambda_{\boldsymbol{a}})$ in our notation.

$$\frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} \leq \frac{1}{2} n \|\boldsymbol{a}\|^{-1/n} \lambda_n(\Lambda_{\boldsymbol{a}}). \tag{3.7}$$

Note also that we have $\#(\widehat{\mathbb{N}}^d \cap T\mathcal{D}) \asymp_{d,\mathcal{D}} T^d$ uniformly over all $T > 0$ for which $\widehat{\mathbb{N}}^d \cap T\mathcal{D} \neq \emptyset$, since $\mathcal{D}$ is bounded with non-empty interior. From these facts together with the fact that $\Lambda_{\boldsymbol{a}} \neq \Lambda_{\boldsymbol{b}}$ for all $\boldsymbol{a} \neq \boldsymbol{b} \in \widehat{\mathbb{N}}^d$ (since $\operatorname{span}_{\mathbb{R}} \Lambda_{\boldsymbol{a}} = \boldsymbol{a}^\perp \neq \boldsymbol{b}^\perp = \operatorname{span}_{\mathbb{R}} \Lambda_{\boldsymbol{b}}$), it follows that

$$\begin{aligned} &\widetilde{P}_d(T,R) \\ &\quad \ll_{d,\mathcal{D}} T^{-d} \#\{\Lambda \in \mathcal{L}_n : \det(\Lambda) \leq \kappa_{\mathcal{D}} T,\ \lambda_n(\Lambda) > 2n^{-1} \det(\Lambda)^{1/n} R\}, \end{aligned} \tag{3.8}$$

where $\mathcal{L}_n$ is the set of all n-dimensional sublattices of $\mathbb{Z}^d$.

Let us set

$$\rho_j(\Lambda) := \lambda_{j+1}(\Lambda)/\lambda_j(\Lambda) \quad \text{for } j = 1, \ldots, n-1. \tag{3.9}$$

(Thus $\rho_j(\Lambda) \geq 1$ for all Λ.) Also, for any $\boldsymbol{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{R}_{\geq 1}^{n-1}$, we set

$$\mathcal{L}_n(\boldsymbol{r}) := \{\Lambda \in \mathcal{L}_n : \rho_j(\Lambda) \geq r_j\ (\forall j)\}. \tag{3.10}$$

Now as a special case of Schmidt's [31, Thm. 5], the number of lattices in $\mathcal{L}_n(\boldsymbol{r})$ with determinant at most T is given by the following asymptotic formula with a precise error term. Let us write $\rho_j(L) = \lambda_{j+1}(L)/\lambda_j(L)$ also for an n-dimensional lattice $L \subset \mathbb{R}^n$, with $\lambda_1(L) \leq \cdots \leq \lambda_n(L)$ being the successive minima of L.

Theorem 5 ([31, Thm. 5]). *For any $\boldsymbol{r} \in \mathbb{R}_{\geq 1}^{n-1}$ and $T > 0$ we have*

$$\begin{aligned} &\#\{\Lambda \in \mathcal{L}_n(\boldsymbol{r}) : \det(\Lambda) \leq T\} \\ &\quad = \frac{\pi^{d/2}}{2\Gamma(1+d/2)} \Big(\prod_{j=2}^{n} \zeta(j)\Big) \mu_n(\{L \in X_n : \rho_j(L) \geq r_j\ \ (\forall j)\}) \cdot T^d \\ &\quad\quad + O_d\Big(\Big(\prod_{j=1}^{n-1} r_j^{-(j-1/n)(n-j)}\Big) T^{d-1/n}\Big). \end{aligned} \tag{3.11}$$

Furthermore,

$$\mu_n(\{L \in X_n : \rho_j(L) \geq r_j\ \ (\forall j)\}) \asymp_d \prod_{j=1}^{n-1} r_j^{-j(n-j)}. \tag{3.12}$$

For our argument we will only make use of the upper bound which follows from the above theorem, viz.

$$\begin{aligned} &\#\{\Lambda \in \mathcal{L}_n(\boldsymbol{r}) : \det(\Lambda) \leq T\} \\ &\quad \ll_d T^d \prod_{j=1}^{n-1} r_j^{-j(n-j)} \Big(1 + T^{-1/n} \prod_{j=1}^{n-1} r_j^{(n-j)/n}\Big). \end{aligned} \tag{3.13}$$

We will now form a finite union of sets $\mathcal{L}_n(\boldsymbol{r})$ which contains the set on the right hand side of (3.8).

