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Construction of normal numbers via pseudo-polynomial prime sequences

by

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1. Introduction. Let $q \geq 2$ be a positive integer. Then every real $\theta \in [0,1)$ admits a unique expansion of the form

$$\theta = \sum_{k\geq 1} a_k q^k \qquad (a_k \in \{0,1,\dots,q-1\})$$

called the q-ary expansion. We denote by $\mathcal{N}(\theta, d_1\cdots d_\ell, N)$ the number of occurrences of the block $d_1\cdots d_\ell$ amongst the first N digits, i.e.

$$\mathcal{N}(\theta, d_1\cdots d_\ell, N) := \#\{0 \leq i < n\colon a_{i+1} = d_1, \dots, a_{i+\ell} = d_\ell\}.$$

Then we call a number *normal of order ℓ in base q* if for each block of length ℓ the frequency of occurrences tends to $q^{-\ell}$. As a qualitative measure of the distance of a number from being normal, for integers N and ℓ we introduce the *discrepancy* of θ by

$$\mathcal{R}_{N,\ell}(\theta) = \sup_{d_1\cdots d_\ell}\left|\frac{\mathcal{N}(\theta, d_1\cdots d_\ell, N)}{N} - q^{-k}\right|,$$

where the supremum is over all blocks of length ℓ. Then a number θ is normal to base q if for each $\ell \geq 1$ we have $\mathcal{R}_{N,\ell}(\theta) = o(1)$ for $N \to \infty$. Furthermore, we call a number *absolutely normal* if it is normal in all bases $q \geq 2$.

Borel [2] used a slightly different, but equivalent (cf. [3, Chapter 4]) definition of normality to show that almost all real numbers are normal with respect to the Lebesgue measure. Despite their omnipresence, it is not known whether numbers such as $\log 2$, π, e or $\sqrt{2}$ are normal to any base. The first construction of a normal number is due to Champernowne [4] who showed that the number

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\dots$$

is normal in base 10.

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The construction of Champernowne laid the base for a class of normal numbers which are of the form

$$\sigma_q = \sigma_q(f) = 0.\lfloor f(1)\rfloor_q \lfloor f(2)\rfloor_q \lfloor f(3)\rfloor_q \lfloor f(4)\rfloor_q \lfloor f(5)\rfloor_q \lfloor f(6)\rfloor_q \ldots,$$

where $\lfloor \cdot \rfloor_q$ denotes the expansion in base q of the integer part. Davenport and Erdős [6] showed that $\sigma(f)$ is normal for f being a polynomial such that $f(\mathbb{N}) \subset \mathbb{N}$. This construction was extended by Schiffer [19] to polynomials with rational coefficients. Furthermore, he showed that for these polynomials, $\mathcal{R}_{N,\ell}(\sigma(f)) \ll (\log N)^{-1}$ and that this is best possible. These results were extended by Nakai and Shiokawa [17] to polynomials having real coefficients. Madritsch, Thuswaldner and Tichy [13] considered transcendental entire functions of bounded logarithmic order. Nakai and Shiokawa [16] used pseudo-polynomial functions, i.e. functions of the form

$$f(x) = \alpha_0 x^{\beta_0} + \alpha_1 x^{\beta_1} + \cdots + \alpha_d x^{\beta_d} \tag{1.1}$$

with $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_d, \beta_d \in \mathbb{R}$, $\alpha_0 > 0$, $\beta_0 > \beta_1 > \cdots > \beta_d > 0$ and at least one $\beta_i \notin \mathbb{Z}$. Since we often only need the leading term, we write $\alpha = \alpha_0$ and $\beta = \beta_0$ for short. They were also able to show that the discrepancy is $\mathcal{O}((\log N)^{-1})$. We refer the interested reader to the books of Kuipers and Niederreiter [11], Drmota and Tichy [7] or Bugeaud [3] for a more complete account on the construction of normal numbers.

The present method of construction by concatenating function values is in close connection with properties of q-additive functions. We call a function f *strictly q-additive* if $f(0) = 0$ and the function operates only on the digits of the q-ary representation, i.e.,

$$f(n) = \sum_{h=0}^{\ell} f(d_h) \quad \text{for } n = \sum_{h=0}^{\ell} d_h q^h.$$

A very simple example of a strictly q-additive function is the sum of digits function s_q, defined by

$$s_q(n) = \sum_{h=0}^{\ell} d_h \quad \text{for } n = \sum_{h=0}^{\ell} d_h q^h.$$

Refining the methods of Nakai and Shiokawa [16] the author obtained the following result.

Theorem 1.1 ([12, Theorem 1.1]). *Let $q \geq 2$ be an integer and f be a strictly q-additive function. If p is a pseudo-polynomial as defined in* (1.1), *then there exists $\eta > 0$ such that*

$$\sum_{n \leq N} f(\lfloor p(n) \rfloor) = \mu_f N \log_q(p(N)) + N F(\log_q(p(N))) + \mathcal{O}(N^{1-\eta}),$$

where

$$\mu_f = \frac{1}{q} \sum_{d=0}^{q-1} f(d)$$

and F is a 1-periodic function depending only on f and p.

In the present paper, however, we are interested in a variant of $\sigma_q(f)$ involving primes. As a first example, Champernowne [4] conjectured and later Copeland and Erdős [5] proved that the number

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\dots$$

is normal in base 10. Similar to the construction above, we want to consider the number

$$\tau_q = \tau_q(f) = 0.\lfloor f(2) \rfloor_q \lfloor f(3) \rfloor_q \lfloor f(5) \rfloor_q \lfloor f(7) \rfloor_q \lfloor f(11) \rfloor_q \lfloor f(13) \rfloor_q \dots,$$

where the arguments of f run through the sequence of primes.

