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An extension of the Khinchin–Groshev theorem

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1. Introduction. In metric Diophantine approximation, the Khinchin–Groshev theorem provides a measure-theoretic characterization of matrices with prescribed approximation properties. In the present paper, we study certain analogues of the theorem in function fields. We begin by recalling some results in Diophantine approximation of complex numbers. In 1854, Ch. Hermite [9] proved that for every $z \in \mathbb{C} \setminus \mathbb{Q}(i)$, there exist infinitely many $(p, q) \in \mathbb{Z}[i]$ such that

$$\left| z - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2}\,|q|}.$$

This theme was developed by several authors including L. Ford, A. L. Schmidt and others. W. J. LeVeque [15] proved a variation of Khinchin's theorem for $\mathbb{Q}(i)$, which was generalized by D. Sullivan [25]. Let d be a positive integer which is not a perfect square and let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$. A function $\psi : (0, \infty) \to (0, \infty)$ is called *quasi-conformal* if there exists $c > 0$ such that

$$\psi(ht) \leq c\psi(t) \quad \text{for all } h \in [1/2, 2] \text{ and } t > 0.$$

THEOREM 1.1 (Sullivan). *Let ψ be a quasi-conformal function as above. Then for almost every complex number z, there exist infinitely many $p, q \in O_d$ so that*

$$\left| z - \frac{p}{q} \right| \leq \frac{\psi(|q|)}{|q|^2} \quad \textit{and} \quad (p, q) = O_d$$

if and only if

$$\int_0^\infty \frac{\psi(x)^2}{x}\, dx = \infty.$$

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Above, (p, q) is the ideal generated by p and q. Further, H. Nakada [19] obtained an asymptotic for the number of solutions. On the other hand, the metric theory of Diophantine approximation over local fields of positive characteristic has also seen several advances. B. de Mathan [17] proved an analogue of Khinchin's theorem in this setting. This was generalized by S. Kristensen to systems of linear forms [13]. See also the related works [10], [11], [5], [4]. We will obtain a version of the Khinchin–Groshev theorem for *imaginary* quadratic extensions of function fields, which as a special case includes an analogue of Sullivan's result above in the function field setting.

1.1. Basic notation. Let $\mathbb{F} = \mathbb{F}_s$ be a field of s elements, where of course s is a prime power. Let $A = \mathbb{F}[T]$ be the ring of polynomials over $\mathbb{F}$ and let $k = \mathbb{F}(T)$ be its field of fractions. Denote by $|\cdot|$ the absolute value function on k generated by

$$|f(T)| = s^{\deg(f)}$$

for $f(T) \in A$. It is easy to see that this valuation is ultrametric. Let k_∞ be the completion of k with respect to this absolute value function. This gives us the field of Laurent polynomials in T^{-1} over $\mathbb{F}$. That is,

$$k_\infty = \mathbb{F}((T^{-1})) = \Big\{ \sum_{i=-\infty}^{N} x_i T^i : N \in \mathbb{Z},\ x_i \in \mathbb{F} \Big\}.$$

Let $\mathfrak{o} = \{x \in k_\infty : |x| \le 1\}$ and denote by $\mathfrak{p}$ its unique prime ideal, i.e. $\mathfrak{p} = \{x \in k_\infty : |x| < 1\}$. For $x \in k_\infty$ denote by $[x]$ the polynomial part of x, and by $\{x\}$ the tail of x, i.e.

$$[x] = \Big[\sum_{i=-\infty}^{N} x_i T^i \Big] = \sum_{i=0}^{N} x_i T^i \in A, \quad \{x\} = \Big\{ \sum_{i=-\infty}^{N} x_i T^i \Big\} = \sum_{i=-\infty}^{-1} x_i T^i \in \mathfrak{p}.$$

For $x \in k_\infty$ define $|\langle x \rangle|$ to be the distance of x to A given by

$$|\langle x \rangle| = \min\{|x - p| : p \in A\}.$$

1.2. ψ-approximable matrices and quadratic extensions. Let $\psi : [1, \infty) \to (0, \infty)$ be a non-increasing continuous function with $\psi \to 0$. Let $f(T) \in A$ be squarefree so that $K = k(\sqrt{f(T)})$ is a quadratic extension of k. Depending on our choice of $f(T)$ it may or may not be true that $\sqrt{f(T)} \in k_\infty$. We are interested in the case when $k_\infty(\sqrt{f(T)})/k_\infty$ is an extension of degree 2. Let d be the degree of $f(T)$. It turns out that $\sqrt{f(T)} \in k_\infty$ exactly when d is even and has square leading coefficient [20, Proposition 14.6]. Following E. Artin, we call K *imaginary* if this is not the case. We fix such an f and set $K_\infty := k_\infty(\sqrt{f(T)})$.

Let $B \subseteq K$ be the integers over A, that is,

$$B = \{x \in K : x \text{ is a root of some monic } h(U) \in A[U]\}.$$

It is not difficult (Lemma 2.1) to see that $B = A[\sqrt{f(T)}]$ provided that $2 \nmid s$. The study of Diophantine approximation of Laurent series in K_∞ by ratios of polynomials in B is thus a function field analogue of Diophantine approximation of complex numbers discussed earlier.

