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On polynomial Gauss sums $(\operatorname{mod} P^n)$, $n \geq 2$

by

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Dedicated to Professor Jing Yu on his 60th birthday

1. Introduction. Let N be a positive integer and let χ be a primitive multiplicative character $(\operatorname{mod} N)$. It is known that the absolute value of the classical Gauss sum

$$\tau(\chi) = \sum_{n=1}^{N-1} \chi(n) \exp\left(\frac{2\pi i n}{N}\right)$$

is $N^{1/2}$. However, it is difficult to determine the argument of this sum. In 1962, C. Chowla [1] and L. J. Mordell [7] independently proved that when N is a prime number, the argument is a root of unity if and only if χ is real. When $N = p^r$ is an odd prime power with $r \geq 2$, R. Odoni [8] gave explicit formulas for the argument of $\tau(\chi)$ by using p-adic analysis. An important role in finding the argument of $\tau(\chi)$ is played by the fact that the group $(\mathbb{Z}/p^r\mathbb{Z})^\times$ is cyclic when p is an odd prime. Finally, T. Funakura [3] computed the classical Gauss sums for all integer n and, further, gave a criterion for the argument of a classical Gauss sum to be a root of unity. Moreover, in 1983, J.-L. Mauclaire [5] provided another elementary proof giving the argument of $\tau(\chi)$ when p is an odd prime. Furthermore, he completed the remaining case of the prime number 2 in [6].

In this paper, we generalize the classical Gauss sums to polynomial Gauss sums in the polynomial ring over the finite field $\mathbb{F}_q$ of q elements. For q odd, we explicitly give the argument of a polynomial Gauss sum. We are then able to generalize the classical Chowla–Mordell theorem to polynomial Gauss sums, providing a necessary and sufficient condition for the argument of a polynomial Gauss sum to be a root of unity.

Throughout this paper, p is an odd prime and $q = p^r$ is a power of p. Let $\mathbb{F}_q$ be the finite field of q elements of characteristic p and let $\operatorname{Tr}_q : \mathbb{F}_q \to \mathbb{F}_p$

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be the trace map onto $\mathbb{F}_p$ (identified with $\mathbb{Z}/p\mathbb{Z}$). Let $e_q : \mathbb{F}_q \to \mathbb{C}^\times$ be the standard additive character of $\mathbb{F}_q$ defined by

$$e_q(\alpha) = \exp\left(\frac{2\pi i \operatorname{Tr}_q(\alpha)}{p}\right) \quad \text{for all } \alpha \text{ in } \mathbb{F}_q.$$

Let $\mathbf{A} = \mathbb{F}_q[T]$ be the polynomial ring in T over $\mathbb{F}_q$ and let $\mathrm{K}_\infty = \mathbb{F}_q(\!(1/T)\!)$ denote the completion field of the rational function field $\mathbb{F}_q(T)$ at the infinite place $1/T$; in other words, every $a \in \mathrm{K}_\infty \setminus \{0\}$ can be expressed as

$$a = \sum_{i=-\infty}^{d} c_i T^i,$$

where $c_i \in \mathbb{F}_q$ and $c_d \neq 0$. The degree and absolute value of a are defined by $\deg a = d$ and $|a| = q^d$. The residue of a at the infinite place is denoted by $\operatorname{res}_\infty a = c_{-1}$. The polynomial exponential map $E : \mathrm{K}_\infty \to \mathbb{C}^\times$ is defined by

$$E(a) = e_q(\operatorname{res}_\infty a) \quad \text{for all } a \text{ in } \mathrm{K}_\infty.$$

Let $Q \in \mathbf{A}$. For any multiplicative character χ of $\mathbf{A}/Q\mathbf{A}$, the *polynomial Gauss sum* of χ is defined by

$$\tau(\chi) = \sum_{[f] \in (\mathbf{A}/Q\mathbf{A})^\times} \chi([f]) E\left(\frac{f}{Q}\right).$$