The paper of Copeland and Erdős [5] corresponds to the function $f(x) = x$. Nakai and Shiokawa [18] showed that the discrepancy for polynomials having rational coefficients is $\mathcal{O}((\log N)^{-1})$. Furthermore, Madritsch, Thuswaldner and Tichy [13] showed that transcendental entire functions of bounded logarithmic order yield normal numbers. Finally, in a recent paper Madritsch and Tichy [14] considered pseudo-polynomials of the special form αx^β with $\alpha > 0$, $\beta > 1$ and $\beta \notin \mathbb{Z}$.

The aim of the present paper is to extend this last construction to arbitrary pseudo-polynomials. Our first main result is the following

THEOREM 1.2. *Let f be a pseudo-polynomial as in* (1.1). *Then*

$$\mathcal{R}_N(\tau_q(f)) \ll (\log N)^{-1}.$$

In our second main result we use the connection of this construction of normal numbers with the arithmetic mean of q-additive functions as described above. Known results are due to Shiokawa [20] and Madritsch and Tichy [14]. Similar results concerning the moments of the sum of digits function over primes have been established by Kátai [10].

Let $\pi(x)$ stand for the number of primes less than or equal to x. Then adapting these ideas to our method we obtain the following

THEOREM 1.3. *Let f be a pseudo-polynomial as in* (1.1). *Then*

$$\sum_{p \le P} s_q(\lfloor f(p) \rfloor) = \frac{q-1}{2} \pi(P) \log_q P^\beta + \mathcal{O}(\pi(P)),$$

where the sum runs over the primes and the implicit $\mathcal{O}$-constant may depend on q and β.

REMARK 1.4. With simple modifications Theorem 1.3 can be extended to completely q-additive functions replacing s_q.

The proof of the two theorems is divided into four parts. In the following section we rewrite both statements in order to obtain as a common base the central theorem—Theorem 2.1. In Section 3 we start with the proof of this theorem by using an indicator function and its Fourier series. These series contain exponential sums which we treat by different methods (with respect to the position in the expansion) in Section 4. Finally, in Section 5 we put the estimates together in order to prove the central theorem and hence our two statements.

2. Preliminaries. Throughout, p will always denote a prime. The implicit constants of $\ll$ and $\mathcal{O}$ may depend on the pseudo-polynomial f and on the parameter $\varepsilon > 0$. Furthermore, we fix a block $d_1 \cdots d_\ell$ of length ℓ, and N, the number of digits we consider.

In the first step we want to know in the expansion of which prime the Nth digit occurs. This can be seen as translation from the digital world to the world of blocks. To this end let $\ell(m)$ denote the length of the q-ary expansion of an integer m. Then we define an integer P by

$$\sum_{p \le P-1} \ell(\lfloor f(p) \rfloor) < N \le \sum_{p \le P} \ell(\lfloor f(p) \rfloor),$$

where the sum runs over all primes. Thus we get the following relation between N and P:

$$\begin{aligned} N &= \sum_{p \le P} \ell(\lfloor f(p) \rfloor) + \mathcal{O}(\pi(P)) + \mathcal{O}(\beta \log_q(P)) \\ &= \frac{\beta}{\log q} P + \mathcal{O}\left(\frac{P}{\log P} \right). \end{aligned} \tag{2.1}$$

Here we have used the prime number theorem in the form (cf. [21, Théorème 4.1])

$$\pi(x) = \operatorname{Li} x + \mathcal{O}\left(\frac{x}{(\log x)^G} \right), \tag{2.2}$$

where G is an arbitrary positive constant and

$$\operatorname{Li} x = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

Now we show that we may neglect the occurrences of the block $d_1 \cdots d_\ell$ between two expansions. We write $\mathcal{N}(f(p))$ for the number of occurrences of this block in the q-ary expansion of $\lfloor f(p) \rfloor$. Then (2.1) implies that

$$\Big| \mathcal{N}(\tau_q(f); d_1 \cdots d_\ell; N) - \sum_{p \le P} \mathcal{N}(f(p)) \Big| \ll \frac{N}{\log N}. \tag{2.3}$$

In the next step we use the polynomial-like behavior of f. In particular, we collect all the values having the same length of expansion. Let j_0 be a sufficiently large integer. Then for each integer $j \geq j_0$ there exists a P_j such that

$$q^{j-2} \leq f(P_j) < q^{j-1} \leq f(P_j+1) < q^j \quad \text{with} \quad P_j \asymp q^{j/\beta}.$$

Furthermore, we set J to be the greatest length of the q-ary expansion of $f(p)$ over the primes $p \leq P$, i.e.,

$$J := \max_{p \leq P} \ell(\lfloor f(p) \rfloor) = \log_q(f(P)) + \mathcal{O}(1) \asymp \log P.$$

Now we show that we may suppose that each expansion has the same length (which we reach by adding leading zeroes). For $P_{j-1} < p \leq P_j$ we may write $f(p)$ in q-ary expansion, i.e.,

$$f(p) = b_{j-1}q^{j-1} + b_{j-2}q^{j-2} + \cdots + b_1 q + b_0 + b_{-1}q^{-1} + \cdots . \tag{2.4}$$

Then we denote by $\mathcal{N}^*(f(p))$ the number of occurrences of the block $d_1 \cdots d_\ell$ in the string $0 \cdots 0 b_{j-1} b_{j-2} \cdots b_1 b_0$, where we filled up the expansion with leading zeroes so that it has length J. The error of doing so can be estimated by

$$\begin{aligned} 0 &\leq \sum_{p \leq P} \mathcal{N}^*(f(p)) - \sum_{p \leq P} \mathcal{N}(f(p)) \\ &\leq \sum_{j=j_0+1}^{J-1} (J-j)(\pi(P_{j+1}) - \pi(P_j)) + \mathcal{O}(1) \\ &\leq \sum_{j=j_0+2}^{J} \pi(P_j) + \mathcal{O}(1) \ll \sum_{j=j_0+2}^{J} \frac{q^{j/\beta}}{j} \ll \frac{P}{\log P} \ll \frac{N}{\log N}. \end{aligned}$$

In the following three sections we will estimate this sum of indicator functions $\mathcal{N}^*$ in order to prove the following theorem.