Let $\mathrm{M}_{m\times n}(K_\infty)$ denote the set of $m\times n$-matrices with real valued entries, $\|\cdot\|$ denote the L^∞ norm and I be the ball $\{z \in \mathrm{M}_{m\times n}(K_\infty) : \|z\| \le 1\}$. We denote by μ the Haar measure on $\mathrm{M}_{m\times n}(K_\infty)$ normalized so that $\mu(I) = 1$. We say that z is *ψ-approximable* if there exist infinitely many $q \in B^n$ and $p \in B^m$ such that

$$\|zq + p\|^{2m} \le \psi(\|q\|^{2n}). \tag{1.1}$$

Denote by $\mathcal{W}_{m\times n}(\psi)$ the set of all ψ-approximable matrices in $\mathrm{M}_{m\times n}(K_\infty)$. Using the Borel–Cantelli lemma we will show (Proposition 2.8) that, provided $2 \nmid s$,

$$\mu(\mathcal{W}_{m\times n}(\psi)) = 0 \quad \text{if} \quad \int_1^\infty \psi(x)\,dx < \infty.$$

We note that the condition $2 \nmid s$ is almost certainly not needed for the convergence case to hold although our proof uses this. The main result of this paper is the converse and does not need any conditions:

THEOREM 1.2. *$\mathcal{W}_{m\times n}(\psi)$ has full measure if*

$$\int_1^\infty \psi(x)\,dx = \infty.$$

Our proof uses the ergodic theory of group actions on homogeneous spaces. We use a strategy due to D. Kleinbock and G. Margulis [12], implemented in the positive characteristic setting by J. Athreya, A. Ghosh and A. Prasad [1]. In §2 we record some facts about function fields and also show the convergence case of Theorem 1.2. In §3, we discuss our main tool, which is a "shrinking target" result from [1], and complete the proof of Theorem 1.2. In fact, an asymptotic formula for the number of solutions also follows from our method. Details, along with Hausdorff and multiplicative versions of Theorem 1.2, will appear in the PhD thesis of the second named author [21].

2. Preliminaries and the convergence case. In this section, we record some preliminary lemmas and also prove the convergence case of Theorem 1.2. We begin with some facts about quadratic function fields.

LEMMA 2.1. *If $2 \nmid s$ then $B = A[\sqrt{f(T)}]$.*

Proof. Let $x \in A[\sqrt{f(T)}]$. We can write x in the form $x = a + b\sqrt{f(T)}$ for some $a, b \in A$. Then x is a root of the monic polynomial

$$U^2 - 2aU + (a^2 - b^2 f(T)) \in A[U]$$

and so $A[\sqrt{f(T)}] \subseteq B$.

Conversely let $x \in K$ be integral over A. Write $x = (a + b\sqrt{f(T)})/c$ with $a, b, c \in A$ not all sharing a common factor, which we can do since $A = \mathbb{F}[T]$ is a unique factorization domain. It is easy to check that x is a root of the following quadratic in $k[U]$:

$$U^2 - \frac{2a}{c}U + \frac{a^2 - b^2 f(T)}{c^2}.$$

If $x \in k$ then, as $A = \mathbb{F}[T]$ is integrally closed in k, we have $x \in A \subset A[\sqrt{f(T)}]$. On the other hand, if $x \notin k$ then the minimal monic polynomial of x must divide the above quadratic, and have degree at least 2, and so is equal to it. Thus $c \,|\, 2a$ and $c^2 \,|\, a^2 - b^2 f(T)$. Now suppose $d \in A$ is an irreducible factor of c. If $d \,|\, 2a$ then $d \,|\, a$ as 2 is a unit. Hence $d^2 \,|\, a^2 - b^2 f(T) \Rightarrow d^2 \,|\, b^2 f(T)$, and as $f(T)$ is squarefree, we have $d \,|\, b$. This contradicts a, b, c sharing no common factor. Therefore c has no irreducible factors and so it is a unit. We conclude that $x \in A[\sqrt{f(T)}]$. ∎

LEMMA 2.2. *Let $f(T) \in A$ be such that $\sqrt{f(T)} \notin k_\infty$. Then for all $x, y \in k$,*

$$|x + y\sqrt{f(T)}|' = \max(|x|, |y|\,|f|^{1/2})$$

for any extension $|\cdot|'$ of $|\cdot|$ to K.

Proof. Any extension $|\cdot|'$ of $|\cdot|$ to K must restrict to $|\cdot|$ on k and satisfy $|\sqrt{f}|' = |f|'^{1/2} = |f|^{1/2}$. The theory of valuations tells us that if $|\cdot|_1, \ldots, |\cdot|_d$ are all extensions of $|\cdot|$ to K then

$$\sum_{i=1}^{d} [K_i : k_\infty] = [K : k]$$

where K_i is the completion of K with respect to the valuation $|\cdot|_i$. Let K_∞ be the completion of K with respect to $|\cdot|'$. Then $[K_\infty : k_\infty] = 2$ because $\sqrt{f} \notin k_\infty$ by assumption. Since $[K : k] = 2$ this means that there are no more extensions of $|\cdot|$ to K. Since the map $K \to K$ sending $\sqrt{f}$ to $-\sqrt{f}$ is a k-automorphism of K, we must have $|x + y\sqrt{f}|' = |x - y\sqrt{f}|'$ for all $x, y \in k$ or else we would generate another valuation lying over $|\cdot|$ for K.

Suppose there exist $x, y \in k$ such that $|x + y\sqrt{f}|' < \max(|x|', |y\sqrt{f}|')$. Then by the ultrametric property we have $|x|' = |y\sqrt{f}|'$. Since there is only one valuation extension, we must have $|x - y\sqrt{f}|' < \max(|x|', |y\sqrt{f}|')$ also. But then

$$|(x + y\sqrt{f}) + (x - y\sqrt{f})|' = |2x|' = |x|'$$

and

$$|(x+y\sqrt{f})+(x-y\sqrt{f})|' \le \max(|x+y\sqrt{f})|', |x-y\sqrt{f}|') = |x+y\sqrt{(f)}|' \\ < \max(|x|', |y\sqrt{f}|') = |x|',$$

a contradiction. ∎

Since the extension $|\cdot|'$ of $|\cdot|$ to K_∞ is unique we will simply write $|\cdot|$ in both cases.

2.1. Haar measures on k_∞ and K_∞. We discuss Haar measures on local fields and their extensions in the context of the present paper. More general constructions can be found in [24, 26].