It is well-known that for any primitive multiplicative character χ of $\mathbf{A}/Q\mathbf{A}$, we have

$$|\tau(\chi)| = |Q|^{1/2},$$

and there is no explicit method to evaluate $\epsilon(\tau)$, the argument of $\tau(\chi)$. In this paper, however, we determine $\epsilon(\tau)$ in the case when $Q = P^n$ $(n \geq 2)$ for any monic irreducible polynomial P in $\mathbf{A}$. It deserves to be mentioned that while the multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ with p an odd prime is always cyclic, the multiplicative group $(\mathbf{A}/P^n\mathbf{A})^\times$ with $n \geq 2$ is not. This makes finding the explicit value of $\tau(\chi)$ more difficult. Now, we give a brief account of the main result of this paper:

Main result. If P is a monic irreducible polynomial in $\mathbf{A}$ and χ is a multiplicative character of $\mathbf{A}/P^n\mathbf{A}$ $(n \geq 2)$, then there exists a specific polynomial a (depending on χ) with $P \nmid a$ and $\deg a < (n/2)\deg P$ such that

$$\tau(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is not primitive,} \\ |P|^{n/2}\chi(-a) & \text{if } \chi \text{ is primitive and } n \text{ is even,} \\ |P|^{n/2}\chi(-a)\epsilon_{4p} & \text{if } \chi \text{ is primitive and } n \text{ is odd,} \end{cases}$$

where ϵ_{4p} is a $4p$th root of unity.

From the main result, when $n \geq 2$, $\epsilon(\tau)$ is a root of unity if and only if χ is primitive. For the remaining case $n = 1$, a criterion for $\epsilon(\tau)$ being a root of unity can be given by the results of Evans [2] and Yang–Zheng [9], since $\mathbf{A}/P\mathbf{A} \cong \mathbb{F}_{q^d}$, where $d = \deg P$: when $n = 1$, the quantity $\epsilon(\tau)$ is a root of unity if and only if

$$\frac{dr(p-1)}{2} \leq \min_u \left\{ S\left(u\, \frac{q^d - 1}{f} \right) \right\}$$

where f is the order of χ, u runs from 1 to f and is coprime to f, and for every positive integer $a < q^d$, $S(a)$ is the sum of the digits appearing in the p-adic representation of a; in other words,

$$S(a) = \sum_{j=0}^{dr-1} a_j \quad \text{for } a = \sum_{j=0}^{dr-1} a_j p^j \quad \text{with } 0 \leq a_j < p.$$

2. Auxiliary lemmas. Throughout this paper, let $n \geq 2$ be a positive integer and $m = \lfloor n/2 \rfloor$, the greatest integer less than or equal to $n/2$. Let P be a monic irreducible polynomial in $\mathbf{A}$, and let $(\mathbf{A}/P^n\mathbf{A})^\times$ denote the unit group of the residue class ring $\mathbf{A}/P^n\mathbf{A}$.

We introduce two types of special subgroups of $(\mathbf{A}/P^n\mathbf{A})^\times$:

$$K_m := \{[1 + fP^{n-m}] \mid \deg f < m \deg P\},$$
$$H_m := \{[1 + fP^m + gP^{2m}] \mid \deg f, \deg g < \deg P\} \quad \text{(only for odd } n\text{)}.$$

Note that K_m is isomorphic to the additive group $\mathbf{A}/P^m\mathbf{A}$. The multiplicative identity

$$[1 + f_1P^m + g_1P^{2m}][1 + f_2P^m + g_2P^{2m}] = [1 + (f_1 + f_2)P^m + (g_1 + f_1f_2 + g_2)P^{2m}]$$

proves that H_m is indeed a subgroup of $(\mathbf{A}/P^n\mathbf{A})^\times$.

In addition, we study the character groups $\widehat{\mathbf{A}/P^m\mathbf{A}}$, $\widehat{K_m}$, and $\widehat{H_m}$ of $\mathbf{A}/P^m\mathbf{A}$, K_m, and H_m, respectively. For any a in $\mathbf{A}$, let $\psi_a : \mathbf{A}/P^m\mathbf{A} \to \mathbb{C}^\times$ be defined by

$$\psi_a([f]) = E\left(\frac{af}{P^m} \right). \tag{2.1}$$

This is an additive character of $\mathbf{A}/P^m\mathbf{A}$, and

$$\widehat{\mathbf{A}/P^m\mathbf{A}} = \{\psi_a \mid a \in \mathbf{A},\, \deg a < m \deg P\}.$$