THEOREM 2.1. *Let f be a pseudo-polynomial as in* (1.1). *Then*

$$\sum_{p \leq P} \mathcal{N}^*(\lfloor f(p) \rfloor) = q^{-\ell} \pi(P) \log_q P^\beta + \mathcal{O}\left(\frac{P}{\log P}\right). \tag{2.5}$$

Using this theorem we can easily deduce our two main results.

Proof of Theorem 1.2. We insert (2.5) into (2.3) and get the desired result. ■

Proof of Theorem 1.3. For this proof we have to rewrite the statement. In particular, we use the fact that the sum of digits function counts the number of 1s, 2s, etc., and assigns weights to them, i.e.,

$$s_q(n)=\sum_{d=0}^{q-1} d\cdot\mathcal{N}(n;d).$$

Thus

$$\begin{aligned}\sum_{p\le P} s_q(\lfloor p^\beta\rfloor) &= \sum_{p\le P}\sum_{d=0}^{q-1} d\cdot\mathcal{N}(p^\beta)=\sum_{p\le P}\sum_{d=0}^{q-1} d\cdot\mathcal{N}^*(p^\beta)+\mathcal{O}\Big(\frac{P}{\log P}\Big)\\ &=\frac{q-1}{2}\pi(P)\log_q(P^\beta)+\mathcal{O}\Big(\frac{P}{\log P}\Big),\end{aligned}$$

and the theorem follows. ∎

In the following sections we will prove Theorem 2.1 in several steps. First we use the "method of little glasses" in order to approximate the indicator function by a Fourier series having smooth coefficients. Then we will apply different methods (depending on the position in the expansion) to estimate the exponential sums that appear in the Fourier series. Finally, we put everything together and get the desired estimate.

3. Proof of Theorem 2.1, part I. We want to ease notation by splitting the pseudo-polynomial f into a polynomial and the rest. There exists a unique decomposition

$$f(x)=g(x)+h(x), \tag{3.1}$$

where $h\in\mathbb{R}[X]$ is a polynomial of degree k (where we set $k=0$ if h is the zero polynomial) and

$$g(x)=\sum_{j=1}^{r}\alpha_j x^{\theta_j}$$

with $r\ge 1$, $\alpha_r\ne 0$, α_j is real, $0<\theta_1<\cdots<\theta_r$ and $\theta_j\notin\mathbb{Z}$ for $1\le j\le r$.

Let γ and ρ be two parameters which we will frequently use. We suppose that

$$0<\gamma<\rho<\min\Big(\frac{1}{4(k+1)},\frac{\theta_r}{2}\Big).$$

The aim of this section is to calculate the Fourier transform of $\mathcal{N}^*$. In order to count the occurrences of the block $d_1\cdots d_\ell$ in the q-ary expansion of $\lfloor f(p)\rfloor$ $(2\le p\le P)$ we define the indicator function

$$\mathcal{I}(t)=\begin{cases}1 & \text{if } \sum_{i=1}^{\ell} d_i q^{-i}\le t-\lfloor t\rfloor<\sum_{i=1}^{\ell} d_i q^{-i}+q^{-\ell},\\ 0 & \text{otherwise,}\end{cases}$$

which is a 1-periodic function. Indeed,

$$\mathcal{I}(q^{-j}f(p))=1 \;\Leftrightarrow\; d_1\cdots d_\ell=b_{j-1}\cdots b_{j-\ell}, \tag{3.2}$$

where $f(p)$ has an expansion as in (2.4). Thus we may write our block counting function as follows:

$$\mathcal{N}^*(f(p)) = \sum_{j=\ell}^{J} \mathcal{I}(q^{-j} f(p)). \tag{3.3}$$

In the following we will use Vinogradov's "method of little glasses" (cf. [23]). We want to approximate $\mathcal{I}$ from above and from below by two 1-periodic functions having small Fourier coefficients. To this end we will use the following

Lemma 3.1 ([23, Lemma 12]). *Let α, β, Δ be real numbers satisfying*

$$0 < \Delta < 1/2, \qquad \Delta \le \beta - \alpha \le 1 - \Delta.$$

Then there exists a periodic function $\psi(x)$ with period 1 satisfying:

(1) $\psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \le x \le \beta - \frac{1}{2}\Delta$,
(2) $\psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \le x \le 1 + \alpha - \frac{1}{2}\Delta$,
(3) $0 \le \psi(x) \le 1$ *in the remainder of the interval* $\alpha - \frac{1}{2}\Delta \le x \le 1 + \alpha - \frac{1}{2}\Delta$,
(4) $\psi(x)$ *has a Fourier series expansion of the form*

$$\psi(x) = \beta - \alpha + \sum_{\substack{\nu=-\infty \\ \nu \ne 0}}^{\infty} A(\nu) e(\nu x),$$

where

$$|A(\nu)| \ll \min\left(\frac{1}{\nu}, \beta - \alpha, \frac{1}{\nu^2 \Delta}\right). \tag{3.4}$$