For any n-dimensional lattice Λ we have

$$\lambda_n(\Lambda)^n = \prod_{j=1}^{n} \lambda_j(\Lambda) \prod_{j=1}^{n-1} \rho_j(\Lambda)^j \asymp_d \det(\Lambda) \prod_{j=1}^{n-1} \rho_j(\Lambda)^j, \tag{3.14}$$

where in the last step we used Minkowski's Second Theorem (cf., e.g., [34, Lectures 3–4]). Hence there exists a constant $c > 0$ which only depends on n (viz., only on d) such that for any n-dimensional lattice Λ and any $R > 0$,

$$\lambda_n(\Lambda) > 2n^{-1} \det(\Lambda)^{1/n} R \ \Rightarrow\ \prod_{j=1}^{n-1} \rho_j(\Lambda)^j > cR^n. \tag{3.15}$$

Note that (1.15) is trivial when $R \ll 1$ (since $\widetilde{P}_d(T, R) \leq 1$ always); hence from now on we may keep $R \geq ec^{-1/n}$ without loss of generality. Set

$$B := \lfloor \log(cR^n) - n \rfloor \in \mathbb{Z}_{\geq 0} \tag{3.16}$$

and

$$\mathcal{R}(n, R) := \Big\{ \boldsymbol{r} = (e^{b_1}, e^{b_2/2}, e^{b_3/3}, \ldots, e^{b_{n-1}/(n-1)}) : \boldsymbol{b} \in \mathbb{Z}_{\geq 0}^{n-1}, \sum_{j=1}^{n-1} b_j = B \Big\}. \tag{3.17}$$

Note that if Λ is any n-dimensional lattice satisfying $\prod_{j=1}^{n-1} \rho_j(\Lambda)^j > cR^n$, then if we set $b_j := \lfloor j \log \rho_j(\Lambda) \rfloor$ we have

$$\begin{aligned} \sum_{j=1}^{n-1} b_j &> \sum_{j=1}^{n-1} (j \log \rho_j(\Lambda) - 1) \\ &> \log(cR^n) - (n-1) > \log(cR^n) - n \geq B. \end{aligned} \tag{3.18}$$

Hence there is a way to decrease some of the b_j's so as to make $\sum_{j=1}^{n-1} b_j = B$, while keeping $\boldsymbol{b} = (b_1, \ldots, b_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. Of course the new vector $\boldsymbol{b} = (b_1, \ldots, b_{n-1})$ still satisfies $b_j \leq j \log \rho_j(\Lambda)$ for each j, i.e. $\rho_j(\Lambda) \geq e^{b_j/j}$. We have thus proved that for any n-dimensional lattice Λ satisfying $\prod_{j=1}^{n-1} \rho_j(\Lambda)^j > cR^n$, there exists some $\boldsymbol{r} \in \mathcal{R}(n, R)$ such that $r_j \leq \rho_j(\Lambda)$ for $j = 1, \ldots, n-1$. This fact together with (3.15) implies that the set on the right hand side of (3.8) is contained in the union of $\mathcal{L}_n(\boldsymbol{r})$ over all $\boldsymbol{r} \in \mathcal{R}(n, R)$. So, by (3.8), for all $T > 0$ with $\widehat{\mathbb{N}}^d \cap T\mathcal{D} \neq \emptyset$ and all $R \geq ec^{-1/n}$ we have

$$\widetilde{P}_d(T,R) \ll_{d,\mathcal{D}} T^{-d} \sum_{\boldsymbol{r}\in\mathcal{R}(n,R)} \#\{\Lambda \in \mathcal{L}_n(\boldsymbol{r}) : \det(\Lambda) \le \kappa_{\mathcal{D}} T\}. \tag{3.19}$$

