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On the mean value of a kind of zeta functions

by

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1. Introduction and main results. Throughout this paper, we always suppose $s = \sigma + it$ and $x \geq 2$. Let

$$d(n) = \sum_{n=kl} 1$$

be the classical divisor function and

$$D(n) = \sum_{n \leq x} d(n)$$

be its summatory function. Dirichlet proved

$$D(x) = x(\log x + 2\gamma - 1) + \Delta(x), \tag{1.1}$$

where $\gamma = \lim_{n\to\infty} \big(\sum_{k=1}^{n} \frac{1}{k} - \log n\big) \approx 0.5721\ldots$ is the Euler constant and

$$\Delta(x) \ll x^{1/2}.$$

Voronoi [13] improved Dirichlet's result to

$$\Delta(x) \ll x^{1/3} \log x.$$

It is conjectured that for any $\varepsilon > 0$, we have

$$\Delta(x) \ll_{\varepsilon} x^{1/4+\varepsilon}.$$

The best result to date is

$$\Delta(x) \ll x^{\frac{131}{416}} (\log x)^{\frac{26947}{8320}},$$

due to Huxley [6].

Let $\zeta(s)$ be the Riemann zeta function. Then the generating function of $d(n)$ is

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad \text{for } \sigma > 1.$$

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Hardy–Littlewood [4] considered the mean square of $\zeta^2(s)$,

$$I_\sigma(T,\zeta^2)=\int_T^{2T}|\zeta(\sigma+it)|^4\,dt \qquad \text{for } 1/2<\sigma<1,$$

and proved

$$I_\sigma(T,\zeta^2)=\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T+o(T). \tag{1.2}$$

Note that their proof is based on the approximation (see e.g. [7, Section 3])

$$\zeta^2(s)=\sum_{n\le x}\frac{d(n)}{n^s}+\chi^2(s)\sum_{n\le y}\frac{d(n)}{n^{1-s}}+O(x^{1/2-\sigma}\log t) \qquad \text{for } 1/2<\sigma<1, \tag{1.3}$$

where $x,y\ge 2$, $4\pi^2xy=t^2$ and

$$\chi(s)=\frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}$$

is the Γ-factor in the functional equation

$$\zeta(s)=\chi(s)\zeta(1-s). \tag{1.4}$$

In this paper, we focus on the following type divisor function:

$$d_{\alpha,\beta}(n)=\sum_{\substack{n=kl\\ \alpha l<k\le\beta l}}1,$$

where α,β are fixed rational numbers satisfying $0<\alpha<\beta$. Define its generating zeta function as

$$\zeta_{\alpha,\beta}(s)=\sum_{n=1}^\infty\frac{d_{\alpha,\beta}(n)}{n^s} \qquad \text{for } \sigma>1.$$

We prove that $\zeta_{\alpha,\beta}(s)$ has an analytic continuation to $\sigma>1/3$ and get an asymptotic formula for the mean square of $\zeta_{\alpha,\beta}(s)$ in the strip $1/2<\sigma<1$.

THEOREM 1.1. *For any $1/2<\sigma<1$ and rational numbers $0<\alpha<\beta$, there exists a constant $\varepsilon(\sigma)>0$ such that*

$$\int_T^{2T}|\zeta_{\alpha,\beta}(\sigma+it)|^2\,dt=T\sum_{n=1}^\infty\frac{d_{\alpha,\beta}^2(n)}{n^{2\sigma}}+O_{\alpha,\beta,\sigma}(T^{1-\varepsilon(\sigma)}). \tag{1.5}$$

Theorem 1.1 can be used to study the distribution of primitive Pythagorean triangles (i.e. triples (a,b,c) with $a,b,c\in\mathbb{N}$, $a^2+b^2=c^2$, $a<b$ and $\gcd(a,b,c)=1$). Let $P(x)$ denote the number of primitive Pythagorean triangles with perimeter $a+b+c\le x$. D. H. Lehmer [10] proved

$$P(x)=\frac{\log 2}{\pi^2}x+O(x^{1/2}\log x).$$

It is difficult to reduce the exponent 1/2 in the error term, which depends on the zero-free region of the Riemann zeta function. However, assuming the Riemann Hypothesis, it was showed in [11] that, for any $\varepsilon > 0$, we have

$$P(x) = \frac{\log 2}{\pi^2} x + O_\varepsilon\big(x^{\frac{5805}{15408}+\varepsilon}\big). \tag{1.6}$$

We improve this result by applying Theorem 1.1 and get

THEOREM 1.2. *If the Riemann Hypothesis is true, then for any $\varepsilon > 0$,*

$$P(x) = \frac{\log 2}{\pi^2} x + O_\varepsilon(x^{4/11+\varepsilon}).$$

Note that $\frac{5805}{15408} = 0.3767\ldots$ and $4/11 = 0.3636\ldots$.

2. Main steps in the proof of Theorem 1.1. First, let us recall the way of getting the asymptotic formula (1.2). In [7, Chapter 3], using the functional equation (1.4), Ivić derives the Voronoi formula for the error term $\Delta(x)$ in (1.1). Then in [7, Chapter 4], he gets the approximation (1.3) by the Voronoi formula, from which we can obtain (1.2) in a standard way.

Now observing that $\zeta_{\alpha,\beta}(s)$ is similar to $\zeta^2(s)$, we may realize the mean square $\int_T^{2T} |\zeta_{\alpha,\beta}(\sigma+it)|^2\,dt$ as an analogue of $\int_T^{2T} |\zeta(\sigma+it)|^4\,dt$. Our main steps in the proof of Theorem 1.1 are similar to the proof of (1.2): In Section 4, we study the asymptotics of the summatory function

$$D_{\alpha,\beta}(x) = \sum_{n\le x} d_{\alpha,\beta}(n). \tag{2.1}$$

In Section 5, we derive a Voronoi type formula for the error term

$$\Delta_{\alpha,\beta}(x) = D_{\alpha,\beta}(x) - \text{main terms}.$$

In Section 6, using the asymptotic formula for $D_{\alpha,\beta}(x)$ and the Voronoi type formula for $\Delta_{\alpha,\beta}(x)$, we obtain the following approximation for $\zeta_{\alpha,\beta}(s)$, which is the key to the proof of Theorem 1.1.

PROPOSITION 2.1. *For fixed rational numbers $\alpha, \beta > 0$, the function $\zeta_{\alpha,\beta}(s)$ can be analytically extended to the half-plane $\sigma > 1/3$ with simple poles at $s = 1/2, 1$. Moreover, suppose $T \ge 2$, $s = \sigma + it$ and $4\pi^2 xy = t^2$. Then for any $1/2 < \sigma < 1$ and $T < t \le 2T$, we have*

$$\zeta_{\alpha,\beta}(s) = \sum_{n\le x} \frac{d_{\alpha,\beta}(n)}{n^s} + \chi^2(s) \sum_{n\le y} \frac{d_{\alpha,\beta}(n)}{n^{s-1}} + E_{\alpha,\beta}(s), \tag{2.2}$$

where $\chi(s)$ is given by (1.4) and $E_{\alpha,\beta}(s)$ satisfies

$$\int_T^{2T} |E_{\alpha,\beta}(\sigma+it)|^2\,dt \ll_{\alpha,\beta,\sigma} (x^{-2\sigma}T^2 + x^{1-\sigma}T^{1/2} + x^{1/2-\sigma}T + x^{-\sigma}T^{3/2}) \log^3 T. \tag{2.3}$$

From (2.2), we can derive Theorem 1.1 in a standard way. Hence the main work is to prove Proposition 2.1.