We note that we could have used Vaaler polynomials [22]; however, we do not gain anything by doing so as the estimates we get are already best possible. Setting

$$\begin{aligned} \alpha_- &= \sum_{\lambda=1}^{\ell} d_\lambda q^{-\lambda} + (2\delta)^{-1}, \qquad & \beta_- &= \sum_{\lambda=1}^{\ell} d_\lambda q^{-\lambda} + q^{-\ell} - (2\delta)^{-1}, \\ \alpha_+ &= \sum_{\lambda=1}^{\ell} d_\lambda q^{-\lambda} - (2\delta)^{-1}, \qquad & \beta_+ &= \sum_{\lambda=1}^{\ell} d_\lambda q^{-\lambda} + q^{-\ell} + (2\delta)^{-1}, \\ \delta &= P^{-\gamma}, \end{aligned} \tag{3.5}$$

and an application of Lemma 3.1 with $(\alpha, \beta, \delta) = (\alpha_-, \beta_-, \delta)$ and $(\alpha, \beta, \delta) = (\alpha_+, \beta_+, \delta)$ provides us with two functions, $\mathcal{I}_-$ and $\mathcal{I}_+$ respectively. By our choice of $(\alpha_\pm, \beta_\pm, \delta)$ it is immediate that

$$\mathcal{I}_-(t) \le \mathcal{I}(t) \le \mathcal{I}_+(t) \qquad (t \in \mathbb{R}). \tag{3.6}$$

Lemma 3.1 also implies that these two functions have Fourier expansions

$$\mathcal{I}_\pm(t) = q^{-\ell} \pm P^{-\gamma} + \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} A_\pm(\nu) e(\nu t) \tag{3.7}$$

satisfying

$$|A_\pm(\nu)| \ll \min(|\nu|^{-1}, P^\gamma |\nu|^{-2}).$$

In a next step we want to replace $\mathcal{I}$ by $\mathcal{I}_+$ in (3.3). For this purpose we deduce, using (3.6) and (3.7), that

$$|\mathcal{I}(t) - q^{-\ell}| \ll P^{-\gamma} + \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} A_\pm(\nu) e(\nu t).$$

Thus setting $t = q^{-j} f(p)$ and summing over $p \le P$ yields

$$\left|\sum_{p\le P} \mathcal{I}(q^{-j} f(p)) - \frac{\pi(P)}{q^\ell}\right| \ll \pi(P) P^{-\gamma} + \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} A_\pm(\nu) \sum_{p\le P} e\left(\frac{\nu}{q^j} f(p)\right). \tag{3.8}$$

Now we consider the coefficients $A_\pm(\nu)$. Noting (3.4) one observes that

$$A_\pm(\nu) \ll \begin{cases} \nu^{-1} & \text{for } |\nu| \le P^\gamma, \\ P^\gamma \nu^{-2} & \text{for } |\nu| > P^\gamma. \end{cases}$$

Estimating all summands with $|\nu| > P^\gamma$ trivially we get

$$\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} A_\pm(\nu) e\left(\frac{\nu}{q^j} f(p)\right) \ll \sum_{\nu=1}^{P^\gamma} \nu^{-1} e\left(\frac{\nu}{q^j} f(p)\right) + P^{-\gamma}.$$

Using this in (3.8) yields

$$\left|\sum_{p\le P} \mathcal{I}(q^{-j} f(p)) - \frac{\pi(P)}{q^\ell}\right| \ll \pi(P) P^{-\gamma} + \sum_{\nu=1}^{P^\gamma} \nu^{-1} S(P, j, \nu),$$

where we have set

$$S(P, j, \nu) := \sum_{p\le P} e\left(\frac{\nu}{q^j} f(p)\right). \tag{3.9}$$

4. Exponential sum estimates. In this section we will focus on the estimation of the sum $S(P, j, \nu)$ for different ranges of j.

If $\theta_r > k \ge 0$, i.e. the leading coefficient of f comes from the pseudo-polynomial part g, then we consider the two ranges

$$1 \le q^j \le P^{\theta_r - \rho} \quad \text{and} \quad P^{\theta_r - \rho} < q^j \le P^{\theta_r}.$$

For the first one we will apply Proposition 4.3, and for the second one Proposition 4.1.

On the other hand, if $k > \theta_r > 0$, meaning that the leading coefficient of f comes from the polynomial part h, then we have an additional part. In particular, in this case we will consider the three ranges

$$1 \leq q^j \leq P^{\theta_r-\rho}, \qquad P^{\theta_r-\rho} < q^j \leq P^{k-1+\rho}, \qquad P^{k-1+\rho} < q^j \leq P^k.$$

Similar to the above, we will treat the first and last ranges by using Propositions 4.3 and 4.1, respectively. For the middle range we will apply Proposition 4.7. Since $2\rho < \theta_r$, the middle range is empty if $k = 1$.

Since the size of j represents the position of the digit in the expansion (cf. (3.2)), in the following subsections we will deal with the "most significant digits", the "least significant digits" and the "digits in the middle", respectively.

4.1. Most significant digits. We start our series of estimates for the exponential sum $S(P, j, \nu)$ for j being in the highest range. In particular, we want to show the following

PROPOSITION 4.1. *Suppose that for some $k \geq 1$ we have $|f^{(k)}(x)| \geq \Lambda$ for any x on $[a, b]$ with $\Lambda > 0$. Then*

$$S(P, j, \nu) \ll \frac{1}{\log P}\Lambda^{-1/k} + \frac{P}{(\log P)^G}.$$

The main idea of the proof is to use Riemann–Stieltjes integration together with

LEMMA 4.2 ([9, Lemma 8.10]). *Let $F : [a, b] \to \mathbb{R}$ and suppose that for some $k \geq 1$ we have $|F^{(k)}(x)| \geq \Lambda$ for any x on $[a, b]$ with $\Lambda > 0$. Then*

$$\left|\int_a^b e(F(x))\, dx\right| \leq k2^k \Lambda^{-1/k}.$$

Proof of Proposition 4.1. We rewrite the sum into a Riemann–Stieltjes integral:

$$S(P, j, \nu) = \sum_{p \leq P} e\left(\frac{\nu}{q^j} f(p)\right) = \int_2^P e\left(\frac{\nu}{q^j} f(t)\right) d\pi(t) + \mathcal{O}(1).$$