Measuring balls in k_∞. Let $v > 0$ and denote by $B_{k_\infty}(v)$ the ball of radius v centered at 0 in k_∞, that is,

$$B_{k_\infty}(v) = \{x \in k_\infty : |x| < v\}.$$

Let μ_{k_∞} be the Haar measure on k_∞ normalized so that

$$\mu_{k_\infty}(B_{k_\infty}(1)) = 1.$$

The range of possible values of $|\cdot|$ on k_∞ is $\{s^n : n \in \mathbb{Z}\}$, so if we let $n_v \in \mathbb{Z}$ be the unique integer such that $s^{n_v-1} < v \le s^{n_v}$, namely $n_v = \lceil \log(v)/\log(s) \rceil$, then $B_{k_\infty}(v) = B_{k_\infty}(s^{n_v})$. Thus it suffices to compute the measure $\mu_{k_\infty}(B_{k_\infty}(v))$ where v is of the form s^n.

LEMMA 2.3. *For all $m \in \mathbb{Z}$ we have $B_{k_\infty}(s^m) = T^m B_{k_\infty}(1)$.*

Proof. Let $x \in B_{k_\infty}(s^m)$, thus $|x| < s^m$ and so $|T^{-m}x| = |T^{-m}|\,|x| = s^{-m}|x| < 1$, and $T^{-m}x \in B_{k_\infty}(1)$. Therefore $x = T^m(T^{-m}x) \in T^m B_{k_\infty}(1)$. Hence $B_{k_\infty}(s^m) \subseteq T^m B_{k_\infty}(1)$. Conversely, if $x \in T^m B_{k_\infty}(1)$ then $x = T^m y$ for some $y \in B_{k_\infty}(1)$ and so $|x| = |T^m|\,|y| < s^m$, thus $x \in B_{k_\infty}(s^m)$. Hence $T^m B_{k_\infty}(1) \subseteq B_{k_\infty}(s^m)$. ∎

LEMMA 2.4. *Let $m, n \in \mathbb{Z}$ and $n \ge 0$. Then*

$$B_{k_\infty}(s^{m+n}) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < n}} (T^m f + B_{k_\infty}(s^m))$$

and the union is disjoint.

Proof. It is clear from the ultrametric property of $|\cdot|$ that the right-hand side is included within the left, so let $x \in B_{k_\infty}(s^{m+n})$. We have $T^{-m}x = [T^{-m}x] + \{T^{-m}x\}$ where $\{T^{-m}x\} \in B_{k_\infty}(1)$, so $T^{-m}x \in [T^{-m}x] + B_{k_\infty}(1)$. Hence $x \in T^m[T^{-m}x] + T^m B_{k_\infty}(1) = T^m[T^{-m}x] + B_{k_\infty}(s^m)$ (Lemma 2.3). Now $\deg([T^{-m}x]) \le \deg(T^{-m}x) = \deg(x) - m < n$, and thus x is contained in the right-hand side of the proposed equality.

To prove that the union is disjoint, suppose that $x \in T^m f + B_{k_\infty}(s^m)$ for some $f \in \mathbb{F}_s[T]$ with $\deg(f) < n$. Then $T^{-m}x \in f + B_{k_\infty}(1)$, and so $x \in (f + B_{k_\infty}(1)) \cap ([T^{-m}x] + B_{k_\infty}(1))$, which implies by the ultrametric property of $|\cdot|$ that $f - [T^{-m}x] \in B_{k_\infty}(1)$. Since $B_{k_\infty}(1) \cap \mathbb{F}_s[T] = \{0\}$, we deduce that $f = [T^{-m}x]$ is uniquely determined by x. ∎

LEMMA 2.5. *For all $n \in \mathbb{Z}$,*

$$\mu_{k_\infty}(B_{k_\infty}(s^n)) = s^n.$$

Proof. If $n \geq 0$ then by Lemma 2.4 we have

$$B_{k_\infty}(s^n) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < n}} (f + B_{k_\infty}(1)).$$

Since the union is disjoint and the Haar measure μ_{k_∞} is translation invariant, it follows that

$$\mu_{k_\infty}(B_{k_\infty}(s^n)) = \mu_{k_\infty}(B_{k_\infty}(1)) \# \{f \in \mathbb{F}_s[T] : \deg(f) < n\} = s^n.$$

If $n < 0$ then again by Lemma 2.4 we have

$$B_{k_\infty}(1) = B_{k_\infty}(s^{n-n}) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < -n}} (T^n f + B_{k_\infty}(s^n)),$$

from which it follows that

$$1 = \mu_{k_\infty}(B_{k_\infty}(s^n)) \# \{f \in \mathbb{F}_s[T] : \deg(f) < -n\} = s^{-n} \mu_{k_\infty}(B_{k_\infty}(s^n))$$

and again $\mu_{k_\infty}(B_{k_\infty}(s^n)) = s^n$. ∎

Since we have been working with powers of s, the following easy corollary will be useful when we are working with a general $v > 0$.

COROLLARY 2.6. *If $v > 0$ and $n \in \mathbb{Z}$ then*

$$\mu_{k_\infty}(B_{k_\infty}(s^n v)) = s^n \mu_{k_\infty}(B_{k_\infty}(v)), \qquad v \leq \mu_{k_\infty}(B_{k_\infty}(v)) < sv.$$

Measuring balls in K_∞. Let $v > 0$. We denote by $B_{K_\infty}(v)$ the ball of radius v about 0 in K_∞, that is,

$$B_{K_\infty}(v) = \{x \in K_\infty : |x| < v\}.$$

Let μ_{K_∞} be the Haar measure on K_∞ normalized such that

$$\mu_{K_\infty}(B_{K_\infty}(1)) = 1.$$

We know that every element of K_∞ is of the form $x + y\sqrt{f}$ for some $x, y \in k_\infty$ and that $|x + y\sqrt{f}| = \max(|x|, |y|\,|f|^{1/2})$ by Lemma 2.2, so

$$B_{K_\infty}(v) = \{x + y\sqrt{f} : x, y \in k_\infty,\ |x| < v,\ |y| < v/|f|^{1/2}\}.$$