3. Preliminary lemmas. Denote the integer part of u by $[u]$. Let $\psi(u) = u - [u] - 1/2$ and $e(x) = e^{2\pi i x}$. It is well known that $\psi(u)$ has a truncated Fourier expansion (see e.g. [5]).

LEMMA 3.1. *For any real number $H > 2$, we have*

$$\psi(u) = -\frac{1}{2\pi i} \sum_{1 \le |h| \le H} \frac{1}{h} e(hu) + O(G(u,H)),$$

where

$$G(u,H) = \min\left(1, \frac{1}{H\|u\|}\right). \tag{3.1}$$

We will use the first derivative test (see e.g. [12, Chapter 21]).

LEMMA 3.2. *Let $G(x)$ and $F(x)$ be real differentiable functions such that $F'(x)/G(x)$ is monotonic and either $F'(x)/G(x) \ge m > 0$ or $F'(x)/G(x) \le -m < 0$. Then*

$$\Big| \int_a^b G(x) e^{iF(x)}\, dx \Big| \le 4m^{-1}.$$

We will also use the following van der Corput B-process (see [8, Lemma 2.2]).

LEMMA 3.3. *Let C_i, $i = 1, \ldots, 7$, be absolute positive constants. Suppose that g is a real-valued function which has four continuous derivatives on the interval $[A,B]$. Let L and W be real parameters not less than 1 such that $C_1 L \le B - A \le C_2 L$,*

$$|g^{(j)}(\omega)| \le -C_{j+2} W L^{1-j} \quad \text{for } \omega \in [A,B],\ j = 1,2,3,4,$$

and

$$g''(\omega) \ge C_7 W L^{-1} \quad \text{or} \quad g''(\omega) \le -C_7 W L^{-1}, \quad \text{for } \omega \in [A,B].$$

Let ϕ denote the inverse function of g'. Define

$$\epsilon_f = \begin{cases} e^{\pi i/4} & \text{if } g''(\omega) > 0 \text{ for all } \omega \in [A,B], \\ e^{-\pi i/4} & \text{if } g''(\omega) < 0 \text{ for all } \omega \in [A,B] \end{cases}$$

and

$$r(x) = \begin{cases} 0 & \text{if } g'(x) \in \mathbb{Z}, \\ \min(1/\|g'(x)\|, \sqrt{L/W}) & \text{else,} \end{cases}$$

with $\|\cdot\|$ *denoting the distance from the nearest integer. Then*

$$\sum_{A<l\le B} e(g(l)) = \epsilon_f \sum_{\min(g'(A),g'(B))\le k\le \max(g'(A),g'(B))}'' \frac{e\big(g(\phi(k)) - k\phi(k)\big)}{\sqrt{|g''(\phi(k))|}} + O\big(r(A) + r(B) + \log(2+W)\big),$$

with the notation

$$\sum_{a\le m\le b}'' \Phi(n) = \tfrac{1}{2}(\chi_{\mathbb{Z}}(a)\Phi(a) + \chi_{\mathbb{Z}}(b)\Phi(b)) + \sum_{a<m<b} \Phi(n),$$

where $\chi_{\mathbb{Z}}(\cdot)$ *is the indicator function of the integers and the* O*-constant depends on the constants* C_i, $i = 1,\dots,7$.

4. Asymptotic formula for the summatory function

PROPOSITION 4.1. *Let* $\alpha = p_1/q_1$ *and* $\beta = p_2/q_2$ *with* $p_1, p_2, q_1, q_2 \in \mathbb{N}$, $\gcd(p_1, q_1) = 1$ *and* $\gcd(q_1, q_2) = 1$. *Then*

$$D_{\alpha,\beta}(x) = c_1 x + c_2\sqrt{x} + \Delta_{\alpha,\beta}(x),$$

where

$$c_1 = c_1(\alpha,\beta) = \frac{\log\alpha - \log\beta}{2}, \qquad c_2 = c_2(\alpha,\beta) = \frac{1}{2}\left(\sqrt{\frac{1}{p_2q_2}} - \sqrt{\frac{1}{p_1q_1}}\right)$$

and

$$\Delta_{\alpha,\beta}(x) = -\sum_{\sqrt{x/\beta}<l\le\sqrt{x/\alpha}} \psi(x/l) + O_{\alpha,\beta}(1). \tag{4.1}$$

Proof. It is enough to consider

$$d_\alpha(n) = \sum_{\substack{n=kl\\ k\le\alpha l}} 1 \quad\text{and}\quad D_\alpha(x) = \sum_{n\le x} d_\alpha(n).$$

Clearly,

$$D_\alpha(x) = \sum_{\substack{kl\le x\\ k\le\alpha l}} 1 = \sum_{l\le x}\ \sum_{k\le\min(x/l,\alpha l)} 1.$$

Write

$$D_\alpha(x) = \sum\nolimits_1 + \sum\nolimits_2 \tag{4.2}$$

with

$$\sum\nolimits_1 = \sum_{l\le\sqrt{x/\alpha}}\ \sum_{k\le\alpha l} 1 \quad\text{and}\quad \sum\nolimits_2 = \sum_{\sqrt{x/\alpha}<l\le x}\ \sum_{k\le x/l} 1.$$

It is easy to see that

$$(4.3)\qquad \sum\nolimits_1 = \sum_{l\le\sqrt{x/\alpha}} (\alpha l - \psi(\alpha l) - 1/2)$$
$$= \frac{x}{2} - \sqrt{\alpha x}\,\psi\Big(\sqrt{\frac{x}{\alpha}}\Big) - \sum_{l\le\sqrt{x/\alpha}} \psi(\alpha l) - \frac{1}{2}\sqrt{\frac{x}{\alpha}} + O_\alpha(1).$$

Similarly,

$$(4.4)\qquad \sum\nolimits_2 = \sum_{\sqrt{x/\alpha}<l\le x} (x/l - \psi(x/l) - 1/2)$$
$$= x\sum_{\sqrt{x/\alpha}<l\le x} 1/l - \sum_{\sqrt{x/\alpha}<l\le x} \psi(x/l) - \frac{1}{2}x + \frac{1}{2}\sqrt{\frac{x}{\alpha}} + O(1).$$

By Euler–Maclaurin summation, we have

$$(4.5)\qquad \sum_{\sqrt{x/\alpha}<l\le x} 1/l = \frac{1}{2}\log x + \frac{1}{2}\log\alpha + \sqrt{\frac{\alpha}{x}}\,\psi\Big(\sqrt{\frac{x}{\alpha}}\Big) + O_a\Big(\frac{1}{x}\Big).$$