Then we apply the prime number theorem in the form (2.2) to pass to the usual integral:

$$S(P, j, \nu) = \int_{P(\log P)^{-G}}^{P} e\left(\frac{\nu}{q^j} f(t)\right) \frac{dt}{\log t} + \mathcal{O}\left(\frac{P}{(\log P)^G}\right).$$

Now we use the second mean-value theorem to get

$$S(P,j,\nu) \ll \frac{1}{\log P} \sup_{\xi} \Big| \int_{P(\log P)^{-G}}^{\xi} e\Big(\frac{\nu}{q^j} f(t)\Big)\,dt \Big| + \frac{P}{(\log P)^G}. \tag{4.1}$$

Finally, an application of Lemma 4.2 proves the proposition. ∎

4.2. Least significant digits. Now we turn our attention to the lowest range of j. In particular, the goal is to prove

PROPOSITION 4.3. *Let P and ρ be positive reals, and f be a pseudo-polynomial as in* (3.1). *If j is such that*

$$1 \le q^j \le P^{\theta_r - \rho}, \tag{4.2}$$

then for $1 \le \nu \le P^\gamma$ there exists $\eta > 0$ (depending only on f and ρ) such that

$$S(P,j,\nu) = (\log P)^8 P^{1-\eta}.$$

Before we launch into the proof we collect some tools that will be necessary. A standard idea for estimating exponential sums over the primes is to rewrite them into ordinary exponential sums over the integers with von Mangoldt's function as weight, and then to apply Vaughan's identity. Recall that *von Mangoldt's function* is

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and an integer } k \ge 1,\\ 0 & \text{otherwise.} \end{cases}$$

For the rewriting process we use the following

LEMMA 4.4. *Let g be a function such that $|g(n)| \le 1$ for all integers n. Then*

$$\Big|\sum_{p \le P} g(p)\Big| \ll \frac{1}{\log P} \max_{t \le P} \Big|\sum_{n \le t} \Lambda(n) g(n)\Big| + \mathcal{O}(\sqrt{P}).$$

Proof. This is Lemma 11 of [15]. However, the proof is short and we need some piece of it later.

We start with a summation by parts yielding

$$\sum_{p \le P} g(p) = \frac{1}{\log P} \sum_{p \le x} (\log p) g(p) + \int_2^P \Big(\sum_{p \le t} (\log p) g(p)\Big) \frac{dt}{t(\log t)^2}.$$

Now we cut the integral at $\sqrt{P}$ and use Chebyshev's inequality (cf. [21, Théorème 1.3]) in the form $\sum_{p \le t} \log p \le \pi(t) \log t \ll t$ for the lower part.

Thus

$$\Big|\sum_{p\le P} g(p)\Big| \le \left(\frac{1}{\log P} + \int_{\sqrt{P}}^{P} \frac{dt}{t(\log t)^2}\right) \max_{\sqrt{P}<t\le P} \Big|\sum_{p\le P} (\log p) g(p)\Big| + \mathcal{O}(\sqrt{P})$$
$$= \frac{2}{\log P} \max_{\sqrt{P}<t\le P} \Big|\sum_{p\le t} (\log p) g(p)\Big| + \mathcal{O}(\sqrt{P}).$$

Finally we again use Chebyshev's inequality $\pi(t) \ll t/\log t$ to obtain

$$\Big|\sum_{n\le t} \Lambda(n) g(n) - \sum_{p\le t} (\log p) g(p)\Big| \le \sum_{p\le\sqrt{t}} (\log p) \sum_{a=2}^{\lfloor \frac{\log t}{\log p}\rfloor} 1 \tag{4.3}$$
$$\le \pi(\sqrt{t}) \log t \ll \sqrt{t}. \blacksquare$$

In the next step we use Vaughan's identity to subdivide this weighted exponential sum into several sums of Type I and II.

Lemma 4.5 ([1, Lemma 2.3]). *Assume that $F(x)$ is any function defined on the real line, supported on $[P/2, P]$ and bounded by F_0. Let further U, V, Z be any parameters satisfying $3 \le U < V < Z < P$, $Z \ge 4U^2$, $P \ge 64Z^2U$, $V^3 \ge 32P$ and $Z - 1/2 \in \mathbb{N}$. Then*

$$\Big|\sum_{P/2<n\le P} \Lambda(n) F(n)\Big| \ll K \log P + F_0 + L(\log P)^8,$$

where K and L are defined by

$$K = \max_M \sum_{m=1}^{\infty} d_3(m) \Big|\sum_{Z<n\le M} F(mn)\Big|,$$
$$L = \sup \sum_{m=1}^{\infty} d_4(m) \Big|\sum_{U<n<V} b(n) F(mn)\Big|,$$

where the supremum is taken over all arithmetic functions $b(n)$ satisfying $|b(n)| \le d_3(n)$.

After subdividing the weighted exponential sum via Vaughan's identity we will use the following lemma in order to estimate the occurring exponential sums.

Lemma 4.6 ([1, Lemma 2.5]). *Let $X, k, q \in \mathbb{N}$ with $k, q \ge 0$ and set $K = 2^k$ and $Q = 2^q$. Let $h(x)$ be a polynomial of degree k with real coefficients. Let $g(x)$ be a real $q+k+2$ times continuously differentiable function on $[X/2, X]$ such that $|g^{(r)}(x)| \asymp FX^{-r}$ $(r = 1, \ldots, q+k+2)$. Then, if $F = o(X^{q+2})$ for*

F and X large enough, we have

$$\Big|\sum_{X/2<x\le X} e(g(x)+h(x))\Big| \ll X^{1-1/K} + X\left(\frac{\log^k X}{F}\right)^{1/K} + X\left(\frac{F}{X^{q+2}}\right)^{\frac{1}{4KQ-2K}}.$$

Proof of Proposition 4.3. An application of Lemma 4.4 yields

$$S(P,j,\nu) \ll \frac{1}{\log P}\max\left|\sum_{n\le P}\Lambda(n)e\left(\frac{\nu}{q^j}(g(n)+h(n))\right)\right| + P^{1/2}.$$

We split the inner sum into $\le \log P$ subsums of the form

$$\left|\sum_{X<n\le 2X}\Lambda(n)e\left(\frac{\nu}{q^j}(g(n)+h(n))\right)\right|$$

with $2X \le P$, and let S be a typical one of them. We may assume that $X \ge P^{1-\rho}$.