Further,

$$\psi_{a_1}\psi_{a_2} = \psi_{a_1+a_2} \tag{2.2}$$

for all a_1 and a_2 in $\mathbf{A}$ with $\deg a_1, \deg a_2 < m \deg P$, and

$$\sum_{\deg f < m \deg P} \psi_a([f]) = \begin{cases} |P^m| & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

For the group K_m, let $\Psi_a : K_m \to \mathbb{C}^\times$ be the multiplicative character defined by

$$\Psi_a([1 + fP^{n-m}]) = \psi_a([f]) \tag{2.4}$$

for all f in $\mathbf{A}$ with $\deg f < m \deg P$. Since K_m is isomorphic to the additive group $\mathbf{A}/P^m\mathbf{A}$, the character group $\widehat{K_m}$ is

$$\widehat{K_m} = \{\Psi_a \mid a \in \mathbf{A},\ \deg a < m \deg P\}. \tag{2.5}$$

When n is an odd integer, $n = 2m + 1$, since q is odd, for any b and c in $\mathbf{A}$, we can define the function $\Psi_{b,c} : H_m \to \mathbb{C}^\times$ by

$$\Psi_{b,c}([1 + fP^m + gP^{2m}]) = E\left(\frac{bf + cg - \frac{1}{2}cf^2}{P}\right). \tag{2.6}$$

Then we have the following lemma.

Lemma 2.1. *If $n \geq 2$ is an odd integer, i.e., $n = 2m + 1$, then the group $\widehat{H_m}$ of the multiplicative characters of the subgroup H_m is*

$$\widehat{H_m} = \{\Psi_{b,c} \mid b, c \in \mathbf{A},\ \deg b, \deg c < \deg P\}.$$

Proof. It is not difficult to check that $\Psi_{b,c}$ is a multiplicative character of H_m. Further, we prove that if $b_1 \not\equiv b_2$ or $c_1 \not\equiv c_2 \pmod{P}$, then $\Psi_{b_1,c_1} \neq \Psi_{b_2,c_2}$. If $\Psi_{b_1,c_1} = \Psi_{b_2,c_2}$ for some b_1, b_2 and c_1, c_2 in $\mathbf{A}$, then

$$\Psi_{b_1,c_1}([1 + fP^m + gP^{2m}]) = \Psi_{b_2,c_2}([1 + fP^m + gP^{2m}])$$

for all polynomials f and g with $\deg f, \deg g < \deg P$. Taking $f = 0$, we have $\Psi_{b_1,c_1}([1 + gP^{2m}]) = \Psi_{b_2,c_2}([1 + gP^{2m}])$ for all g with $\deg g < \deg P$, that is,

$$E\left(\frac{c_1 g}{P}\right) = E\left(\frac{c_2 g}{P}\right).$$

This implies that

$$E\left(\frac{(c_1 - c_2)g}{P}\right) = 1$$

for all g with $\deg g < \deg P$. Hence, $c_1 \equiv c_2 \pmod{P}$. Moreover, taking $g = 0$, we get $\Psi_{b_1,c_1}([1 + fP^m]) = \Psi_{b_2,c_2}([1 + fP^m])$ for all f with $\deg f < \deg P$, that is,

$$E\left(\frac{b_1 f - \frac{1}{2}c_1 f^2}{P}\right) = E\left(\frac{b_2 f - \frac{1}{2}c_2 f^2}{P}\right).$$

It follows that

$$E\left(\frac{(b_1 - b_2)f - \frac{1}{2}(c_1 - c_2)f^2}{P}\right) = 1$$

for all f with $\deg f < \deg P$. Since we know that $c_1 \equiv c_2 \pmod P$, the above is equivalent to

$$E\left(\frac{(b_1 - b_2)f}{P}\right) = 1$$

for all f with $\deg f < \deg P$. Hence, $b_1 \equiv b_2 \pmod P$. Thus, we proved that $\Psi_{b_1,c_1} \neq \Psi_{b_2,c_2}$ if $b_1 \not\equiv b_2 \pmod P$ or $c_1 \not\equiv c_2 \pmod P$.