Combining (4.2)–(4.5), we get

$$D_\alpha(x) = \frac{x}{2}\log x + \frac{\log\alpha}{2}x - \sum_{\sqrt{x/\alpha}<l\le x} \psi(x/l) - \sum_{l\le\sqrt{x/\alpha}} \psi(\alpha l) + O_\alpha(1).$$

Note that

$$-\sum_{l\le\sqrt{x/\alpha}} \psi(\alpha l) = -\sum_{l\le\sqrt{q_1 x/p_1}} \psi\Big(\frac{p_1 l}{q_1}\Big) = \frac{1}{2}\sqrt{\frac{x}{p_1 q_1}} + O_\alpha(1).$$

Hence

$$(4.6)\qquad D_\alpha(x) = \frac{x}{2}\log x + \frac{\log\alpha}{2}x - \sum_{\sqrt{x/\alpha}<l\le x} \psi(x/l)$$
$$+ \frac{1}{2}\sqrt{\frac{x}{p_1 q_1}} + O_\alpha(1).$$

Similarly, for

$$d_\beta(n) = \sum_{\substack{n=kl\\ k\le\beta l}} 1 \quad\text{and}\quad D_\beta(x) = \sum_{n\le x} d_\beta(n),$$

we have

$$(4.7)\qquad D_\beta(x) = \frac{x}{2}\log x + \frac{\log\beta}{2}x - \sum_{\sqrt{x/\beta}<l\le x} \psi(x/l) + \frac{1}{2}\sqrt{\frac{x}{p_2 q_2}} + O_\beta(1).$$

Now Proposition 4.1 follows from (4.6), (4.7) and

$$D_{\alpha,\beta}(x) = D_\beta(x) - D_\alpha(x). \ \blacksquare$$

COROLLARY 4.2. *We have*

$$D_{\alpha,\beta}(x) = c_1 x + c_2\sqrt{x} + O_{\alpha,\beta}(x^{1/3}),$$

where c_1, c_2 *are as in Proposition* 4.1.

Proof. This can be proved easily (even with a better upper bound for the error term) by applying Lemma 3.1 and exponential pairs (see [3]) to Proposition 4.1. ■

5. A Voronoi type formula. In this section, we will use the technique of [8] to derive a Voronoi type formula for $\Delta_{\alpha,\beta}(x)$. Define

$$d_{\alpha,\beta}(n,H) = \sum_{1\le h\le H} \sideset{}{''}\sum_{\substack{h\alpha\le k\le h\beta \\ n=hk}} 1,$$

where $\sum''$ is as in Lemma 3.3. Using the van der Corput B-process and the same argument as in [14, Section 6.2], we can derive the following Voronoi type formula for $\Delta_{\alpha,\beta}(x)$.

LEMMA 5.1. *Suppose* $\alpha,\beta > 0$ *are fixed rational numbers and* $G(u,H)$ *is given by* (3.1). *Then for any* $H \ge 2$, *we have*

$$\Delta_{\alpha,\beta}(x) = M_{\alpha,\beta}(x,H) + E_{\alpha,\beta}(x,H) + F_{\alpha,\beta}(x,H),$$

where

$$M_{\alpha,\beta}(x,H) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\le\beta H^2} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4), \tag{5.1}$$

$$E_{\alpha,\beta}(x,H) \ll \sum_{\sqrt{x/\alpha}<l\le\sqrt{x/\beta}} G\biggl(\frac{x}{l}, H\biggr), \tag{5.2}$$

$$F_{\alpha,\beta}(x,H) \ll_{\alpha,\beta} \log H. \tag{5.3}$$

Proof. Applying Lemma 3.1 to (4.1), we get

$$\Delta_{\alpha,\beta}(x) = \frac{1}{2\pi i} \sum_{1\le|h|\le H} \frac{1}{h} \sum_{\sqrt{x/\beta}<l\le\sqrt{x/\alpha}} e\biggl(\frac{hx}{l}\biggr) + E_{\alpha,\beta}(x,H) + O_{\alpha,\beta}(1)$$

with

$$E_{\alpha,\beta}(x,H) \ll \sum_{\sqrt{x/\alpha}<l\le\sqrt{x/\beta}} G\biggl(\frac{x}{l}, H\biggr). \tag{5.4}$$

Let

$$S_{\alpha,\beta}(x,H)=\frac{1}{2\pi i}\sum_{1\le h\le H}\frac{1}{h}\sum_{\sqrt{x/\beta}<l\le\sqrt{x/\alpha}}e\Big(\frac{hx}{l}\Big). \tag{5.5}$$

Then we can write

$$\Delta_{\alpha,\beta}(x)=\frac{1}{2\pi i}\big(S_{\alpha,\beta}(x,H)-\overline{S_{\alpha,\beta}(x,H)}\big)+E_{\alpha,\beta}(x,H)+O_{\alpha,\beta}(1). \tag{5.6}$$

To treat the inner sum

$$\sum_{\sqrt{x/\beta}<l\le\sqrt{x/\alpha}}e\Big(\frac{hx}{l}\Big)\quad\text{for } 1\le h\le H$$

in (5.5), we apply Lemma 3.3. Let

$$A=\sqrt{\frac{x}{\beta}},\qquad B=\sqrt{\frac{x}{\alpha}},\qquad g(l)=\frac{hx}{l}.$$

Then

$$g'(l)=-\frac{hx}{l^2},\quad g''(l)=\frac{2hx}{l^3},\quad g^{(3)}(l)=-\frac{6hx}{l^4},\quad g^{(4)}(l)=\frac{24hx}{l^5},$$

$$g'(B)=-h\alpha,\quad g'(A)=-h\beta,\quad \frac{2\alpha^{3/2}h}{\sqrt{x}}<g''(l)\le\frac{2\beta^{3/2}h}{\sqrt{x}},\quad |g'''(l)|\ll_{\alpha,\beta}\frac{h}{x}.$$

Hence we can take

$$W=1,\qquad L=\frac{\sqrt{x}}{h},\qquad \phi(k)=\sqrt{-\frac{hx}{k}},$$

$$g(\phi(k))-k\phi(k)=2\sqrt{-hkx},\qquad g''\left(\phi\left(k\right)\right)=2\sqrt{\frac{(-k)^3}{hx}}.$$

Noting that α,β are rational numbers, we have

$$r(A),r(B)\ll_{\alpha,\beta}1. \tag{5.7}$$

Now for $1\le h\le H$, by Lemma 3.3, we get

$$\begin{aligned}\sum_{\sqrt{x/\alpha}<l\le\sqrt{x/\beta}}e\Big(\frac{hx}{l}\Big)\\ &=\frac{e^{\pi i/4}}{\sqrt{2}}\sideset{}{''}\sum_{-h\beta\le k\le -h\alpha}\frac{h^{1/4}x^{1/4}}{(-k)^{3/4}}e\big(2\sqrt{-hkx}\big)+O_{\alpha,\beta}(1)\\ &=\frac{1}{\sqrt{2}}\sideset{}{''}\sum_{h\alpha\le k\le h\beta}\frac{h^{1/4}x^{1/4}}{k^{3/4}}e\big(2\sqrt{hkx}+1/8\big)+O_{\alpha,\beta}(1).\end{aligned} \tag{5.8}$$