Treating K_∞ as the product k_∞^2, the product measure $\mu_{k_\infty^2}$ is a Haar measure for K_∞ so there exists some constant $c > 0$ such that

$$c\mu_{k_\infty^2} = \mu_{K_\infty}$$

and hence

$$\mu_{K_\infty}(B_{K_\infty}(v)) = c\mu_{k_\infty^2}(B_{K_\infty}(v)) = c\mu_{k_\infty}(B_{k_\infty}(v))\mu_{k_\infty}(B_{k_\infty}(v/|f|^{1/2})).$$

The value c must be such that

$$c\mu_{k_\infty}(B_{k_\infty}(1))\mu_{k_\infty}(B_{k_\infty}(1/|f|^{1/2})) = c\mu_{k_\infty}(B_{k_\infty}(1/|f|^{1/2})) = 1$$

thus $c = s^{\lfloor \deg(f)/2 \rfloor}$ and

$$\mu_{K_\infty}(B_{K_\infty}(v)) = \mu_{k_\infty}(B_{k_\infty}(v))\mu_{k_\infty}(B_{k_\infty}(v/|f|^{1/2}))s^{\lfloor \deg(f)/2 \rfloor}.$$

Corollary 2.6 then easily implies

COROLLARY 2.7. *For all $v > 0$ and $n \in \mathbb{Z}$:*

(1) $\mu_{K_\infty}(B_{K_\infty}(s^n v)) = s^{2n}\mu_{K_\infty}(B_{K_\infty}(v))$,
(2) $\mu_{K_\infty}(B_{K_\infty}(s^n)) = s^{2n}$,
(3) $v^2/s^2 \le \mu_{K_\infty}(B_{K_\infty}(v)) < v^2 s^2$.

2.2. The convergence case. We now prove the convergence case of Theorem 1.2. Firstly we note that $\mathcal{W}_{m\times n}(\psi)$ is invariant under translation by $M_{m\times n}(B)$. Indeed, if $z \in \mathcal{W}_{m\times n}(\psi)$ and $z' \in M_{m\times n}(B)$ then for each $q \in B^n$ and $p \in B^m$ such that $\|zq + p\|^{2m} \le \psi(\|q\|^{2n})$ we have

$$\|(z + z')q + (p - z'q)\|^{2m} = \|zq + p\|^{2m} \le \psi(\|q\|^{2n}),$$

and so $z + z' \in \mathcal{W}_{m\times n}(\psi)$. Consider the additive subgroup $P \subset K_\infty$ defined by

$$P = \{x + y\sqrt{f} : x, y \in k_\infty \text{ and } \max(|x|, |y|) < 1\}.$$

Clearly P is a fundamental domain for K_∞/B. That is, for every $x + y\sqrt{f} \in K_\infty$ there is a unique $z \in P$ and $b \in B$ such that $x + y\sqrt{f} = z + b$. In particular, $b = [x] + [y]\sqrt{f}$ and $z = \{x\} + \{y\}\sqrt{f}$. So $M_{m\times n}(P)$ is a fundamental domain for $M_{m\times n}(K_\infty)/M_{m\times n}(B)$.

Let $\mathscr{P} = M_{m\times n}(P)$. Since $\mathcal{W}_{m\times n}(\psi)$ is invariant under translation by $M_{m\times n}(B)$, this means that for any $z \in M_{m\times n}(K_\infty)$,

$$\mu(\mathscr{P} \cap \mathcal{W}_{m\times n}(\psi)) = \mu((\mathscr{P} + z) \cap \mathcal{W}_{m\times n}(\psi)).$$

Since $M_{m\times n}(K_\infty)$ is the countable union of translations $\bigcup_{z\in M_{m\times n}(B)}(z + \mathscr{P})$, to prove that $\mu(\mathcal{W}_{m\times n}(\psi)) = 0$ we only need to show that $\mu(\mathscr{P} \cap \mathcal{W}_{m\times n}(\psi)) = 0$. We now fix some non-zero $q \in B^n$ and consider the maps

$$M_{m\times n}(K_\infty) \xrightarrow{q} K_\infty^m \xrightarrow{\pi} P^m$$

where q represents right multiplication by q, and π is the quotient map $K_\infty^m \to P^m$ given by reduction modulo B^m.

Given a measurable $\mathscr{B} \subseteq P^m$ define

$$\widetilde{\mu}(\mathscr{B}) = \mu(\mathscr{P} \cap q^{-1}\pi^{-1}(\mathscr{B})).$$

We will show that $\widetilde{\mu}$ is translation invariant. Let $y \in P^m$ and consider the set $q^{-1}\pi^{-1}(\mathscr{B} + y)$. As q is non-zero the map $M_{m\times n}(K_\infty) \xrightarrow{q} K_\infty^m$ is surjective so there exists some $Y \in M_{m\times n}(K_\infty)$ such that $\pi(Yq) = y$. It is easily verified that $q^{-1}\pi^{-1}(\mathscr{B}) + Y = q^{-1}\pi^{-1}(\mathscr{B} + y)$. Thus

$$\widetilde{\mu}(\mathscr{B} + y) = \mu(\mathscr{P} \cap (q^{-1}\pi^{-1}(\mathscr{B}) + Y)).$$

Since μ is translation invariant, we see that

$$\widetilde{\mu}(\mathscr{B} + y) = \mu\big((\mathscr{P} - Y) \cap (q^{-1}\pi^{-1}(\mathscr{B}))\big).$$

The set $q^{-1}\pi^{-1}(\mathscr{B})$ is invariant under translation by elements of $M_{m\times n}(B)$ so we have

$$\mu\big(\mathscr{P} \cap (q^{-1}\pi^{-1}(\mathscr{B}))\big) = \mu\big((\mathscr{P} - Y) \cap (q^{-1}\pi^{-1}(\mathscr{B}))\big)$$

and thus $\widetilde{\mu}(\mathscr{B}) = \widetilde{\mu}(\mathscr{B}+y)$. It is also easy to see that $\widetilde{\mu}$ is a measure on P^m. Up to multiplication by a positive constant, the only translation invariant measure of P^m is a Haar measure. Therefore $\widetilde{\mu}$ is the Haar measure on P^m and $\widetilde{\mu}(P^m) = \mu(\mathscr{P} \cap (q^{-1}\pi^{-1}(P^m))) = \mu(\mathscr{P}) = \mu_{K_\infty}(P)^{mn}$. Thus we can relate $\widetilde{\mu}$ to the Haar measure $\mu_{K_\infty^m}$ by

$$\widetilde{\mu}(\mathscr{B}) = \frac{\mu_{K_\infty}(P)^{mn}}{\mu_{K_\infty}(P)^m}\mu_{K_\infty^m}(\mathscr{B}).$$