Therefore, via (3.13),

$$\widetilde{P}_d(T,R) \ll_{d,\mathcal{D}} \sum_{\substack{\boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{n-1}\\ b_1+\cdots+b_{n-1}=B}} \exp\Big\{-\sum_{j=1}^{n-1}(n-j)b_j\Big\} + T^{-1/n} \sum_{\substack{\boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{n-1}\\ b_1+\cdots+b_{n-1}=B}} \exp\Big\{-\sum_{j=1}^{n-1}(1-(nj)^{-1})(n-j)b_j\Big\}. \tag{3.20}$$

If $n=2$ then each sum above has exactly one term, and we conclude

$$\widetilde{P}_3(T,R) \ll_{\mathcal{D}} R^{-2} + T^{-1/2}R^{-1}. \tag{3.21}$$

If $R < \kappa_{\mathcal{D}}^{1/2} T^{1/2}$ then this gives $\widetilde{P}_3(T,R) \ll_{\mathcal{D}} R^{-2}$. On the other hand if $R \ge \kappa_{\mathcal{D}}^{1/2} T^{1/2}$ then $\widetilde{P}_3(T,R) = 0$ by (3.1). Hence the proof of (1.15) is complete in the case $n=2$.

We now assume $n \ge 3$. We set

$$\begin{aligned} \gamma_1(j) &:= n-j,\\ \gamma_2(j) &:= (1-(nj)^{-1})(n-j) = n + n^{-1} - (j+j^{-1}). \end{aligned} \tag{3.22}$$

Now for any $\boldsymbol{b} \in \mathbb{Z}_{\ge 0}^{n-1}$ with $b_1+\cdots+b_{n-1} = B$ and $b_1+\cdots+b_{n-2} =: s$ we have, since $\gamma_1(j)$ is a decreasing function of j,

$$\begin{aligned} \sum_{j=1}^{n-1}\gamma_1(j)b_j &\ge \gamma_1(n-2)\sum_{j=1}^{n-2} b_j + \gamma_1(n-1)b_{n-1}\\ &= 2s + (B-s) = B+s. \end{aligned} \tag{3.23}$$

Similarly, since also $\gamma_2(j)$ is a decreasing function of j for $j \ge 1$,

$$\begin{aligned} \sum_{j=1}^{n-1}\gamma_2(j)b_j &\ge \gamma_2(n-2)s + \gamma_2(n-1)(B-s)\\ &= \Big(1-\frac{1}{n(n-1)}\Big)B + \Big(1-\frac{1}{(n-1)(n-2)}\Big)s. \end{aligned} \tag{3.24}$$

Note also that for any $s \in \{0,1,\ldots,B\}$ there are exactly $\binom{s+n-3}{n-3}$ vectors $\boldsymbol{b} \in \mathbb{Z}_{\ge 0}^{n-1}$ satisfying $b_1+\cdots+b_{n-1} = B$ and $b_1+\cdots+b_{n-2} = s$. Hence

$$\text{(3.25)}\quad \widetilde{P}_d(T,R) \ll_{d,\mathcal{D}} \sum_{s=0}^{B} \binom{s+n-3}{n-3} e^{-B-s} + T^{-1/n} \sum_{s=0}^{B} \binom{s+n-3}{n-3} e^{-(1-\frac{1}{n(n-1)})B-(1-\frac{1}{(n-1)(n-2)})s}$$
$$\ll_{d,\mathcal{D}} e^{-B} + T^{-1/n} e^{-(1-\frac{1}{n(n-1)})B} \ll_d R^{-n}(1+T^{-1/n}R^{1/(n-1)}).$$

If $R < \kappa_{\mathcal{D}}^{1-1/n} T^{1-1/n}$ then this gives $\widetilde{P}_d(T,R) \ll_{d,\mathcal{D}} R^{-n}$. On the other hand if $R \geq \kappa_{\mathcal{D}}^{1-1/n} T^{1-1/n}$ then $\widetilde{P}_d(T,R) = 0$ by (3.1). Hence the proof of (1.15) is complete. ∎

Remark 2. Note that our proof makes crucial use of the precise error terms which Schmidt has worked out for the asymptotic formulas in [31, Sec. 2]. In this vein, note that the proof of the bound $\widetilde{P}_d(T,R) \ll_d R^{-2}$ in [2, Thm. 1.1] is correct as it stands only when T is sufficiently large in a way which may depend on R (as well as d); this is because the proof in [2] uses Schmidt's [31, Thm. 2], in which the rate of convergence may depend in an unspecified way on the chosen set $\mathscr{D}$ of lattice similarity classes.