Inserting (5.8) into (5.5) gives

$$\begin{aligned} S_{\alpha,\beta}(x,H) &= \frac{1}{\sqrt{2}}\sum_{1\le h\le H}\frac{1}{h}\sum\nolimits''_{h\alpha\le k\le h\beta}\frac{h^{1/4}x^{1/4}}{k^{3/4}}e\big(2\sqrt{hkx}+1/8\big)+O_{\alpha,\beta}(\log H)\\ &= \frac{x^{1/4}}{\sqrt{2}}\sum_{1\le h\le H}\sum\nolimits''_{h\alpha\le k\le h\beta}\frac{1}{(hk)^{3/4}}e\big(2\sqrt{hkx}+1/8\big)+O_{\alpha,\beta}(\log H)\\ &= \frac{x^{1/4}}{\sqrt{2}}\sum_{n\le\beta H^2}\frac{d_{\alpha,\beta}(n,H)}{n^{3/4}}e\big(2\sqrt{nx}+1/8\big)+O_{\alpha,\beta}(\log H). \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{2\pi i}\big(S_{\alpha,\beta}(x,H)-\overline{S_{\alpha,\beta}(x,H)}\big)\\ &\qquad= \frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\le\beta H^2}\frac{d_{\alpha,\beta}(n,H)}{n^{3/4}}\cos(4\pi\sqrt{nx}-\pi/4)+O_{\alpha,\beta}(\log H). \end{aligned}$$

This combined with (5.6) and (5.4) yields Lemma 5.1. ∎

REMARK 5.2. The bound (5.3) is important in the proof of Theorem 1.1. If α,β are not rational numbers, the author has not been able to get the estimate (5.3) because in this case (5.7) does not hold.

6. Proof of Proposition 2.1. First, let us show that $\zeta_{\alpha,\beta}(s)$ can be analytically extended to $\sigma>1/3$. For $\sigma>1$ and any $N\ge 1$, write

$$\begin{aligned} \zeta_{\alpha,\beta}(s) &= \sum_{n\le N}\frac{d_{\alpha,\beta}(n)}{n^s}+\sum_{n>N}\frac{d_{\alpha,\beta}(n)}{n^s}\\ &= \sum_{n\le N}\frac{d_{\alpha,\beta}(n)}{n^s}+\int_{N^+}^{\infty}u^{-s}\,dD_{\alpha,\beta}(u), \end{aligned}$$

where $D_{\alpha,\beta}(u)$ is defined by (2.1). Applying Proposition 4.1, we get

$$\begin{aligned} \zeta_{\alpha,\beta}(s) &= \sum_{n\le N}\frac{d_{\alpha,\beta}(n)}{n^s}+\int_{N^+}^{\infty}u^{-s}\,d\big(c_1u+c_2\sqrt{u}+\Delta_{\alpha,\beta}(u)\big)\\ &= \sum_{n\le N}\frac{d_{\alpha,\beta}(n)}{n^s}+c_1\int_{N^+}^{\infty}u^{-s}\,du+\frac{c_2}{2}\int_{N^+}^{\infty}u^{-s-1/2}\,du\\ &\quad+\int_{N^+}^{\infty}u^{-s}\,d\Delta_{\alpha,\beta}(u). \end{aligned}$$

By partial integration, we have

$$\zeta_{\alpha,\beta}(s) = \sum_{n\le N} \frac{d_{\alpha,\beta}(n)}{n^s} - \frac{c_1 N^{1-s}}{1-s} - \frac{c_2 N^{1/2-s}}{1-2s} + s\int_{N^+}^{\infty} \Delta_{\alpha,\beta}(u)u^{-s-1}\,du + O(N^{1/3-\sigma}). \tag{6.1}$$

From Corollary 4.2, we can see that the integral in (6.1) is absolutely convergent for $\sigma > 1/3$, hence (6.1) gives an analytic continuation of $\zeta_{\alpha,\beta}(s)$ for $\sigma > 1/3$. This proves the first assertion of Proposition 2.1.

Now suppose that $\sigma > 1/3$ and $2 \le T < t \le 2T$. From now on, we take $N = T^A$ with $A > 0$ being a constant, sufficiently large. Break the sum in (6.1) into

$$\sum_{n\le N} \frac{d_{\alpha,\beta}(n)}{n^s} = \sum_{n\le x} \frac{d_{\alpha,\beta}(n)}{n^s} + \sum_{x<n\le N} \frac{d_{\alpha,\beta}(n)}{n^s}. \tag{6.2}$$

For the second sum, applying Proposition 4.1 again, we have

$$\begin{aligned}\sum_{x<n\le N} \frac{d_{\alpha,\beta}(n)}{n^s} &= \int_x^N u^{-s}\,dD_{\alpha,\beta}(u)\\ &= \int_x^N u^{-s}\,d\big(c_1(\alpha,\beta)u + c_2(\alpha,\beta)\sqrt{u} + \Delta_{\alpha,\beta}(u)\big).\end{aligned}$$

By partial integration, we have

$$\begin{aligned}\sum_{x<n\le N} \frac{d_{\alpha,\beta}(n)}{n^s} &= c_1(\alpha,\beta)\bigg(\frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s}\bigg)\\ &\quad + c_2(\alpha,\beta)\bigg(\frac{N^{1/2-s}}{1-2s} - \frac{x^{1/2-s}}{1-2s}\bigg) + N^{-s}\Delta_{\alpha,\beta}(N)\\ &\quad - x^{-s}\Delta_{\alpha,\beta}(x) + s\int_x^N \Delta_{\alpha,\beta}(u)u^{-s-1}\,du.\end{aligned} \tag{6.3}$$

Combining (6.1)–(6.3), we get

$$\begin{aligned}\zeta_{\alpha,\beta}(s) &= \sum_{n\le x} \frac{d_{\alpha,\beta}(n)}{n^s} + s\int_x^N \Delta_{\alpha,\beta}(u)u^{-s-1}\,du\\ &\quad + O_{\alpha,\beta,\sigma}(x^{1-\sigma}t^{-1} + x^{1/3-\sigma})\end{aligned} \tag{6.4}$$

for any $\sigma > 1/3$.

Our tool to prove Proposition 2.1 is the Voronoi formula for $\Delta_{\alpha,\beta}(x)$. Using Lemma 5.1, we can write

$$s\int_x^N \Delta_{\alpha,\beta}(u)u^{-s-1}\,du = \mathfrak{M}(s)+\mathfrak{E}(s)+\mathfrak{F}(s), \tag{6.5}$$

where

$$\mathfrak{M}(s) = \mathfrak{M}_{\alpha,\beta}(s,H,x,N) = s\int_x^N M_{\alpha,\beta}(u,H)u^{-s-1}\,du, \tag{6.6}$$

$$\mathfrak{E}(s) = \mathfrak{E}_{\alpha,\beta}(s,H,x,N) = s\int_x^N E_{\alpha,\beta}(u,H)u^{-s-1}\,du, \tag{6.7}$$

$$\mathfrak{F}(s) = \mathfrak{F}_{\alpha,\beta}(s,H,x,N) = s\int_x^N F_{\alpha,\beta}(u,H)u^{-s-1}\,du. \tag{6.8}$$

In Lemmas 7.2 and 7.3, we will show that the upper bound of $\mathfrak{E}(s)$ is small when H is large compared to N. We will also show that the mean square of $\mathfrak{F}(s)$ has an acceptable estimate. In Lemma 8.1, we will pick out the second term in (2.2) from $\mathfrak{M}(s)$. Combining (6.4) and (6.5) with Lemmas 7.2, 7.3 and 8.1, we get Proposition 2.1.