Proposition 2.8. *If* $\int_1^\infty \psi(x)\,dx < \infty$ *then*

$$\mu(\mathcal{W}_{m\times n}(\psi)) = 0.$$

Proof. Fix some non-zero $q \in B^n$. Let

$$S_q = \{z \in M_{m\times n}(K_\infty) : \exists p \in B^m \text{ such that } \|qz + p\|^{2m} < \psi(\|q\|^{2n})\}.$$

Given some $x \in K_\infty^m$ and $v > 0$ the condition

$$\exists p \in B^m \text{ such that } \|x + p\| < v$$

is equivalent to

$$\exists p \in B^m \text{ such that } x + p \in B_{K_\infty^m}(v)$$

where $B_{K_\infty^m}(v)$ denotes the open box of radius v about some $0 \in K_\infty^m$ in the sup-metric. This is again equivalent to

$$x \in \bigcup_{p\in B^m} (B_{K_\infty^m}(v) + p).$$

Now the pre-image $\pi^{-1}(y)$ of some $y \in P^m$ is the coset $\{y + b : b \in B^m\}$. Also, given some $z \in K_\infty^m$ the point $\pi(z) \in P^m$ is equivalent to z modulo B^m

and so the cosets $\{z+b : b \in B^m\}$ and $\{\pi(z)+b : b \in B^m\}$ are equal. Hence $\{z+b : b \in B^m\} = \pi^{-1}\{\pi(z)\}$ and our condition is equivalent to

$$x \in \pi^{-1}(\pi(B_{K_\infty^m}(v))).$$

We deduce that the set S_q consists of elements $z \in M_{m\times n}(K_\infty)$ such that $zq \in \pi^{-1}(\pi(B_{K_\infty^m}(v_q)))$ where $v_q = \psi(\|q\|^{2n})^{1/2m}$, and so

$$S_q = q^{-1}\pi^{-1}(\pi(B_{K_\infty^m}(v_q))).$$

Therefore we can compute the measure

$$\begin{aligned}\mu(S_q \cap \mathscr{P}) &= \mu(q^{-1}\pi^{-1}(\pi(B_{K_\infty^m}(v_q)) \cap \mathscr{P})\\ &= \widetilde{\mu}(\pi(B_{K_\infty^m}(v_q))\\ &= c\mu_{K_\infty^m}(\pi(B_{K_\infty^m}(v_q))\end{aligned}$$

where $c = \mu_{K_\infty}(P)^{m(n-1)}$. Now suppose that $v_q \le 1$. In this case $B_{K_\infty^m}(v_q) \subseteq \mathscr{P}$ and thus $\pi(B_{K_\infty^m}(v_q)) = B_{K_\infty^m}(v_q)$, so

$$\begin{aligned}\mu(S_q \cap \mathscr{P}) &= c\mu_{K_\infty^m}(B_{K_\infty^m}(v_q)) = c\mu_{K_\infty}(B_{K_\infty}(v_q))^m\\ &< c(v_q^2 s^2)^m = cs^{2m}\psi(\|q\|^{2n}).\end{aligned}$$

Next let $d > 0$ and define

$$S_d = \bigcup_{\substack{q\in B^n\\ \|q\|=s^d}} S_q.$$

This union can only be non-empty for $d \in \frac{1}{2}\mathbb{N}$. If for some $d \in \frac{1}{2}\mathbb{N}$ we have $v_d = \psi(s^{2dn})^{1/2m} \le 1$, then any $q \in B$ with $\|q\| = s^d$ satisfies $v_q \le 1$ and hence

$$\begin{aligned}\mu(S_d \cap \mathscr{P}) &< \#\{q \in B^n : \|q\| = s^d\}cs^{2m}\psi(s^{2dn})\\ &\le s^{2(d+1)n}cs^{2m}\psi(s^{2dn}) = c's^{2dn}\psi(s^{2dn}),\end{aligned}$$

where $c' = cs^{2n+2m}$. We know that $z \in \mathcal{W}_{m\times n}(\psi)$ if and only if

$$z \in \limsup_{d\in\frac{1}{2}\mathbb{N}} S_d.$$

There can only be finitely many d with $v_d > 1$ as ψ is monotone non-increasing to zero, thus the Borel–Cantelli lemma tells us that if

$$\sum_{d\in\frac{1}{2}\mathbb{N}} s^{2dn}\psi(s^{2dn}) = \sum_{e=1}^{\infty} s^{en}\psi(s^{en}) < \infty$$

then

$$\mu(\mathcal{W}_{m\times n}(\psi) \cap \mathscr{P}) = 0.$$

The convergence of the sum follows from the fact that ψ is non-increasing and that $\int_{x=1}^{\infty} \psi(x)\,dx < \infty$. ∎

3. Shrinking targets

3.1. Background. In this section, we will prove the divergence counterpart ([1]) of the Khinchin–Groshev theorem. We will use dynamics on a certain homogeneous space. Let $G = \mathrm{SL}_{m+n}(K_\infty)$, $\Gamma = \mathrm{SL}_{m+n}(B)$ and denote G/Γ by Υ. Then, by a theorem of H. Behr [2] and G. Harder [8] (which is a function field analogue of a theorem of Borel and Harish-Chandra), Γ is a non-uniform lattice in G. In fact, Υ can be identified with the space of full rank free B-modules of covolume 1 in K_∞^{m+n}, and it is straightforward to see that the latter space is non-compact. The compact subsets of this space can be described using a positive characteristic analogue of Mahler's compactness criterion, a proof of which, using a function field analogue of the geometry of numbers due to K. Mahler [16], can be found in [7].