3.2. Proof of Corollary 1. Let us first note that (1.18) is again a direct consequence of the classical bound by Schur, (3.3). Indeed, for any $\boldsymbol{a} \in \widehat{\mathbb{N}}^d$ satisfying $a_1 \leq \cdots \leq a_d$, by (3.3) we have

$$\text{(3.26)}\quad \frac{f(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}} < d \cdot \frac{a_1}{(a_1\cdots a_{d-1})^{1/(d-1)}} \cdot a_d^{1-1/(d-1)} \leq d a_d^{1-1/(d-1)} < d\|\boldsymbol{a}\|^{1-1/(d-1)},$$

and this implies (1.18).

The following lemma refines [3, Thm. 2 and Remark 1]. Recall that $n = d-1 \geq 2$. Let us write $\|\boldsymbol{x}\|_\infty := \max(|x_1|,\ldots,|x_n|)$ for the maximum norm of a vector $\boldsymbol{x} \in \mathbb{R}^n$.

Lemma 8. *For any $T>0$ and $\alpha>0$ we have*

$$\#\left\{\boldsymbol{x} = (x_1,\ldots,x_n) \in \mathbb{N}^n : \|\boldsymbol{x}\|_\infty \leq T,\ \frac{\|\boldsymbol{x}\|_\infty}{(x_1\cdots x_n)^{1/n}} > \alpha\right\} \ll_n T^n \alpha^{-n} (\log(2+\alpha))^{n-2}.$$

Remark 3. For any fixed $\varepsilon > 0$ the above bound is in fact sharp in the range $1 \leq \alpha \leq T^{1-1/n-\varepsilon}$, in the sense that the cardinality on the left hand side is also $\gg_{n,\varepsilon} T^n\alpha^{-n}(\log(2+\alpha))^{n-2}$ uniformly over all $T \geq T_0(n,\varepsilon)$ and all $1 \leq \alpha \leq T^{1-1/n-\varepsilon}$. However we do not need this fact and we will not prove it here.

Proof of Lemma 8. It suffices to prove

$$\#\Big\{\boldsymbol{x}\in\mathbb{N}^n : \frac{1}{2}T<\|\boldsymbol{x}\|_\infty\leq T,\ \frac{\|\boldsymbol{x}\|_\infty}{(x_1\cdots x_n)^{1/n}}>\alpha\Big\} \ll_n T^n\alpha^{-n}(\log(2+\alpha))^{n-2}, \tag{3.27}$$

since the lemma then follows by dyadic decomposition in the T-variable. Of course we may assume $T\geq 1$ since otherwise the set on the left hand side is empty. We may also assume $\alpha\geq 1$ since otherwise the right hand side is $\gg_n T^n$ and (3.27) is trivial. Now note that if $\boldsymbol{x}$ belongs to the set on the left hand side of (3.27) then for every real vector $\boldsymbol{y}$ in the unit box $\boldsymbol{x}+[0,1]^n$ we have $\frac{1}{2}T<\|\boldsymbol{y}\|_\infty\leq T+1\leq 2T$ and (since all $x_j\geq 1$)

$$\prod_{j=1}^n y_j\leq\prod_{j=1}^n(x_j+1)\leq\prod_{j=1}^n(2x_j)=2^n\prod_{j=1}^n x_j < 2^n(\|\boldsymbol{x}\|_\infty)^n\alpha^{-n}\leq 2^nT^n\alpha^{-n}. \tag{3.28}$$