Finally, since the cardinality of $\widehat{H_m}$ is

$$|\widehat{H_m}| = |H_m| = |P|^2$$

and the number of characters $\Psi_{b,c}$ with $\deg b, \deg c < \deg P$ is also equal to $|P|^2$, the desired conclusion follows. ∎

3. The arguments of polynomial Gauss sums. In this section, we prove our main result. Let the integer m, the subgroups K_m, H_m, and the characters $\psi_a, \Psi_a, \Psi_{b,c}$ be defined as in Section 2. Let $\widehat{(\mathbf{A}/P^n\mathbf{A})^\times}$ be the group of multiplicative characters χ of $(\mathbf{A}/P^n\mathbf{A})^\times$. For convenience, we use $\chi(f)$ to represent the complex value $\chi([f])$. Recall that a multiplicative character χ of $\mathbf{A}/P^n\mathbf{A}$ is called *primitive* if χ does not factor through $(\mathbf{A}/P^k\mathbf{A})^\times$ for any integer k with $0 \leq k < n$.

Consider the restriction $\chi|_{K_m}$ of the multiplicative character χ to the subgroup K_m. Since $\chi|_{K_m}$ is a multiplicative character of K_m, by (2.5) there exists a unique polynomial a in $\mathbf{A}$ with $\deg a < m \deg P$ such that $\chi|_{K_m} = \Psi_a$, that is,

$$(3.1)\quad \chi(1 + fP^{n-m}) = \chi|_{K_m}(1 + fP^{n-m}) = \Psi_a([1 + fP^{n-m}]) = \psi_a([f])$$

for all f in $\mathbf{A}$ with $\deg f < m \deg P$. Moreover, if P divides a then χ factors through $(\mathbf{A}/P^{n-1}\mathbf{A})^\times$. Conversely, if χ is not primitive, then χ factors through $(\mathbf{A}/P^{n-1}\mathbf{A})^\times$. Hence,

$$(3.2)\qquad \chi \text{ is primitive if and only if } P \nmid a.$$

When n is odd, i.e., $n = 2m + 1$, consider the restriction $\chi|_{H_m}$. Since $\chi|_{H_m}$ is a multiplicative character of H_m, by Lemma 2.1 there exist unique polynomials b and c in $\mathbf{A}$ with $\deg b, \deg c < \deg P$ such that $\chi|_{H_m} = \Psi_{b,c}$, that is,

$$\chi(1 + fP^m + gP^{2m}) = \chi|_{H_m}(1 + fP^m + gP^{2m}) = \Psi_{b,c}([1 + fP^m + gP^{2m}])$$

for all f and g in $\mathbf{A}$ with $\deg f, \deg g < \deg P$. Moreover, if $c = 0$ then χ factors through $(\mathbf{A}/P^{n-1}\mathbf{A})^\times$. Hence, if χ is primitive then $c \neq 0$.

To abbreviate our proof of the main theorem, we prove Lemma 3.1 below first. In the proof of this lemma, we use a result of Hsu [4] saying that when

P is a monic polynomial in $\mathbf{A}$, then

$$\sum_{\deg f<\deg P} E\left(\frac{f^2}{P}\right) = |P|^{1/2} i^{(|P|-1)^2/4}. \tag{3.3}$$

LEMMA 3.1. *Let $n \geq 2$ be an odd integer, i.e., $n = 2m+1$, and let χ be a primitive multiplicative character of $(\mathbf{A}/P^n\mathbf{A})^\times$. If $\chi|_{H_m} = \Psi_{b,c}$ for some b, c in $\mathbf{A}$ with $\deg b, \deg c < \deg P$, then $c \neq 0$ and*

$$\sum_{\deg f<\deg P} \chi(1+fP^m) = |P|^{1/2} E\left(\frac{\frac{1}{2}b^2c'}{P}\right)\left(\frac{-2c}{P}\right) i^{(|P|-1)^2/4},$$

where c' denotes the polynomial in $\mathbf{A}$ such that $\deg c' < \deg P$, $c'c \equiv 1 \pmod P$, and $\left(\frac{-2c}{P}\right)$ is the polynomial quadratic residue symbol.

