Infinite 2-class field towers of some imaginary quadratic number fields

by

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1. Introduction. Let $K$ be an imaginary quadratic number field with discriminant $d$, and $C_K$ denote the ideal class group of $K$. We mean by the 2-class field tower of $K$ the sequence of fields $K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_i \subseteq \ldots$, where $K_{i+1}$ is the Hilbert 2-class field (i.e. the maximal unramified abelian 2-extension) of $K_i$. If $K_{i+1} \neq K_i$ for all $i$, then we say that the 2-class field tower of $K$ is infinite.

By the results of Golod–Shafarevich [3] and Vinberg–Gaschütz [12, 15], the 2-class field tower of $K$ is infinite if 2-rank $C_K \geq 5$. On the other hand, Koch [6] and Hajir [4, 5] proved that the 2-class field tower of $K$ is infinite if 4-rank $C_K \geq 3$. When 2-rank $C_K = 3$ and 4-rank $C_K = 0$, there are some examples of infinite families of $K$ with infinite (resp. finite) 2-class field towers [4, 7]. However, when 2-rank $C_K = 4$, no example of $K$ with finite 2-class field tower has ever been known. It has been conjectured [9] that the 2-class field tower of such a $K$ is always infinite. In this direction, Benjamin [1, 2] proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and 4-rank $C_K = 2$, except some type of Rédei matrix of $K$.

In this paper, we study the case where 2-rank $C_K = 4$ and exactly one negative prime discriminant divides $d$, and prove that the 2-class field tower of such a $K$ is infinite, except for one type of Rédei matrix of $K$. To prove our theorem, we use Martinet’s inequalities [9] and their corollaries. We also use some properties of Rédei matrices [10, 11, 13, 14]. A similar problem for real quadratic number fields is treated by Maire [8], by a different method.

2. Martinet’s inequalities and their corollaries. Let $K$ be an imaginary quadratic number field.

Martinet’s inequality (general case, [9]). Let $E/F$ be a quadratic extension of number fields. Denote by $r_1$ (resp. $r_2$) the number of real (resp.
imaginary) places of \( F \). Also denote by \( t \) (resp. \( \varrho \)) the number of finite (resp. infinite) places of \( F \) which ramify in \( E \). If

\[
t \geq r_1 + r_2 - \varrho + 3 + 2\sqrt{2(r_1 + r_2) - \varrho + 1},
\]

then the 2-class field tower of \( E \) is infinite.

**Martinet’s inequality I.** Let \( F \) be a totally real number field of degree \( n \), and \( E \) a totally imaginary quadratic extension of \( F \). Let \( t \) be the number of prime ideals of \( F \) which ramify in \( E \). If

\[
t \geq 3 + 2\sqrt{n + 1},
\]

then the 2-class field tower of \( E \) is infinite.

**Proof.** Since \( r_1 = \varrho = n \) and \( r_2 = 0 \), the assertion follows from the general case of Martinet’s inequality.

**Corollary 1.** Let \( F \) be a real quadratic number field. Suppose that four rational primes split in \( F \) and ramify in \( K \), or that a rational prime remains prime in \( F \) and three other rational primes split in \( F \) and these four rational primes ramify in \( K \). Then the 2-class field tower of \( E = FK \) is infinite.

**Proof.** Since \( n = 2 \) and \( t \geq 7 \geq 3 + 2\sqrt{2 + 1} = 6.464 \ldots \) in these cases, the 2-class field tower of \( E = FK \) is infinite by Martinet’s inequality I.

**Corollary 2.** Let \( F \) be a totally real number field of degree 4. Suppose that two rational primes split completely in \( F \) and ramify in \( K \), or that a rational prime splits completely in \( F \) and two other rational primes are unramified and split into at least two primes in \( F \) and these three rational primes ramify in \( K \). Then the 2-class field tower of \( E = FK \) is infinite.