Hence the left hand side of (3.27) is

$$\begin{aligned}&\leq \operatorname{vol}\Big(\Big\{\boldsymbol{y}\in\mathbb{R}^n_{\geq 1} : T/2<\|\boldsymbol{y}\|_\infty\leq 2T,\ \prod_{j=1}^n y_j<2^nT^n\alpha^{-n}\Big\}\Big)\\ &\leq n\int_1^{2T}\cdots\int_1^{2T}\int_{T/2}^{2T} I\Big(\prod_{j=1}^n y_j<2^nT^n\alpha^{-n}\Big)\,dy_n\,dy_{n-1}\cdots dy_1\\ &\leq 2nT\int_1^{2T}\cdots\int_1^{2T} I\Big(\prod_{j=1}^{n-1} y_j<2^{n+1}T^{n-1}\alpha^{-n}\Big)\,dy_{n-1}\cdots dy_1\\ &= 2^nnT^n\int_0^{\log(2T)}\cdots\int_0^{\log(2T)} I\Big(\sum_{j=1}^{n-1}u_j>\log(\alpha^n/4)\Big)e^{-\sum_{j=1}^{n-1}u_j}\,du_{n-1}\cdots du_1,\end{aligned} \tag{3.29}$$

where in the last step we substituted $y_j=2Te^{-u_j}$. If $n=2$ then the last expression is clearly $\ll T^2\alpha^{-2}$, as desired. From now on we assume $n\geq 3$. Set $u_{n-1}=s+\log(\alpha^n/4)-\sum_{j=1}^{n-2}u_j$; then the conditions $\sum_{j=1}^{n-1}u_j>\log(\alpha^n/4)$ and $u_{n-1}>0$ are equivalent to $s>0$ and $\sum_{j=1}^{n-2}u_j<s+\log(\alpha^n/4)$, respectively. Hence the last expression in (3.29) is

$$\leq 2^{n+2}nT^n\alpha^{-n}\int_0^\infty e^{-s}\Big(\int_0^\infty\cdots\int_0^\infty I\Big(\sum_{j=1}^{n-2}u_j<s+\log(\alpha^n/4)\Big) \times du_{n-2}\cdots du_1\Big)\,ds \tag{3.30}$$

$$\leq \frac{2^{n+2}n}{(n-2)!}T^n\alpha^{-n}\int_0^\infty e^{-s}(s+n\log\alpha)^{n-2}\,ds \ll_n T^n\alpha^{-n}(\log(2+\alpha))^{n-2},$$

where we used $\alpha \geq 1$. ■

We now give the proof of (1.17) in Corollary 1. We may assume $R \geq 10$ since otherwise (1.17) follows immediately from $P_d(T,R) \leq 1$. We keep $R' \in [1,R]$, to be fixed later. Now

$$\begin{aligned}(3.31)\quad & P_d(T,R)\\ &\ll_{d,\mathcal{D}} T^{-d}\#\left\{\boldsymbol{a}\in\widehat{\mathbb{N}}^d\cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} > R' \text{ or } \frac{s(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}} > \frac{R}{R'}\right\}\\ &\leq T^{-d}\#\left\{\boldsymbol{a}\in\widehat{\mathbb{N}}^d\cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} > R'\right\}\\ &\quad + T^{-d}\#\left\{\boldsymbol{a}\in\mathbb{N}^d : \|\boldsymbol{a}\|_\infty \leq \kappa'_{\mathcal{D}}T,\ \frac{s(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}} > \frac{R}{R'}\right\},\end{aligned}$$

where $\kappa'_{\mathcal{D}} := \sup_{\boldsymbol{x}\in\mathcal{D}}\|\boldsymbol{x}\|_\infty$. In the last term, at the price of an extra factor d we may impose the extra assumption $a_d = \max(a_1,\ldots,a_d)$. For such vectors $\boldsymbol{a}$, we have

$$\begin{aligned}(3.32)\quad \frac{s(\boldsymbol{a})}{(a_1\cdots a_d)^{1/(d-1)}} &< \frac{d^{3/2}a_d\max(a_1,\ldots,a_n)}{\|\boldsymbol{a}\|^{1-1/n}(a_1\cdots a_d)^{1/n}}\\ &< \frac{d^{3/2}a_d\max(a_1,\ldots,a_n)}{a_d^{1-1/n}(a_1\cdots a_d)^{1/n}} = d^{3/2}\frac{\|(a_1,\ldots,a_n)\|_\infty}{(a_1\cdots a_n)^{1/n}}.\end{aligned}$$