Using Vaughan's identity (Lemma 4.5) with $U = \frac14 X^{1/5}$, $V = 4X^{1/3}$ and Z the unique number in $1/2+\mathbb{N}$ which is closest to $\frac14 X^{2/5}$, we obtain

$$S \ll 1 + (\log X)S_1 + (\log X)^8 S_2, \tag{4.4}$$

where

$$S_1 = \sum_{x<2X/Z} d_3(x) \sum_{y>Z,\, X/x<y<2X/x} e\left(\frac{\nu}{q^j}(g(xy)+h(xy))\right),$$

$$S_2 = \sum_{X/V<x\le 2X/U} d_4(x) \sum_{U<y<V,\, X/x<y\le 2X/x} b(y)e\left(\frac{\nu}{q^j}(g(xy)+h(xy))\right).$$

We start with the estimation of S_1. Since $d_3(x) \ll x^{\varepsilon}$, we have

$$|S_1| \ll X^{\varepsilon}\sum_{x\le 2X/Z}\left|\sum_{\substack{X/x<y2X/x\\ y>Z}} e\left(\frac{\nu}{q^j}(g(xy)+h(xy))\right)\right|.$$

To estimate the inner sum we fix x and denote $Y = X/x$. Since $\theta_r \notin \mathbb{Z}$ and $\theta_r > k \ge 0$, we have

$$\left|\frac{\partial^\ell g(xy)}{\partial y^\ell}\right| \asymp X^{\theta_r}Y^{-\ell}.$$

Now on the one hand, since $q^j \le P^{\theta_r-\rho}$, we have $\nu q^{-j}X^{\theta_r} \gg X^{\rho}$. On the other hand for $\ell \ge 5(\lfloor\theta_r\rfloor+1)$ we get

$$\frac{\nu}{q^j}X^{\theta_r}Y^{-\ell} \le P^{\gamma}X^{\theta_r-2\ell/5} \ll X^{-1/2}.$$

Thus an application of Lemma 4.6 yields

$$(4.5)\qquad |S_1| \ll X^{\varepsilon} \sum_{x\le 2X/Z} Y\big[Y^{-1/K} + (\log Y)^k X^{-\rho/K} + X^{-\frac{1}{2}\frac{1}{4K\cdot 8L^5-2K}}\big] \\ \ll X^{1+\varepsilon}(\log X)\big(X^{-\rho} + X^{-\frac{1}{64L^5-4}}\big)^{1/K},$$

where we have used the inequalities $k/K<1$ and $\rho<1/3$.

For the second sum S_2 we start by splitting the interval $(X/V, 2X/U]$ into $\le \log X$ subintervals of the form $(X_1, 2X_1]$. Thus

$$|S_2| \le (\log X)X^{\varepsilon} \sum_{X_1<x\le 2X_1} \Bigg| \sum_{\substack{U<y<V\\ X/x<y\le 2X/x}} b(y) e\bigg(\frac{\nu}{q^j}(g(xy)+h(xy))\bigg)\Bigg|.$$

Now an application of Cauchy's inequality together with $|b(y)| \ll X^{\varepsilon}$ yields

$$|S_2|^2 \le (\log X)^2 X^{2\varepsilon} X_1 \sum_{X_1<x\le 2X_1} \Bigg| \sum_{\substack{U<y<V\\ X/x<y\le 2X/x}} b(y) e\bigg(\frac{\nu}{q^j}(g(xy)+h(xy))\bigg)\Bigg|^2$$
$$\ll (\log X)^2 X^{4\varepsilon} X_1 \times \\ \Bigg(X_1\frac{X}{X_1} + \Bigg|\sum_{X_1<x\le 2X_1}\sum_{A<y_1<y_2\le B} e\bigg(\frac{\nu}{q^j}\big(g(xy_1)-g(xy_2)+h(xy_1)-h(xy_2)\big)\bigg)\Bigg|\Bigg),$$

where $A=\max\{U, X/x\}$ and $B=\min\{U, 2X/x\}$. Changing the order of summation, we get

$$|S_2|^2 \ll (\log X)^2 X^{4\varepsilon} X_1 \\ \times\Bigg(X + \sum_{A<y_1<y_2\le B}\Bigg|\sum_{X_1<x\le 2X_1} e\bigg(\frac{\nu}{q^j}\big(g(xy_1)-g(xy_2)+h(xy_1)-h(xy_2)\big)\bigg)\Bigg|\Bigg).$$

As above we want to apply Lemma 4.6. To this end we fix y_1 and $y_2 \ne y_1$. Similarly to the above we get

$$\left|\frac{\partial^\ell\big(g(xy_1)-g(xy_2)+h(xy_1)-h(xy_2)\big)}{\partial x^\ell}\right| \asymp \frac{|y_1-y_2|}{y_1} X^{\theta_r} X_1^{-\ell}.$$