**Proof.** Since \( n = 4 \) and \( t \geq 8 \geq 3 + 2\sqrt{4 + 1} = 7.472 \ldots \) in these cases, the 2-class field tower of \( E = FK \) is infinite by Martinet’s inequality I.

**Martinet’s inequality II.** Let \( F \) be a totally imaginary number field of degree \( n \), and \( E \) a quadratic extension of \( F \). Let \( t \) be the number of prime ideals of \( F \) which ramify in \( E \). If

\[
t \geq n/2 + 3 + 2\sqrt{n + 1},
\]

then the 2-class field tower of \( E \) is infinite.

**Proof.** Since \( r_1 = \varrho = 0 \) and \( r_2 = n/2 \), the assertion follows from the general case of Martinet’s inequality.

**Corollary 3.** Let \( F \) be an imaginary quadratic number field. Suppose that four rational primes split in \( F \) and ramify in \( K \). Then the 2-class field tower of \( E = FK \) is infinite.

**Proof.** Since \( n = 2 \) and \( t \geq 8 \geq 2/2 + 3 + 2\sqrt{2 + 1} = 7.464 \ldots \), the 2-class field tower of \( E = FK \) is infinite by Martinet’s inequality II.
3. The case with one negative prime discriminant. Let $K$ be an imaginary quadratic number field with discriminant $d$. First we recall some properties of Rédei matrices of quadratic number fields [10, 11, 13, 14].

A rational integer is called a discriminant if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a prime discriminant. Prime discriminants are denoted by $p^* = (-1)^{(p-1)/2}p$ (if $p$ is an odd prime), or $p^* = -4, 8$ or $-8$ (if $p$ is equal to 2). Let $d = p_1^*p_2^*\ldots p_t^*$ be the unique factorization of $d$ into a product of prime discriminants. By genus theory, we have 2-rank $C_K = t - 1$.

Using Kronecker symbols $(\frac{D}{p})$, where $D$ is a discriminant and $p$ is a prime number satisfying $p \nmid D$, we define the Rédei matrix $R_K = (a_{ij}) \in \mathbb{M}_{t \times t}(\mathbb{Z}/2\mathbb{Z})$ of $K$ by

$$
(-1)^{a_{ij}} = \begin{cases}
\begin{pmatrix}
p_i^* \\
p_j^*
\end{pmatrix} & (i \neq j), \\
\begin{pmatrix}d/p_i^* \\
p_i^*
\end{pmatrix} & (i = j).
\end{cases}
$$

By the definition of the Kronecker symbol, we have $a_{ij} = 0$ ($i \neq j$) if and only if the rational prime $p_j$ splits in $\mathbb{Q}(\sqrt{p_i^*})$. Note that the sum of all row vectors of $R_K$ is equal to the zero vector $\mathbf{0}$ in $(\mathbb{Z}/2\mathbb{Z})^t$ so that rank $R_K \leq t - 1$ and the solution space $X$ of the linear equations $xR_K = \mathbf{0}$ ($x \in (\mathbb{Z}/2\mathbb{Z})^t$) contains the vector $\mathbf{1} = (1, 1, \ldots, 1)$ ($t$ times). By the results of Rédei and Rédei–Reichardt, we have 4-rank $C_K = t - 1 - \text{rank } R_K$.

In the case where $p_i^* \neq -4$ and $p_j^* \neq -4$, we have $a_{ij} = a_{ji}$ ($i \neq j$) if and only if $p_i^* > 0$ or $p_j^* > 0$, by the quadratic reciprocity law. Therefore, if exactly one negative prime discriminant ($\neq -4$) divides $d$, then $R_K$ is a symmetric matrix.