Before starting the proof, we provide a brief description of the results and techniques of [1]. The main aim of that paper was to provide a function field analogue of the Khinchin–Groshev theorem using homogeneous dynamics. In fact, a general 0-1 law is proved for cuspidal excursions of multi-parameter diagonal flows on G/Γ where G denotes the k-points of a semisimple, simply connected linear algebraic group, k is a local field of positive characteristic, and Γ is a non-uniform lattice in G. Another interesting corollary of such a 0-1 law is a function field analogue of Sullivan's *logarithm law* for geodesic excursions on quotients of Bruhat–Tits buildings. This provides analogues in the setting of function fields of results of Kleinbock and Margulis [12]. For the rest of the paper, we revert to $G = \mathrm{SL}_{m+n}(K_\infty)$, $\Gamma = \mathrm{SL}_{m+n}(B)$ and set μ to be the probability measure on Υ which descends from Haar measure on G.

Given a matrix $z \in \mathrm{M}_{m\times n}(K_\infty)$, we associate to it the module $\Lambda_z \in \Upsilon$ defined by

$$\Lambda_z := \begin{pmatrix} I_m & z \\ 0 & I_n \end{pmatrix} B^{m+n},$$

a typical vector of which looks like

$$\begin{pmatrix} I_m & z \\ 0 & I_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} zq+p \\ q \end{pmatrix}.$$

For $t \in \mathbb{Z}$ we define

$$f_t = \mathrm{diag}(\underbrace{T^{nt},\dots,T^{nt}}_{m\text{ times}},\underbrace{T^{-mt},\dots,T^{-mt}}_{n\text{ times}}), \tag{3.1}$$

([1]) In fact the results in this section can be used to prove the convergence part as well, but we feel that our earlier explicit calculation enhances the exposition of this paper.

and on $\varUpsilon$, define the function

$$\Delta(\Lambda) := \max_{v\in\Lambda\setminus\{0\}} \log_s \frac{1}{\|v\|}. \tag{3.2}$$

Note that Δ takes values in the non-negative real numbers. Then we show below that if there are infinitely many $t > 0$ such that

$$\Delta(f_t\Lambda_z) \geq r(t), \tag{3.3}$$

then $z \in \mathcal{W}_{m\times n}(\psi)$. Here $r(t)$ is an explicitly constructed function (the *Dani function*) which is closely related to ψ. By Mahler's compactness criterion, the sets $\{\Lambda \in \varUpsilon : \Delta(f_t\Lambda) \geq r(t)\}$ form a sequence of decreasing neighbourhoods of infinity, i.e. are complements of larger and larger compact sets as $t \to \infty$. If the semiorbit $\{f_t\Lambda_z : t > 0\}$ meets these neighbourhoods infinitely often, then the matrix z is ψ-approximable. This scenario has been widely studied in dynamics and is known as the *shrinking target property*.

The process of associating Diophantine approximation to cusp excursions in this particular instance has been christened the *Dani correspondence* by Kleinbock and Margulis. The strategy of the proof is then to show that the sets $\{\Lambda \in \varUpsilon : \Delta(f_t\Lambda) \geq r(t)\}$ are quasi-independent on average. The quantitative Borel–Cantelli lemma stated below then ensures that their $\limsup$ has full measure, which will prove the theorem. This strategy is again due to Kleinbock and Margulis, who created an abstract setup incorporating these ideas in the setting of real numbers. In [1], we formulated and proved an analogue of their main theorem for general semisimple groups in positive characteristic. This result will be our main tool and is stated below. The quasi-independence in this setting is provided by the exponential decay of matrix coefficients for semisimple groups in positive characteristic (Theorem 2.1 in [1]). For the purposes of the present paper, we simply need to prove an analogue of the Dani correspondence stated above for quadratic extensions. A direct application of Theorem 3.2 below then completes the argument.

3.2. 0-1 laws. Let $\mathcal{B}$ be a family of measurable subsets of $\varUpsilon$ and let $\mathfrak{F} = \{f_n\}$ denote a sequence of μ-preserving transformations of $\varUpsilon$. The following terminology is taken from [12].

Definition 3.1 (Borel–Cantelli families). We say that $\mathcal{B}$ is *Borel–Cantelli for* $\mathfrak{F}$ if for every sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of sets from $\mathcal{B}$,

$$\mu(\{x \in \varUpsilon : f_n(x) \in \mathcal{A}_n \text{ for infinitely many } n \in \mathbb{N}\}) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(\mathcal{A}_n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(\mathcal{A}_n) = \infty. \end{cases}$$

For a function Δ on $\varUpsilon$ and an integer $n \in \mathbb{N}$, denote by Φ_Δ the *tail distribution function*, defined by

$$\Phi_\Delta(n) := \mu(\{x \in \varUpsilon : \Delta(x) \geq s^n\}). \tag{3.4}$$

We say that a function $\varDelta$ on $\varUpsilon$ is *smooth* if there exists a compact open subgroup U of G such that $\varDelta$ is U-invariant. For $\kappa > 0$, say that $\varDelta$ is *κ-UDL* (an abbreviation for *κ-ultra distance like*) if it is smooth and

$$\varPhi_{\varDelta}(n) \asymp s^{-\kappa n} \quad \forall n \in \mathbb{Z}. \tag{3.5}$$

Finally, we say that $\varDelta$ is *UDL* if it is smooth and there exists $\kappa > 0$ such that (3.5) holds.

A key example of a UDL function is the one in (3.2). This can be seen using reduction theory on $\varUpsilon$, specifically a generalization of Siegel's mean value theorem due to M. Morishita [18] followed by a verbatim repetition of Theorem 7.3 in [12]. See §4.2 in [1]. We are now ready for Theorem 1.6 from [1], which will be our main tool.