Now, on the one hand we have $\frac{\nu}{q^j}\frac{|y_1-y_2|}{y_1}X^{\theta_r} \gg X^{\rho}$, and on the other hand

$$\frac{\nu}{q^j}\,\frac{|y_1-y_2|}{y_1} X^{\theta_r} X_1^{-\ell} \ll X^{\gamma+\theta_r}\left(\frac{X}{V}\right)^{-\ell} \ll X^{\gamma+\theta_r-2\ell/3} \ll X^{-1/2}$$

if $\ell \geq 2\lfloor\theta_r\rfloor + 3$. Thus again an application of Lemma 4.6 yields

$$\begin{aligned}(4.6)\quad &|S_2|^2\\ &\ll (\log X)^2 X^{4\varepsilon} X_1\Big(X + \sum_{A<y_1<y_2\leq B} X_1(X_1^{-1/K} + X^{-\rho/K} + X^{-\frac{1}{2}\frac{1}{4K\cdot 2L^2-2K}})\Big)\\ &\ll (\log X)^2 X^{4\varepsilon}\big(X^{5/3} + X^{2-\rho/K} + X^{2-\frac{1}{16KL^2-4K}}\big).\end{aligned}$$

Plugging the two estimates (4.5) and (4.6) into (4.4) proves the proposition. ∎

4.3. The digits in the middle. For those j leading to a position between θ_r and k, the situation is getting more involved. These sums correspond to the "digits in the middle" in the proof of Theorem 2.1. We want to prove the following

PROPOSITION 4.7. *Let P and ρ be positive reals and f be a pseudo-polynomial as in* (3.1). *If $2\rho < \theta_r < k$ and j is such that*

$$P^{\theta_r-\rho} < q^j \leq P^{k-1+\rho}, \tag{4.7}$$

then for $1 \leq \nu \leq P^{\gamma}$ we have

$$S(P,j,\nu) = \sum_{p\leq P} e\bigg(\frac{\nu f(p)}{q^j}\bigg) \ll P^{1-\rho/4^k}.$$

The main idea in this range is to use the fact that the dominant part of f comes from the polynomial h. Therefore after getting rid of the function g we will estimate the sum over the polynomial by applying

LEMMA 4.8. *Let $h \in \mathbb{R}[X]$ be a polynomial of degree $k \geq 2$. Suppose α is the leading coefficient of h and there are integers a and q such that*

$$|q\alpha - a| < 1/q \quad \text{with} \quad (a,q) = 1.$$

Then for any $\varepsilon > 0$ and $H \leq X$,

$$\sum_{X<p\leq X+H} (\log p) e(h(p)) \ll H^{1+\varepsilon}\bigg(\frac{1}{q} + \frac{1}{H^{1/2}} + \frac{q}{H^k}\bigg)^{4^{1-k}}.$$

Proof. This is a slight variation of [8, Theorem 1], where we sum over an interval of the form $]X, X+H]$ instead of one of the form $]0, X]$. ∎

Now we have enough tools for

Proof of Proposition 4.7. As in the proof of Proposition 4.3 we start from an application of Lemma 4.4 yielding

$$S(P,j,\nu) \ll \frac{1}{\log P} \max \bigg|\sum_{n\leq P} \Lambda(n) e\bigg(\frac{\nu}{q^j}(g(n) + h(n))\bigg)\bigg| + P^{1/2}.$$

We split the inner sum into $\le \log P$ subsums of the form

$$S := \sum_{X<n\le X+H} \Lambda(n) e\left(\frac{\nu}{q^j}(g(n)+h(n))\right)$$

with $P^{1-2\rho} \le X \le P$ and

$$H = \min(P^{1-\theta_r}|\nu|^{-1}q^j, X).$$

Now we want to separate the function parts g and h. Therefore we define two functions T and φ by

$$T(x) = \sum_{X<n\le X+x} \Lambda(n) e\left(\frac{\nu}{q^j} h(n)\right) \quad \text{and} \quad \varphi(x) := e\left(\frac{\nu}{q^j} g(X+x)\right).$$

Then summation by parts yields

$$\text{(4.8)} \qquad \sum_{X<n\le X+H} \Lambda(n) e\left(\frac{\nu}{q^j}(g(n)+h(n))\right) = \sum_{n=1}^{H} \varphi(n)(T(n)-T(n-1))$$
$$= \sum_{n=1}^{H} T(n)(\varphi(n)-\varphi(n+1)) + \varphi(H-1)T(H)$$
$$\ll |T(H)| + \sum_{n=1}^{H-1} |\varphi(n)-\varphi(n+1)|\,|T(n)|.$$

Let α_k be the leading coefficient of P. Then by Diophantine approximation there always exists a rational a/b with $b>0$, $(a,b)=1$,

$$1 \le b \le H^{k-\rho} \quad \text{and} \quad \left|\frac{\nu\alpha_k}{q^j} - \frac{a}{b}\right| \le \frac{H^{\rho-k}}{b}.$$

We distinguish three cases according to the size of b.

Case 1: $H^\rho < b$. In this case we may apply Lemma 4.8 together with (4.3) to get

$$T(h) \ll H^{1-\rho/4^{k-1}+\varepsilon}.$$

Case 2: $2 \le b < H^\rho$. In this case we get

$$\left|\frac{\nu\alpha_k}{q^j}\right| \ge \left|\frac{a}{b}\right| - \frac{1}{b^2} \ge \frac{1}{2b} \ge \frac{1}{2}H^{-\rho} \ge \frac{1}{2}P^{-\rho}.$$

Since $2\rho < \theta_r$, this contradicts our lower bound $q^j \ge P^{\theta_r-\rho}$.

Case 3: $b=1$. This case requires a further distinction according to whether $a=0$ or not.

Case 3.1: $|\nu\alpha_k/q^j| \ge 1/2$. It follows that

$$q^j \le 2|\nu\alpha_k|,$$

again contradicting our lower bound $q^j \ge P^{\theta_r-\rho}$.

Case 3.2: $|\nu\alpha_k/q^j| < 1/2$. This implies that $a = 0$, which yields

$$q^j \ge |\nu\alpha_k| H^{k-\rho}. \tag{4.9}$$

We distinguish two further cases according to whether $P^{1-\theta_r}|\nu|^{-1}q^j \le X$ or not.