Proof. Since χ is primitive, we know that $c \neq 0$. Since $\chi|_{H_m} = \Psi_{b,c}$ and q is odd, from (2.6) we have

$$\begin{aligned}
\sum_{\deg f<\deg P} \chi(1+fP^m) &= \sum_{\deg f<\deg P} \Psi_{b,c}([1+fP^m]) \\
&= \sum_{\deg f<\deg P} E\left(\frac{bf - \frac{1}{2}cf^2}{P}\right) \\
&= E\left(\frac{\frac{1}{2}b^2c'}{P}\right) \sum_{\deg f<\deg P} E\left(\frac{-\frac{1}{2}c(f-bc')^2}{P}\right) \\
&= E\left(\frac{\frac{1}{2}b^2c'}{P}\right) \sum_{\deg f<\deg P} E\left(\frac{-\frac{1}{2}cf^2}{P}\right).
\end{aligned}$$

Furthermore, since $-\frac{1}{2}c \neq 0$, we have

$$\begin{aligned}
\sum_{\deg f<\deg P} \chi(1+fP^m) &= E\left(\frac{\frac{1}{2}b^2c'}{P}\right) \sum_{\deg f<\deg P} \left(\frac{-\frac{1}{2}c}{P}\right) E\left(\frac{f^2}{P}\right) \\
&= E\left(\frac{\frac{1}{2}b^2c'}{P}\right)\left(\frac{-\frac{1}{2}c}{P}\right) \sum_{\deg f<\deg P} E\left(\frac{f^2}{P}\right).
\end{aligned}$$

It follows directly from (3.3) that

$$\begin{aligned}
\sum_{\deg f<\deg P} \chi(1+fP^m) &= E\left(\frac{\frac{1}{2}b^2c'}{P}\right)\left(\frac{-\frac{1}{2}c}{P}\right) \cdot |P|^{1/2} i^{(|P|-1)^2/4} \\
&= |P|^{1/2} E\left(\frac{\frac{1}{2}b^2c'}{P}\right)\left(\frac{-2c}{P}\right) i^{(|P|-1)^2/4}. \quad \blacksquare
\end{aligned}$$

The formula for the argument of $\tau(\chi)$ is given in

THEOREM 3.2. *Let $n \geq 2$ be an integer, let χ be a multiplicative character of $\mathbf{A}/P^n\mathbf{A}$, and let $m = \lfloor n/2 \rfloor$. Let a be the polynomial in $\mathbf{A}$ with $\deg a < m \deg P$ such that $\chi|_{K_m} = \Psi_a$. Then*

(1) *if χ is not primitive, then $\tau(\chi) = 0$;*
(2) *if χ is primitive and n is even, then $P \nmid a$ and*

$$\tau(\chi) = |P|^{n/2}\chi(-a);$$

(3) *if χ is primitive, n is odd, and $\chi|_{H_m} = \Psi_{b,c}$ with b, c in $\mathbf{A}$ and $\deg b, \deg c < \deg P$, then $P \nmid a$, $c \neq 0$, and*

$$\tau(\chi) = |P|^{n/2}\chi(-a)E\left(\frac{\frac{1}{2}b^2c'}{P}\right)\left(\frac{-2c}{P}\right)i^{(|P|-1)^2/4}.$$

Proof. Every element $[f]$ in $(\mathbf{A}/P^n\mathbf{A})^\times$ can be uniquely represented as

$$[g][1 + hP^m],$$

where $g \in \mathbf{A}$ with $\deg g < m \deg P$, $P \nmid g$, and $h \in \mathbf{A}$ with $\deg h < (n-m)\deg P$. From the definition of the polynomial Gauss sum, we have

$$\begin{aligned}\tau(\chi) &= \sum_{[f]\in(\mathbf{A}/P^n\mathbf{A})^\times} \chi(f)E\left(\frac{f}{P^n}\right) \\ &= \sum_{\deg g<m\deg P,\, P\nmid g}\ \sum_{\deg h<(n-m)\deg P} \chi(g(1+hP^m))E\left(\frac{g(1+hP^m)}{P^n}\right).\end{aligned}$$