THEOREM 3.2. *Let $\mathfrak{F} = \{f_n : n \in \mathbb{N}\}$ be a sequence of elements of G satisfying*

$$\sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} \|f_n f_m^{-1}\|^{-\beta} < \infty \quad \forall \beta > 0, \tag{3.6}$$

and let $\varDelta$ be a UDL function on $\varUpsilon$. Then

$$\mathcal{B}(\varDelta) := \{\{x \in \varUpsilon : \varDelta(x) \geq s^n\} : n \in \mathbb{Z}\} \tag{3.7}$$

is Borel–Cantelli for $\mathfrak{F}$.

For the convenience of the reader, we outline the main steps in the divergence half of the proof of Theorem 3.2. Note that the converse of the Borel–Cantelli lemma, i.e. the lemma used in the convergence case of the Khinchin–Groshev theorem (Proposition 2.8) is clearly false without additional conditions on the sets ensuring some form of weak independence between the sets. A very general statement to this effect, i.e. positive measure of a lim sup set in the presence of quasi-independence on average, has been abstracted from work by W. M. Schmidt by V. G. Sprindžuk. For a sequence $\mathfrak{H} = \{h_n\}$ of functions on $(\varUpsilon, \mu)$, we define

$$S_{\mathfrak{H},N}(x) := \sum_{n=1}^{N} h_n(x) \quad \text{and} \quad E_{\mathfrak{H},N} := \int_{\varUpsilon} S_{\mathfrak{H},N}\, d\mu. \tag{3.8}$$

The following theorem can be found in [23].

PROPOSITION 3.3. *Let $\mathfrak{H} = \{h_n : n \in \mathbb{N}\}$ be a sequence of functions on $(\varUpsilon, \mu)$ satisfying the following two conditions:*

$$\int_{\varUpsilon} h_n\, d\mu \leq 1 \quad \textit{for every } n \in \mathbb{N}, \tag{3.9}$$

and there exists $C > 0$ such that

$$\sum_{m,n=M}^{N} \Big(\int_{\Upsilon} h_m h_n \, d\mu - \int_{\Upsilon} h_m \, d\mu \int_{\Upsilon} h_n \, d\mu \Big) \le C \sum_{n=M}^{N} \int_{\Upsilon} h_n \, d\mu \tag{3.10}$$

for all $N > M \ge 1$. Then for μ-almost every $x \in \Upsilon$,

$$\lim_{N\to\infty} \frac{S_{\mathfrak{H},N}(x)}{E_{\mathfrak{H},N}} = 1 \tag{3.11}$$

whenever $E_{\mathfrak{H},\infty}$ diverges.

Note that the above proposition in fact offers a quantitative form of the Borel–Cantelli lemma. Let h_n be the characteristic function of the set $\{x \in \Upsilon : \Delta(x) \ge s^n\}$. To prove Theorem 3.2, one applies Lemma 3.3 to the *twisted* sequence

$$\mathfrak{H}^{\mathfrak{F}} := \{f_n^{-1} h_n : n \in \mathbb{N}\}.$$

To verify (3.10) in Proposition 3.3, one uses quantitative estimates for decay of matrix coefficients. Denote by $L_0^2(\Upsilon)$ the subspace of $L^2(\Upsilon)$ orthogonal to constant functions, and by ρ_0 the regular representation of G on $L_0^2(\Upsilon)$. The following is Theorem 2.1 from [1] for the special case of Υ.

Theorem 3.4. *There exist constants $a > 0$ and $b \in \mathbb{Z}_+$ such that for any $g \in G$ and any smooth functions $\phi, \psi \in L_0^2(\Upsilon)$,*

$$|\langle \rho_0(g)\phi, \psi\rangle| \le a\|\phi\|_2 \|\psi\|_2 \|g\|^{-1/b}. \tag{3.12}$$

The constants depend on G, Γ and the choice of compact subgroup U in the definition of the smooth vectors.

We now proceed to complete the proof of the Khinchin–Groshev theorem. Recall that we have a sequence $\{f_t\}$ (as in (3.1)) which clearly satisfies (3.6), a UDL function Δ (as in (3.2)) on Υ and an associated family $\mathcal{B}(\Delta)$ of sets (as in (3.7)). In order to complete the proof, we simply need to explain how these sets are related to the Diophantine inequalities we wish to analyze. In other words, all that remains is to prove a version of the Dani correspondence for quadratic extensions of function fields. This is accomplished below. In fact, we prove a result which is slightly more general than the Khinchin–Groshev theorem.

For a vector v in K_∞^{m+n} we denote by $v^{(m)}$ the vector comprising its first m coordinates and by $v_{(n)}$ the vector comprising the last n. Furthermore, a module $\Lambda \in \Upsilon$ is called (ψ, n)-*approximable* if there exist infinitely many v with arbitrarily large $\|v_{(n)}\|$ such that

$$\|v^{(m)}\|^{2m} \le \psi(\|v_{(n)}\|^{2n}). \tag{3.13}$$

We will show:

THEOREM 3.5. *Almost every lattice Λ in Υ is (ψ, n)-approximable provided*

$$\int_{1}^{\infty} \psi(x)\, dx \tag{3.14}$$

diverges.

Theorem 1.2 follows from the above using a simple Fubini type argument (cf. [12, 8.7] and [3, 2.1]) which we will not repeat. To deduce Theorem 3.5 from Theorem 3.2 we note that f_t satisfies (3.6) and, as mentioned above, Δ is a UDL function. The final link is provided by a function field version of the Dani correspondence. First we record a version of Lemma 8.3 in [12] suitable for our needs.