7. An upper bound and a mean square estimate. To bound $\mathfrak{E}(s)$, we need the following mean value estimate for $G(u,H)$ defined by (3.1).

LEMMA 7.1. *For any $N\ge 1$ and $H\ge 2$, we have*

$$\int_0^N G(u,H)\,du \ll \frac{N\log H}{H}.$$

Proof. Noting that $G(u,H)$ is a positive 1-periodic function, we have

$$\int_0^N G(u,H)\,du \le \sum_{k=0}^{[N]}\int_k^{k+1} G(u,H)\,du \ll N\int_0^1 G(u,H)\,du$$
$$= N\int_{-1/2}^{1/2}\min\Big(1,\frac{1}{H\|u\|}\Big)\,du.$$

Noting that $\|u\|=|u|$ for $u\in[-1/2,1/2]$, we get

$$\int_0^N G(u,H)\,du \ll N\int_{-1/2}^{1/2}\min\Big(1,\frac{1}{H|u|}\Big)\,du \ll N\int_0^{1/2}\min\Big(1,\frac{1}{Hu}\Big)\,du$$
$$\ll N\int_0^{1/H} du + \frac{N}{H}\int_{1/H}^{1/2}\frac{1}{u}\,du,$$

which yields

$$\int_0^N G(u,H)\,du \ll \frac{N\log H}{H}. \quad \blacksquare$$

By Lemma 7.1, we can get

LEMMA 7.2. *For any $\sigma > 1/2$, we have*

$$\mathfrak{E}(s) \ll \frac{tx^{-\sigma-1}N^2\log H}{H}.$$

Proof. By (6.7) and trivial estimates, we get

$$\begin{aligned}\mathfrak{E}(s) &\ll t\int_x^N \sum_{\sqrt{u/\alpha}<l\le\sqrt{u/\beta}} G\Big(\frac{u}{l},H\Big)u^{-\sigma-1}\,du\\ &\ll tx^{-\sigma-1}\sum_{\sqrt{x/\alpha}<l\le\sqrt{N/\beta}}\int_x^N G\Big(\frac{u}{l},H\Big)\,du\\ &= tx^{-\sigma-1}\sum_{\sqrt{x/\alpha}<l\le\sqrt{N/\beta}} l\int_{x/l}^{N/l} G(u,H)\,du.\end{aligned}$$

This combined with Lemma 7.1 yields

$$\mathfrak{E}(s) \ll tx^{-\sigma-1}\sum_{l\le\sqrt{N/\beta}} l\int_0^N G(u,H)\,du \ll \frac{tx^{-\sigma-1}N^2\log H}{H}. \quad \blacksquare$$

Now we consider the mean square of $\mathfrak{F}(s)$.

LEMMA 7.3. *For $\sigma > 1/2$, we have*

$$\int_T^{2T} |\mathfrak{F}(s)|^2\,dt \ll_{\alpha,\beta,\sigma} x^{-2\sigma}T^2\log^2 H\log N.$$

Proof. Noting that $F_{\alpha,\beta}(u) \ll_{\alpha,\beta} \log H$ and unfolding the square in the integral, we get

$$\begin{aligned}\int_T^{2T} |\mathfrak{F}(s)|^2\,dt &\ll T^2\int_T^{2T}\Big|\int_x^N F_{\alpha,\beta}(u)u^{-s-1}\,du\Big|^2\,dt\\ &\ll_{\alpha,\beta} T^2\log^2 H\int_x^N\int_x^N (u_1u_2)^{-\sigma-1}\Big|\int_T^{2T}\Big(\frac{u_2}{u_1}\Big)^{it}\,dt\Big|\,du_1\,du_2.\end{aligned}$$

Applying Lemma 3.2 to the above integral over t, we have

$$\int_T^{2T} |\mathfrak{F}(s)|^2\,dt \ll_{\alpha,\beta} T^2\log^2 H \int_x^N\int_x^N (u_1u_2)^{-\sigma-1}\min\left(T, \frac{1}{|\log\frac{u_2}{u_1}|}\right) du_1\,du_2$$
$$\ll_{\alpha,\beta} T^2\log^2 H \int_x^N\int_{u_1}^N (u_1u_2)^{-\sigma-1}\min\left(T, \frac{1}{\log\frac{u_2}{u_1}}\right) du_1\,du_2.$$

Write this as

$$(7.1)\qquad \int_T^{2T} |\mathfrak{F}(s)|^2\,dt \ll_{\alpha,\beta} \int_1 + \int_2 + \int_3,$$

where

$$\int_1 = T^3\log^2 H \int_x^N u_1^{-\sigma-1}\int_{u_1}^{e^{1/T}u_1} u_2^{-\sigma-1}\,du_2\,du_1,$$
$$\int_2 = T^2\log^2 H \int_x^N u_1^{-\sigma-1}\int_{e^{1/T}u_1}^{\frac{3}{2}u_1} u_2^{-\sigma-1}\frac{1}{\log\frac{u_2}{u_1}}\,du_2\,du_1,$$
$$\int_3 = T^2\log^2 H \int_x^N u_1^{-\sigma-1}\int_{\frac{3}{2}u_1}^{N} u_2^{-\sigma-1}\frac{1}{\log\frac{u_2}{u_1}}\,du_2\,du_1.$$

Let us deal with $\int_i$, $i=1,2,3$, separately. For $\int_1$, we have

$$\int_1 \ll T^3\log^2 H \int_x^N u_1^{-2\sigma-2}\int_{u_1}^{e^{1/T}u_1} du_2\,du_1$$
$$\ll T^3\log^2 H \int_x^N u_1^{-2\sigma-2}(e^{1/T}u_1-u_1)\,du_1$$
$$\ll T^3(e^{1/T}-1)\log^2 H \int_x^N u_1^{-2\sigma-1}\,du_1,$$

which yields

$$(7.2)\qquad \int_1 \ll_\sigma x^{-2\sigma}T^2\log^2 H.$$

For $\int_2$, we have

$$\begin{aligned}\int_2 &= T^2 \log^2 H \int_x^N u_1^{-\sigma-1} \int_{e^{1/T}u_1}^{\frac{3}{2}u_1} u_2^{-\sigma-1} \frac{1}{\log \frac{u_2}{u_1}}\, du_2\, du_1 \\ &= T^2 \log^2 H \int_x^N u_1^{-\sigma-1} \int_{e^{1/T}u_1}^{\frac{3}{2}u_1} u_2^{-\sigma-1} \frac{1}{\log\bigl(1+\frac{u_2-u_1}{u_1}\bigr)}\, du_2\, du_1 \\ &\ll T^2 \log^2 H \int_x^N u_1^{-2\sigma-1} \int_{e^{1/T}u_1}^{\frac{3}{2}u_1} \frac{1}{u_2-u_1}\, du_2\, du_1 \\ &\ll T^2 \log^2 H \int_x^N u_1^{-2\sigma-1} \log u_1\, du_1,\end{aligned}$$

which yields

$$\int_2 \ll_\sigma x^{-2\sigma} T^2 \log^2 H \log N. \tag{7.3}$$

For $\int_3$, we have

$$\int_3 \ll T^2 \log^2 H \Bigl(\int_x^N u^{-\sigma-1}\, du\Bigr)^2 \ll_\sigma x^{-2\sigma} T^2 \log^2 H. \tag{7.4}$$