Case 3.2.1: $P^{1-\theta_r}|\nu|^{-1}q^j \le X$. This implies that $q^j \le P^{\theta_r}|\nu|$ and

$$H = P^{1-\theta_r}|\nu|^{-1}q^j \ge P^{1-\rho}|\nu|^{-1} \ge P^{1-2\rho}.$$

Plugging these estimates into (4.9) gives

$$P^{\theta_r} \ge |\alpha_k| P^{(1-2\rho)(k-\rho)}.$$

However, since $4(k+1)\rho < 1$, we have

$$(1-2\rho)(k-\rho) > k-1+2\rho \ge \theta_r,$$

yielding a contradiction.

Case 3.2.2: $P^{1-\theta_r}|\nu|^{-1}q^j > X$. Then $H = X \ge P^{1-2\rho}$ and (4.9) becomes

$$P^{k-1+\rho} \ge |\nu\alpha_k| P^{(1-2\rho)(k-\rho)},$$

yielding a similar contradiction to that in Case 3.2.1.

Therefore Case 1 is the only one possible, and we may always apply Lemma 4.8 together with (4.3). Plugging this into (4.8) yields

$$\sum_{X<n\le X+H} \Lambda(n) e\biggl(\frac{\nu}{q^j}(g(n)+h(n))\biggr) \ll H^{1-\rho/4^{k-1}+\varepsilon}\Bigl(1 + \sum_{X<n\le X+H} |\varphi(n)-\varphi(n+1)|\Bigr).$$

Now by our choice of H together with an application of the mean value theorem we have

$$\sum_{X\le n\le X+H} |\varphi(n)-\varphi(n+1)| \ll H\frac{\nu}{q^j}P^{\theta-1} \ll 1.$$

Thus

$$\sum_{X\le n\le X+H} \Lambda(n) e\biggl(\frac{\nu}{q^j}(g(n)+h(n))\biggr) \ll H^{1-\rho/4^{k-1}+\varepsilon}. \ \blacksquare$$

5. Proof of Theorem 2.1, part II. Now we use all the tools from the section above in order to estimate

$$\sum_{j=\ell}^{J} \biggl|\sum_{p\le P} \mathcal{I}(q^{-j}f(p)) - \frac{\pi(P)}{q^\ell}\biggr| \ll \pi(P)H^{-1}J + \sum_{\nu=1}^{H} \nu^{-1} \sum_{j=\ell}^{J} S(P,j,\nu). \tag{5.1}$$

As indicated in the section above, we split the sum over j into two or three parts according to whether $\theta_r > k$ or not. In any case an application of Proposition 4.3 yields, for the least significant digits,

$$\sum_{1\le\nu\le P^\gamma} \nu^{-1} \sum_{1\le q^j\le P^{\theta_r-\rho}} S(P,j,\nu) \ll (\log P)^9 JP^{1-\eta}. \tag{5.2}$$

Now suppose that $\theta_r > k$. Then an application of Proposition 4.1 yields

$$\begin{aligned}\sum_{1\le\nu\le P^\gamma} \nu^{-1} &\sum_{P^{\theta_r-\rho}<q^j\le P^{\theta_r}} S(P,j,\nu) \\ &\ll \sum_{1\le\nu\le P^\gamma} \nu^{-1} \sum_{P^{\theta_r-\rho}<q^j\le P^{\theta_r}} \frac{1}{\log P}\left(\frac{\nu}{q^j}\right)^{-1/\lfloor\theta_r\rfloor} + \frac{P}{(\log P)^{G-2}} \ll \frac{P}{\log P}.\end{aligned} \tag{5.3}$$

Plugging the estimates (5.2) and (5.3) into (5.1) we get

$$\sum_{j=\ell}^{J}\left|\sum_{p\le P}\mathcal{I}(q^{-j}f(p)) - \frac{\pi(P)}{q^\ell}\right| \ll \frac{P}{\log P},$$

which together with (3.3) proves Theorem 2.1 in the case $\theta_r > k$.

On the other hand, if $\theta_r < k$, then we consider the two ranges

$$P^{\theta_r-\rho} < q^j \le P^{k-1+\rho} \quad\text{and}\quad P^{k-1+\rho} < q^j \le P^k.$$

For the "digits in the middle" an application of Proposition 4.7 yields

$$\begin{aligned}\sum_{1\le\nu\le P^\gamma} \nu^{-1} &\sum_{P^{\theta_r-\rho}<q^j\le P^{k-1+\rho}} S(P,j,\nu) \\ &\ll \sum_{1\le\nu\le P^\gamma} \nu^{-1} \sum_{P^{\theta_r-\rho}<q^j\le P^{k-1+\rho}} P^{1-\rho/4^k} \ll (\log P)JP^{1-\rho/4^k}.\end{aligned} \tag{5.4}$$

Finally, we consider the most significant digits. By Proposition 4.1 we have

$$\begin{aligned}\sum_{1\le\nu\le P^\gamma} \nu^{-1} &\sum_{P^{k-1+\rho}<q^j\le P^{k}} S(P,j,\nu) \\ &\ll \sum_{1\le\nu\le P^\gamma} \nu^{-1} \sum_{P^{k-1+\rho}<q^j\le P^{k}} \frac{1}{\log P}\left(\frac{\nu}{q^j}\right)^{-1/k} + \frac{P}{(\log P)^{G-2}} \ll \frac{P}{\log P}.\end{aligned} \tag{5.5}$$

Plugging the estimates (5.2), (5.4) and (5.5) into (5.1) we get

$$\sum_{j=\ell}^{J}\left|\sum_{p\le P}\mathcal{I}(q^{-j}f(p)) - \frac{\pi(P)}{q^\ell}\right| \ll \frac{P}{\log P},$$

which together with (3.3) proves Theorem 2.1 in the case $\theta_r < k$.

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