LEMMA 3.6. *Fix $m, n \in \mathbb{N}$, $u > 1$ and $x_0 > 0$, and let $\psi : [x_0, \infty) \to (0, \infty)$ be non-increasing and continuous. Then there exists a pair of continuous functions $\lambda, L : [t_0, \infty) \to \mathbb{R}$, where*

$$t_0 = \frac{\log_u x_0}{n(m+n)} - \frac{\log_u \psi(x_0)}{m(m+n)},$$

such that

$$\begin{aligned} &\lambda(t) \text{ is strictly increasing to } \infty,\\ &L(t) \text{ is non-decreasing } (\text{to } \infty \text{ if } \psi \to 0), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\psi(u^{\lambda(t)}) = u^{-L(t)},\\ &\qquad L(t) = mt(m+n) - \frac{m}{n}\lambda(t) \qquad \forall t \geq t_0. \end{aligned} \tag{3.16}$$

Define the Dani function $r(t) : [t_0, \infty) \to \mathbb{R}$ by $r(t) = \frac{L(t)-\lambda(t)}{m+n}$. Then

$$\int_{x_0}^{\infty} \psi(x)\, dx < \infty \;\Leftrightarrow\; \int_{t_0}^{\infty} s^{-(m+n)r(t)}\, dt < \infty. \tag{3.17}$$

Proof. For a fixed $t \in \mathbb{R}$ consider the functions

$$L_1(\lambda) = L_1^t(\lambda) = mt(m+n) - \frac{m}{n}\lambda \quad \text{and} \quad L_2(\lambda) = -\log_u \psi(u^\lambda).$$

The first function L_1 is a decreasing line of gradient $-m/n$, and the second L_2 is a continuous non-decreasing function. Due to the definition of ψ, L_2 is only defined for $\lambda \geq \log_u x_0$. Notice that if we have functions L and λ as desired in the lemma, then

$$L_1(\lambda(t)) = L_2(\lambda(t)) = L(t).$$

That is, $(\lambda(t), L(t))$ is a point of intersection of L_1 and L_2. Now L_1 and L_2 have at most one point of intersection, so if there is one we define $(\lambda(t), L(t))$ to be that point. There is an intersection if

$$L_1(\log_u x_0) \geq L_2(\log_u x_0),$$

that is,

$$mt(m+n) - \frac{m}{n}\log_u x_0 \geq -\log_u \psi(u^{\log_u x_0}),$$
$$mt(m+n) \geq \frac{m}{n}\log_u x_0 - \log_u \psi(x_0),$$
$$t \geq \frac{\log_u x_0}{n(m+n)} - \frac{\log_u \psi(x_0)}{m(m+n)} = t_0.$$

Thus we have defined $(\lambda(t), L(t))$ for all $t \geq t_0$. Note that we have forced the equalities (3.16); now we must check that the other conditions hold.

The unique value of t such that L_1^t intersects a given point (x, y) is $t = \frac{x}{n(m+n)} + \frac{y}{m(m+n)}$. Now let $t_0 \leq t < t'$. If $\lambda(t') \leq \lambda(t)$ then

$$L(t') = L_2(\lambda(t')) \leq L_2(\lambda(t)) = L(t).$$

Since $L_1^{t'}$ intersects $(\lambda(t'), L(t'))$ we have

$$t' = \frac{\lambda(t')}{n(m+n)} + \frac{L(t')}{m(m+n)} \leq \frac{\lambda(t)}{n(m+n)} + \frac{L(t)}{m(m+n)} = t.$$

This is a contradiction. Thus $\lambda(t') > \lambda(t)$.

Now let $T \geq \log_u(x_0)$ and

$$t = \frac{T}{n(m+n)} + \frac{L_2(T)}{m(m+n)}$$

so that L_1^t intersects the point $(T, L_2(T))$ and so $\lambda(t) = T$. We see that λ is strictly increasing and takes arbitrarily large values, and thus $\lambda \to \infty$. From $L(t) = -\log_u \psi(u^{\lambda(t)})$ we immediately deduce that L is non-decreasing and $L \to \infty$ if $\psi \to 0$. Finally, (3.17) follows from a simple change of coordinates just as in [12]. ∎

Note that

$$L(t) = mnt + mr(t) \quad \text{and} \quad \lambda(t) = mnt - nr(t).$$

To complete the proof of the theorem we now only need:

Proposition 3.7. *Fix $m, n \in \mathbb{Z}$, let $u = s^{1/2}$ and let $\psi : [x_0, \infty) \to (0, \infty)$ be as above. Let $\Lambda \in \Upsilon$. Define $r : [t_0, \infty) \to \mathbb{R}$ using Lemma 3.6 using the values $2m$ and $2n$ for m and n. If there exist arbitrarily large $t \in \mathbb{N}$ such that*

$$\Delta(f_t\Lambda) \geq r(t)$$

then Λ is (ψ, n)-approximable.

Proof. Assume that there exists some $v \in \Lambda$ such that $-\log_u \|f_t v\| \geq r(t)$. This is the same as

$$-\log_u |T^{nt}| \|v^{(m)}\| \geq r(t) \quad \text{and} \quad -\log_u |T^{-mt}| \|v_{(n)}\| \geq r(t).$$

Rearranging we get

$$\begin{cases} \|v^{(m)}\|^{2m} \leq u^{-2mr(t)} s^{-2mnt} = u^{-2mr(t)-4mnt} = u^{-L(t)}, \\ \|v_{(n)}\|^{2n} \leq s^{2mnt} u^{-2nr(t)} = u^{4mnt-2nr(t)} = u^{\lambda(t)}. \end{cases}$$

Now if $v^{(m)} = 0$ for some $v \in \Lambda$ then for all integer multiples w of v we have w satisfying $0 = \|w^{(m)}\|^{2m} \leq \psi(\|w_{(n)}\|^{2n})$. So we ignore this case. From the above we have

$$\|v^{(m)}\|^{2m} \leq u^{-L(t)} = \psi(u^{\lambda(t)}).$$

By decreasing monotonicity of ψ we have $\psi(u^{\lambda(t)}) \leq \psi(\|v_n\|^{2n})$ and thus

$$\|v^{(m)}\|^{2m} \leq \psi(\|v_n\|^{2n}).$$

As $v^{(m)} \neq 0$ and the above inequalities hold for arbitrarily large t, the fact that $\|v^{(m)}\|^m \leq s^{-L(t)}$ (here $L(t) \to \infty$ if $\psi \to 0$) gets arbitrarily small implies that $\|v_{(n)}\|$ gets arbitrarily large, and thus we are done. ■

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