Hence for any $T>0$ with $\widehat{\mathbb{N}}^d\cap T\mathcal{D}\neq\emptyset$,

$$\begin{aligned}(3.33)\quad & P_d(T,R)\\ &\ll_{d,\mathcal{D}} T^{-d}\#\left\{\boldsymbol{a}\in\widehat{\mathbb{N}}^d\cap T\mathcal{D} : \frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} > R'\right\}\\ &\qquad + T^{-n}\#\left\{\boldsymbol{a}\in\mathbb{N}^n : \|\boldsymbol{a}\|_\infty \leq \kappa'_{\mathcal{D}}T,\ \frac{\|(a_1,\ldots,a_n)\|_\infty}{(a_1\cdots a_n)^{1/n}} > \frac{1}{d^{3/2}}\frac{R}{R'}\right\}\\ &\ll_{d,\mathcal{D}} R'^{-n} + R^{-n}R'^{n}\left(\log\left(2+\frac{R}{R'}\right)\right)^{n-2},\end{aligned}$$

where we used Theorem 3 and Lemma 8. The bound in (1.17) now follows by choosing $R' = \sqrt{R}(\log(R+2))^{1/n-1/2}$. ■

4. Lattice coverings of space with convex bodies. According to a theorem of Schmidt ([30, Thm. 11*]), sharpening a previous result by Rogers ([25, Thm. 2]), if n is sufficiently large, then for any n-dimensional convex body K of volume

$$\text{(4.1)} \qquad \mathrm{vol}_n(K) \geq (1+\eta_0)^n \quad \text{(with } \eta_0 = 0.756\ldots \text{ as in Theorem 4)},$$

there exists a lattice $L \in X_n$ such that the translates of K by L cover $\mathbb{R}^n$, viz. $K + L = \mathbb{R}^n$. The lower bound (4.1) was shortly afterwards improved by Rogers to a subexponential bound, in [27]. However, our purpose in this section is to point out that the argument in [30], [25] can fairly easily be modified to show that $K + L = \mathbb{R}^n$ holds not just for *some* lattice $L \in X_n$, but in fact for a subset of large measure in X_n:

THEOREM 6. *Let $\eta_0 = 0.756\ldots$ be the unique real root of $e\log\eta + \eta = 0$. For every dimension n larger than a certain absolute constant, if a is any real number satisfying*

$$\text{(4.2)} \qquad n\eta_0^{n/2} \leq a < 1,$$

and K is any n-dimensional convex body of volume

$$\text{(4.3)} \qquad \mathrm{vol}_n(K) \geq n(1+\eta_0 a^{-1/n})^n,$$

then

$$\text{(4.4)} \qquad \mu_n(\{L \in X_n : K + L = \mathbb{R}^n\}) \geq 1 - a.$$

In particular, for any given constant $\alpha > 1 + \eta_0$ there exists $c < 1$ such that for any sufficiently large n, and for any convex body $K \subset \mathbb{R}^n$ of volume $\geq \alpha^n$, the probability that K fails to give a covering with respect to a random lattice $L \in X_n$ is $\leq c^n$, i.e. exponentially small in n.

We obtain Theorem 4 as a special case of this by taking $n = d-1$ and $K = \alpha(d-1)!^{1/(d-1)}\Delta$.

4.1. Proof of Theorem 6. We start by recalling another result of Rogers ([26]) which is used in the proof of [30, Thm. 11*]. For any (Lebesgue) measurable set $M \subset \mathbb{R}^n$ and any lattice $L \in X_n$ we write $\epsilon(M, L)$ for the density of the set of points in $\mathbb{R}^n$ left uncovered by the translates of M by the vectors of L. In other words,

$$\text{(4.5)} \qquad \epsilon(M, L) = 1 - \mathrm{vol}_n((M+L)/L).$$

(Note that $(M+L)/L$ is a well-defined measurable subset of the torus $\mathbb{R}^n/L$.)

THEOREM 7 ([26, Thm. 1] (3)). *For any measurable set $M \subset \mathbb{R}^n$ ($n \geq 2$) of volume V,*

$$\text{(4.6)} \qquad \int_{X_n} \epsilon(M, L)\, d\mu_n(L) \leq 1 - V + \tfrac{1}{2}V^2.$$

Let us note the following corollary.

(3) The boundedness assumption in Rogers' statement of [26, Thm. 1] can be disposed of, cf. [26, p. 211]. Note also that we do not have to require $V \leq 1$, although if $V > 1$ then the bound in (4.6) is subsumed by the bound $\int \epsilon(M, L)\, d\mu_n \leq 1/2$, which follows by applying Theorem 7 to an arbitrary subset $M' \subset M$ of volume 1.