From (7.1)–(7.4), we get Lemma 7.3. ∎

8. Picking out the second term in Proposition 2.1. The second term of (2.2) in Proposition 2.1 is hidden in $\mathfrak{M}(s)$. In this section, we will pick it out and prove

LEMMA 8.1. *For $\sigma > 1/2$, we have*

$$\begin{aligned}\mathfrak{M}(s) = \chi^2(s) \sum_{n\le x} d_{\alpha,\beta}(n) n^{s-1} + O\bigl(t^{-1/2} x^{1-\sigma} \log H \\ + x^{1/2-\sigma}\log H + x^{1/2-\sigma}\log t + x^{-\sigma} t^{1/2}\log t\bigr).\end{aligned} \tag{8.1}$$

The idea of the proof of Lemma 8.1 comes from [7, Chapter 4]. By (6.6) and (5.1), we have

$$\mathfrak{M}(s) = \frac{s}{\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{n\le \beta H^2} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} \cos(4\pi\sqrt{nu} - \pi/4)\, du.$$

Let $\eta > 0$ be a fixed, sufficiently small constant. Using $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, we can write

$$\mathfrak{M}(s) = \mathfrak{M}_1(s) + \mathfrak{M}_2(s) + \mathfrak{M}_3(s) + \mathfrak{M}_4(s) \tag{8.2}$$

with

$$\mathfrak{M}_1(s) = \frac{s}{2\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{n \le (1+\eta)y} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} e(2\sqrt{nu} - 1/8)\, du,$$

$$\mathfrak{M}_2(s) = \frac{s}{2\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{(1+\eta)y < n \le \beta H^2} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} e(2\sqrt{nu} - 1/8)\, du,$$

$$\mathfrak{M}_3(s) = \frac{s}{2\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{n \le y} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} e(-2\sqrt{nu} + 1/8)\, du,$$

$$\mathfrak{M}_4(s) = \frac{s}{2\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{y < n \le \beta H^2} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} e(-2\sqrt{nu} + 1/8)\, du.$$

We will bound $\mathfrak{M}_2(s)$, $\mathfrak{M}_3(s)$ and $\mathfrak{M}_4(s)$ in the following Lemmas 8.2–8.4 and pick out the first term on the right side of (8.1) in Lemma 8.5. From Lemmas 8.2–8.5 and (8.2), we get Lemma 8.1.

LEMMA 8.2. *For $\sigma > 1/2$, we have*

$$\mathfrak{M}_2(s) \ll t^{-1/2} x^{1-\sigma} \log H.$$

Proof. Write

$$\mathfrak{M}_2(s) = \frac{s}{2\pi\sqrt{2}} \sum_{(1+\eta)y < n \le \beta H^2} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \times \int_x^N u^{-\sigma-3/4} e\left(-\frac{t}{2\pi}\log u + 2\sqrt{nu} - 1/8\right) du.$$

In Lemma 3.2, taking

$$G(u) = u^{-\sigma-3/4}, \qquad F_t(u) = -\frac{t}{2\pi}\log u + 2\sqrt{nu} - 1/8,$$

we obtain

$$F_t'(u) = -\frac{t}{2\pi u} + \sqrt{\frac{n}{u}}, \qquad \frac{F_t'(u)}{G(u)} = -\frac{t}{2\pi} u^{\sigma-1/4} + \sqrt{n}\, u^{\sigma+1/4}.$$

Since $n > (1+\eta)y$, $u > x$ and $4\pi^2 xy = t^2$, we get

$$\left(\frac{F_t'(u)}{G(u)}\right)' = -(\sigma - 1/4)\frac{t}{2\pi} u^{\sigma-5/4} + (\sigma + 1/4)\sqrt{n}\, u^{\sigma-3/4} > 0.$$

Thus $F'(u)/G(u)$ is monotonic and

$$\begin{aligned}\frac{F_t'(u)}{G(u)} &= -\left(\frac{t^2}{4\pi^2 nu}\right)^{1/2}\sqrt{n}\,u^{\sigma+1/4} + \sqrt{n}\,u^{\sigma+1/4} \\ &\geq -\left(\frac{t^2}{4\pi^2(1+\eta)yx}\right)^{1/2}\sqrt{n}\,x^{\sigma+1/4} + \sqrt{n}\,x^{\sigma+1/4} \\ &\geq \left(1 - \frac{1}{\sqrt{1+\eta}}\right)\sqrt{n}\,x^{\sigma+1/4} \gg \sqrt{n}\,x^{\sigma+1/4}.\end{aligned}$$

Hence Lemma 3.2 gives

$$\int_x^N u^{-\sigma-3/4} e\left(-\frac{t}{2\pi}\log u - \sqrt{nu} + 1/8\right) du \ll x^{-\sigma-1/4} n^{-1/2},$$

which yields

$$\mathfrak{M}_2(s) \ll x^{-\sigma+3/4} \sum_{(1+\eta)y<n\leq\beta H^2} \frac{d_{\alpha,\beta}(n;H)}{n^{5/4}} \ll t^{-1/2}x^{1-\sigma}\log H. \ \blacksquare$$

LEMMA 8.3. *For $\sigma > 1/2$, we have*

$$\mathfrak{M}_3(s) \ll_\sigma (x^{1/2-\sigma} + x^{-\sigma}t^{1/2})\log t.$$

Proof. Write

$$\begin{aligned}\mathfrak{M}_3(s) &= \frac{s}{2\pi\sqrt{2}} \int_x^N u^{-s-3/4} \sum_{n\leq y} \frac{d_{\alpha,\beta}(n,H)}{n^{3/4}} e(-2\sqrt{nu} + 1/8)\,du \\ &= -\frac{1}{2\pi\sqrt{2}} \int_x^N (-s+1/4) \sum_{n\leq y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \\ &\qquad\qquad \times e(-2\sqrt{nu}+1/8)u^{-s-3/4}\,du \\ &\quad + \frac{1}{8\pi\sqrt{2}} \int_x^N \sum_{n\leq y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} e(-2\sqrt{nu}+1/8)u^{-s-3/4}\,du \\ &= -\frac{1}{2\pi\sqrt{2}} \sum_{n\leq y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \int_x^N e(-2\sqrt{nu}+1/8)\,du^{-s+1/4} \\ &\quad + \frac{1}{8\pi\sqrt{2}} \sum_{n\leq y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \int_x^N e(-2\sqrt{nu}+1/8)u^{-s-3/4}\,du.\end{aligned}$$