Since χ is a multiplicative character of $\mathbf{A}/P^n\mathbf{A}$, we have

$$\chi(g(1+hP^m)) = \chi(g)\chi(1+hP^m). \tag{3.4}$$

Applying the definition of res_∞ and $\deg g < m \deg P$, we get $E\left(\frac{g}{P^n}\right) = 1$ and

$$\begin{aligned}E\left(\frac{g(1+hP^m)}{P^n}\right) &= E\left(\frac{g+ghP^m}{P^n}\right) \\ &= E\left(\frac{g}{P^n}\right)E\left(\frac{ghP^m}{P^n}\right) = E\left(\frac{gh}{P^{n-m}}\right).\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\tau(\chi) = \sum_{\deg g<m\deg P,\, P\nmid g} \chi(g) \sum_{\deg h<(n-m)\deg P} \chi(1+hP^m)E\left(\frac{gh}{P^{n-m}}\right). \tag{3.6}$$

When n is even, $n = 2m$, we have

$$\tau(\chi) = \sum_{\deg g<m\deg P,\, P\nmid g} \chi(g) \sum_{\deg h<m\deg P} \chi(1+hP^{n-m})E\left(\frac{gh}{P^m}\right).$$

Applying (3.1), (2.1), and (2.2) to the above expression, we get

$$\begin{aligned}\tau(\chi) &= \sum_{\deg g < m \deg P,\, P \nmid g} \chi(g) \sum_{\deg h < m \deg P} \psi_a([h])\psi_g([h]) \\ &= \sum_{\deg g < m \deg P,\, P \nmid g} \chi(g) \sum_{\deg h < m \deg P} \psi_{a+g}([h]).\end{aligned}$$

According to (2.3), we have

$$\sum_{\deg h < m \deg P} \psi_{a+g}([h]) = \begin{cases} |P^m| & \text{if } g = -a, \\ 0 & \text{otherwise.} \end{cases}$$

From (3.2),

$$\tau(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is not primitive,} \\ |P|^m \chi(-a) & \text{if } \chi \text{ is primitive.} \end{cases}$$

When n is odd, $n = 2m+1$, we write h in (3.6) as $h_0 + h_1 P$, where $h_0, h_1 \in \mathbf{A}$ with $\deg h_0 < \deg P$ and $\deg h_1 < m \deg P$. Hence, (3.6) becomes

$$\begin{aligned}\tau(\chi) &= \sum_{\substack{\deg g < m \deg P \\ P \nmid g}} \chi(g) \sum_{\substack{\deg h_0 < \deg P \\ \deg h_1 < m \deg P}} \chi(1 + h_0 P^m + h_1 P^{m+1}) E\left(\frac{g(h_0 + h_1 P)}{P^{n-m}}\right) \\ &= \sum_{\deg g < m \deg P,\, P \nmid g} \chi(g) \sum_{\deg h_1 < m \deg P} \chi(1 + h_1 P^{m+1}) E\left(\frac{g h_1}{P^{n-m-1}}\right) \\ &\qquad\qquad \times \sum_{\deg h_0 < \deg P} \chi(1 + h_0 P^m) E\left(\frac{g h_0}{P^{n-m}}\right).\end{aligned}$$

Since

$$\deg g h_0 = \deg g + \deg h_0 \le (n-m)\deg P - 2,$$

we have $E\left(\frac{g h_0}{P^{n-m}}\right) = 1$ and so

$$\begin{aligned}\tau(\chi) = &\sum_{\deg g < m \deg P,\, P \nmid g} \chi(g) \sum_{\deg h_1 < m \deg P} \chi(1 + h_1 P^{n-m}) E\left(\frac{g h_1}{P^m}\right) \\ &\qquad\qquad \times \sum_{\deg h_0 < \deg P} \chi(1 + h_0 P^m).\end{aligned}$$

Similar to the discussion in the case of even n, we obtain

$$\tau(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is not primitive,} \\ |P|^m \chi(-a) \displaystyle\sum_{\deg h_0 < \deg P} \chi(1 + h_0 P^m) & \text{if } \chi \text{ is primitive,} \end{cases}$$

and Lemma 3.1 yields

$$\tau(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is not primitive,} \\ |P|^{n/2}\chi(-a)E\left(\dfrac{\frac{1}{2}b^2c}{P}\right)\left(\dfrac{-2c}{P}\right)i^{(|P|-1)^2/4} & \text{if } \chi \text{ is primitive.} \end{cases} \blacksquare$$

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