COROLLARY 3. *For any $C > 0$ and any measurable set $M \subset \mathbb{R}^n$ $(n \geq 2)$ of volume V,*

$$\mu_n(\{L \in X_n : \epsilon(M, L) \geq 1 - V + CV^2\}) \leq \frac{1}{2C}. \tag{4.7}$$

Proof. Clearly, for *any* lattice $L \in X_n$ we have $\mathrm{vol}_n((M + L)/L) \leq V$, and thus

$$\epsilon(M, L) \geq 1 - V. \tag{4.8}$$

Hence if p denotes the measure on the left hand side of (4.7) then

$$\begin{aligned} \int_{X_n} \epsilon(M, L)\, d\mu_n(L) &\geq p(1 - V + CV^2) + (1 - p)(1 - V) \\ &= 1 - V + pCV^2, \end{aligned} \tag{4.9}$$

and thus Theorem 7 implies $pC \leq 1/2$. ∎

Proof of Theorem 6. Let a and K be as in the statement of the theorem. Let $r = 0.278\ldots$ be the root of the equation $1+r+\log r = 0$; then $\eta_0 = e^{-r}$. We set $K' = \rho K$, where $\rho > 0$ is chosen so that the volume of K' is

$$V = \mathrm{vol}_n(K') = rn. \tag{4.10}$$

We also set

$$\eta = e^{-r} a^{-1/n} = \eta_0 a^{-1/n}. \tag{4.11}$$

Now by Schmidt [30, Thm. 10*] (applied with $\varepsilon = 1$), if n is larger than a certain absolute constant then

$$\int_{X_n} \epsilon(K', L)\, dL \leq 2(1 + V^{n-1} n^{-n+1} e^{V+n}) e^{-V} = 2(1 + r^{-1}) e^{-rn}, \tag{4.12}$$

and thus

$$\mu_n(\{L \in X_n : \epsilon(K', L) \geq 4(1 + r^{-1}) e^{-rn} a^{-1}\}) \leq \tfrac{1}{2} a. \tag{4.13}$$

Also, by Corollary 3,

$$\mu_n(\{L \in X_n : \epsilon(\eta K', L) \geq 1 - \eta^n V + a^{-1} \eta^{2n} V^2\}) \leq \tfrac{1}{2} a. \tag{4.14}$$

Note that $e^{-rn} a^{-1}/(\eta^n V) = 1/V = r^{-1} n^{-1} \to 0$ as $n \to \infty$, and also

$$\frac{a^{-1} \eta^{2n} V^2}{\eta^n V} = a^{-1} \eta^n V = a^{-2} e^{-rn} rn \leq rn^{-1} \to 0 \quad \text{as } n \to \infty, \tag{4.15}$$

where we used (4.2). Hence for n larger than a certain absolute constant,

$$1 - \eta^n V + a^{-1} \eta^{2n} V^2 + 4(1 + r^{-1}) e^{-rn} a^{-1} < 1. \tag{4.16}$$

It follows from (4.13), (4.14) and (4.16) that

$$\mu_n(\{L \in X_n : \epsilon(\eta K', L) + \epsilon(K', L) < 1\}) \geq 1 - a. \tag{4.17}$$

However, for any $L \in X_n$ that satisfies $\epsilon(\eta K', L) + \epsilon(K', L) < 1$ we have $(1+\eta)K' + L = \mathbb{R}^n$, since K' is convex (cf. [25, Sec. 1.3]), and thus also $\alpha K' + L = \mathbb{R}^n$ for any $\alpha \geq 1+\eta$. In particular, since $K = \rho^{-1}K'$, we have $K+L = \mathbb{R}^n$ for any such L, provided that $\rho^{-1} \geq 1+\eta$. But $\mathrm{vol}_n(K) = \rho^{-n}V$; hence $\rho^{-1} \geq 1+\eta$ is equivalent to $\mathrm{vol}_n(K) \geq (1+\eta)^n V$, and this inequality certainly holds, because of $V < n$ and our assumption (4.3). Hence (4.4) follows from (4.17). ∎

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