By partial integration, we have

$$\mathfrak{M}_3(s) = \mathfrak{M}_{31}(s) + \mathfrak{M}_{32}(s) + \mathfrak{M}_{33}(s) + \mathfrak{M}_{34}(s), \tag{8.3}$$

where

$$\mathfrak{M}_{31}(s) = -\frac{N^{-s+1/4}}{2\pi\sqrt{2}} \sum_{n\le y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} e(-2\sqrt{nN}+1/8),$$

$$\mathfrak{M}_{32}(s) = \frac{x^{-s+1/4}}{2\pi\sqrt{2}} \sum_{n\le y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} e(-2\sqrt{nx}+1/8),$$

$$\mathfrak{M}_{33}(s) = -\frac{i}{\sqrt{2}} \sum_{n\le y} \frac{d_{\alpha,\beta}(n;H)}{n^{1/4}} \int_x^N u^{-\sigma-1/4} \times e\left(-\frac{t}{2\pi}\log u - 2\sqrt{nu} - 1/8\right) du,$$

$$\mathfrak{M}_{34}(s) = \frac{1}{8\pi\sqrt{2}} \sum_{n\le y} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \int_x^N e(-2\sqrt{nu}-1/8) u^{-s-3/4}\, du.$$

Using $d_{\alpha,\beta}(n;H) \le d(n)$ and trivial estimates, it is easy to get

$$\mathfrak{M}_{31}(s), \mathfrak{M}_{32}(s) \ll_\sigma x^{1/2-\sigma}\log t, \tag{8.4}$$

$$\mathfrak{M}_{34}(s) \ll_\sigma x^{-\sigma+1/4} y^{1/4} \log y \ll x^{-\sigma} t^{1/2} \log t. \tag{8.5}$$

Now we deal with $\mathfrak{M}_{33}(s)$. In Lemma 3.2, let

$$H(u) = 1, \qquad G(u) = u^{-\sigma-1/4}, \qquad F_t(u) = -\frac{t}{2\pi}\log u - 2\sqrt{nu} - 1/8.$$

Then we have

$$F_t'(u) = -\frac{t}{2\pi u} - \sqrt{\frac{n}{u}}, \qquad \frac{F_t'(u)}{G(u)} = -\frac{t}{2\pi} u^{\sigma-3/4} - \sqrt{n} u^{\sigma-1/4}.$$

Obviously,

$$\frac{F_t'(u)}{G(u)} < -\sqrt{n}\, u^{\sigma-1/4} \le -\sqrt{n} x^{\sigma-1/4}.$$

Noting that

$$\left(\frac{F_t'(u)}{G(u)}\right)' = -\left(\sigma - \frac{3}{4}\right)\frac{t}{2\pi} u^{\sigma-7/4} - \left(\sigma - \frac{1}{4}\right)\sqrt{n}\, u^{\sigma-5/4},$$

let $u_0 = \frac{(3/4-\sigma)t}{(\sigma-1/4)2\pi\sqrt{n}}$ be the root of $(F_t'(u)/G(u))' = 0$. If $u_0 \in [x, N]$, then $F_t'(u)/G(u)$ is monotonic in $[x, u_0]$ and $[u_0, N]$ respectively; otherwise $F_t'(u)/G(u)$ is monotonic in $[x, N]$. In either case, Lemma 3.2 is valid and gives

$$\int_x^N u^{-\sigma-1/4} e\left(-\frac{t}{2\pi}\log u - 2\sqrt{nu} - 1/8\right) du \ll n^{-1/2} x^{1/4-\sigma},$$

which yields

$$(8.6)\quad \mathfrak{M}_{33}(s) \ll \sum_{n\le y} \frac{d(n)}{n^{1/4}} n^{-1/2} x^{1/4-\sigma} \ll x^{1/4-\sigma} y^{1/4}\log y \ll x^{-\sigma} t^{1/2}\log t.$$

Then Lemma 8.3 follows by collecting (8.3)–(8.6). ∎

LEMMA 8.4. *For $\sigma > 1/2$, we have*

$$\mathfrak{M}_4(s) \ll x^{1/2-\sigma}\log H.$$

Proof. Write

$$\mathfrak{M}_4(s) = \frac{s}{2\pi\sqrt{2}} \sum_{y<n\le\beta H^2} \frac{d_{\alpha,\beta}(n;H)}{n^{3/4}} \int_x^N u^{-\sigma-3/4} \times e\Big(-\frac{t}{2\pi}\log u - 2\sqrt{nu} + 1/8\Big)\,du.$$

In Lemma 3.2, taking

$$G(u) = u^{-\sigma-3/4}, \qquad F_t(u) = -\frac{t}{2\pi}\log u - 2\sqrt{nu} + 1/8,$$

we have

$$F_t'(u) = -\frac{t}{2\pi u} - \sqrt{\frac{n}{u}}, \qquad \frac{F_t'(u)}{G(u)} = -\frac{t}{2\pi} u^{\sigma-1/4} - \sqrt{n}\,u^{\sigma+1/4}.$$

Thus $F'(u)/G(u)$ is monotonic and

$$\frac{F_t'(u)}{G(u)} = -\frac{t}{2\pi}u^{\sigma-1/4} - \sqrt{n}\,u^{\sigma+1/4} < -\sqrt{n}\,x^{\sigma+1/4}.$$

Hence Lemma 3.2 gives

$$\int_x^N u^{-\sigma-3/4} e\Big(-\frac{t}{2\pi}\log u - \sqrt{nu} + 1/8\Big)\,du \ll x^{-\sigma-1/4} n^{-1/2},$$

which yields

$$\mathfrak{M}_4(s) \ll x^{-\sigma+3/4} \sum_{y<n\le\beta H^2} \frac{d_{\alpha,\beta}(n;H)}{n^{5/4}} \ll x^{1/2-\sigma}\log H. \ \blacksquare$$

LEMMA 8.5. *For $\sigma > 1/2$, we have*

$$\mathfrak{M}_1(s) = \chi^2(s) \sum_{n\le y} d_{\alpha,\beta}(n) n^{s-1} + O(x^{1/2-\sigma}\log t).$$

Proof. Similar to the the proof of Lemma 8.3, we rewrite $\mathfrak{M}_1(s)$ as

$$\begin{aligned}\mathfrak{M}_1(s) &= \frac{s}{2\pi\sqrt{2}}\int_x^N u^{-s-3/4}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n,H)}{n^{3/4}}e(2\sqrt{nu}-1/8)\,du\\ &= -\frac{1}{2\pi\sqrt{2}}\int_x^N(-s+1/4)\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}\\ &\qquad\qquad\times e(2\sqrt{nu}-1/8)u^{-s-3/4}\,du\\ &\quad+\frac{1}{8\pi\sqrt{2}}\int_x^N\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}e(2\sqrt{nu}-1/8)u^{-s-3/4}\,du\\ &= -\frac{1}{2\pi\sqrt{2}}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}\int_x^N e(2\sqrt{nu}-1/8)\,du^{-s+1/4}\\ &\quad+\frac{1}{8\pi\sqrt{2}}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}\int_x^N e(2\sqrt{nu}-1/8)u^{-s-3/4}\,du.\end{aligned}$$

By partial integration, we have

$$\mathfrak{M}_1(s)=\mathfrak{M}_{11}(s)+\mathfrak{M}_{12}(s)+\mathfrak{M}_{13}(s)+\mathfrak{M}_{14}(s), \tag{8.7}$$

where

$$\mathfrak{M}_{11}(s)=-\frac{1}{i\sqrt{2}}\sum_{n\le(1+\eta)y}d_{\alpha,\beta}(n;H)n^{-1/4}I_n$$

with

$$I_n=\int_x^N u^{-\sigma-1/4}e\left(-\frac{t}{2\pi}\log u+2\sqrt{nu}-1/8\right)du,$$

$$\mathfrak{M}_{12}(s)=-\frac{1}{2\pi\sqrt{2}}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}e(2\sqrt{nN}-1/8)N^{-s+1/4},$$

$$\mathfrak{M}_{13}(s)=\frac{1}{2\pi\sqrt{2}}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}e(2\sqrt{nx}-1/8)x^{-s+1/4},$$

$$\mathfrak{M}_{14}(s)=\frac{1}{8\pi\sqrt{2}}\sum_{n\le(1+\eta)y}\frac{d_{\alpha,\beta}(n;H)}{n^{3/4}}\int_x^N e(2\sqrt{nu}-1/8)u^{-s-3/4}\,du.$$

Note that $\eta>0$ is a fixed, sufficiently small constant. Then from $d_{\alpha,\beta}(n;H)\le d(n)$ and trivial estimates, we get

$$\mathfrak{M}_{12}(s),\mathfrak{M}_{13}(s),\mathfrak{M}_{14}(s)\ll_\sigma x^{1/4-\sigma}y^{1/4}\log y\ll_\sigma x^{-\sigma}t^{1/2}\log t. \tag{8.8}$$

Now only $\mathfrak{M}_{11}(s)$ is left. In [7, Chapter 4, pp. 108–110], Ivić discussed I_n and showed that

$$-\frac{1}{i\sqrt{2}}\sum_{n\le(1+\eta)y} d(n)n^{-1/4}I_n = \chi^2(s)\sum_{n\le y} d(n)n^{s-1} + O(x^{1/2-\sigma}\log t),$$

where $\chi(s)$ is given by (1.4). Replacing $d(n)$ by $d_{\alpha,\beta}(n;H)$, the same argument is also valid, which gives

$$\begin{aligned}\mathfrak{M}_{11}(s) &= -\frac{1}{i\sqrt{2}}\sum_{n\le(1+\eta)y} d_{\alpha,\beta}(n;H)n^{-1/4}I_n\\ &= \chi^2(s)\sum_{n\le y} d_{\alpha,\beta}(n;H)n^{s-1} + O(x^{1/2-\sigma}\log t).\end{aligned}$$

Take $H = T^B$ with $B > 3A > 0$ being a constant, sufficiently large. Then

$$\begin{aligned}\mathfrak{M}_{11}(s) &= \chi^2(s)\sum_{n\le y}\Big(\sum_{\substack{1\le h\le H\\ n=hk}}\ \sideset{}{''}\sum_{h\alpha\le k\le h\beta} 1\Big)n^{s-1} + O(x^{1/2-\sigma}\log t)\\ &= \chi^2(s)\sum_{n\le y}\Big(\sum_{\substack{1\le h\le H\\ n=hk}}\ \sum_{h\alpha\le k\le h\beta} 1\Big)n^{s-1}\\ &\quad + O\Big(|\chi(s)|^2\sum_{h\ll\sqrt{y}} h^{2\sigma-2}\Big) + O(x^{1/2-\sigma}\log T)\\ &= \chi^2(s)\sum_{n\le y} d_{\alpha,\beta}(n)n^{s-1} + O(t^{1-2\sigma}y^{\sigma-1/2} + x^{1/2-\sigma}\log T)\\ &= \chi^2(s)\sum_{n\le y} d_{\alpha,\beta}(n)n^{s-1} + O(x^{1/2-\sigma}\log T),\end{aligned}\tag{8.9}$$

where we have used

$$\chi(\sigma+it) = (2\pi/t)^{\sigma+it-1/2}e^{i(t+\pi/4)}(1+O(t^{-1})) \quad \text{for } t\ge 2.\tag{8.10}$$

Combining (8.7)–(8.9) gives Lemma 8.5. ∎

9. Outline of the proof of Theorem 1.2. A primitive Pythagorean triangle is a triple (a,b,c) of natural numbers with $a^2+b^2=c^2, a<b$ and $\gcd(a,b,c)=1$. Let $P(x)$ denote the number of primitive Pythagorean triangles with perimeter less than x. D. H. Lehmer [10] showed that

$$P(x) = \frac{\log 2}{\pi^2}x + O(x^{1/2}\log x),$$

which was revisited by J. Lambek and L. Moser in [9]. The exponents 1/2 in the error term cannot be reduced because the current technique depends on the best zero-free regions of the Riemann zeta function, which is hard to improve. In [11], the author showed that if the Riemann Hypothesis is true, then (1.6) holds. Let

$$r(n) = \sum_{\substack{2d^2+2dl=n \\ l<d}} 1 = \sum_{\substack{2dl=n \\ d<l<2d}} 1$$

and

$$Z(s) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s} \qquad \text{for } \sigma > 1.$$

We can prove that $Z(s)$ has an analytic continuation to $\sigma > 1/3$ and has two simple poles at $s = 1, 1/2$. The exponent $\frac{5805}{15408}$ in (1.6) depends on the estimate of the exponential sum:

$$\sum_{m\sim M} \mu(m) \sum_{n\sim N} a_n e\biggl(\frac{cx^{1/2}n^{1/2}}{m}\biggr) \tag{9.1}$$

with $a_n \ll 1$ and c being a constant. Here the ranges of M and N are determined by the smallest σ such that

$$\int_T^{2T} |Z(\sigma + it)|\, dt \ll_{\sigma,\varepsilon} T^{1+\varepsilon} \tag{9.2}$$

for any $\varepsilon > 0$. In [11], the present author showed that $\sigma > \frac{1064}{1644} = 0.6472\ldots$ is admissible. Then by estimating the exponential sum (9.1) for $M \le x^{\frac{651}{1926}}$, $N \le x^{\frac{3798}{15408}}$, he got (1.6). In the MathSciNet review of [11], R. C. Baker mentioned that using the method in his paper [2], it is possible to prove that $\sigma > 3/5 = 0.6$, which implies an improvement of (1.6). Now by Theorem 1.1, we see that (9.2) holds for any $\sigma > 1/2$, which forces us to deal with the exponential sum (9.1) for $M, N \le x^{1/4+\varepsilon}$. However, the estimate in this range has been investigated carefully by R. C. Baker [1], which yields Theorem 1.2.

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