**Theorem.** Let $K$ be an imaginary quadratic number field with discriminant $d$. Suppose that 2-rank $C_K = 4$ and exactly one negative prime discriminant divides $d$. Let $d = p_1^*p_2^*p_3^*p_4^*p_5^*$ ($p_1^* < 0$) be the unique factorization of $d$ into a product of prime discriminants. Then the 2-class field tower of $K$ is infinite, except the case where

$$
R_K = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix},
$$

by changing the order of $p_i$’s ($2 \leq i \leq 5$). In the exceptional case, $p_1^* \neq -4$ and the 4-rank of $C_K$ is equal to 0.
Proof. First, suppose that \( p_1^* = -4 \). Then we have \( p_j \equiv 1 \mod 4 \) for any \( j \) \((2 \leq j \leq 5)\). Put \( F = \mathbb{Q}(\sqrt{p_1^*}) = \mathbb{Q}(\sqrt{-1}) \). Then the four rational primes \( p_j \) \((2 \leq j \leq 5)\) split in \( F \) and ramify in \( K \). Hence, the 2-class field tower of \( E = FK = K(\sqrt{-1}) \) is infinite by Corollary 3. Since \( E/K \) is an unramified 2-extension, the 2-class field tower of \( K \) is also infinite.

In the following, we assume that \( p_1^* \neq -4 \). Therefore, \( R_K \) is a symmetric matrix. For each Rédei matrix \( R_K \), if we could find a subfield \( F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j}) \), \( \mathbb{Q}(\sqrt{p_i}, \sqrt{p_jp_k}) \) or \( \mathbb{Q}(\sqrt{p_ip_j}, \sqrt{p_ip_k}) \) \((i, j, k \in \{2, 3, 4, 5\})\) of the genus field \( \mathbb{Q}(\sqrt{p_1^*}, \ldots, \sqrt{p_5^*}) \) of \( K \) which satisfies the condition of Corollary 2, then the 2-class field tower of \( E = FK \) would be infinite. Since \( E/K \) is an unramified 2-extension, we conclude that the 2-class field tower of \( K \) is also infinite, in those cases.

First, suppose that there exists a column vector \( a_j = (a_{ij}) \) \((1 \leq j \leq 5)\) of \( R_K \) for which at least two of \( a_{ij} \)'s \((2 \leq i \leq 5, i \neq j)\) are 0. Then, assuming that \( a_{ij} = a_{kj} = 0 \) \((i \neq j, k \neq j)\), we put \( F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_k}) \). Since the rational prime \( p_j \) splits completely in \( F \) and ramifies in \( K \), and two rational primes \( p_i, p_m \) \(\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}\) are unramified and split into at least two primes in \( F \) and ramify in \( K \), the 2-class field tower of \( E = FK \) is infinite by Corollary 2. Hence, the 2-class field tower of \( K \) is also infinite.

In the following, we assume that at most one of \( a_{ij} \)'s \((2 \leq i \leq 5, i \neq j)\) is 0 for each column vector \( a_j = (a_{ij}) \) \((1 \leq j \leq 5)\) of \( R_K \).

(i) One of \( a_{i1} \)'s \((2 \leq i \leq 5)\) is 0: In this case, we may assume that \( a_{21} = 0 \) and \( a_{31} = a_{41} = a_{51} = 1 \) without loss of generality. If \( a_{32} = a_{42} = a_{52} = 1 \), then we put \( F = \mathbb{Q}(\sqrt{p_3p_4}, \sqrt{p_3p_5}) \). Since \( \left( \frac{p_ip_k}{p_j} \right) = (-1)(-1) = 1 \) for each \( j \in \{1, 2\} \) and \( i, k \in \{3, 4, 5\} \), the two rational primes \( p_1 \) and \( p_2 \) split completely in \( F \) and ramify in \( K \). Therefore, the 2-class field tower of \( E = FK \) is infinite by Corollary 2, and the 2-class field tower of \( K \) is also infinite. On the other hand, if one of \( a_{i2} \)'s \((3 \leq i \leq 5)\) is 0, then we may assume that \( a_{32} = 0 \) and \( a_{42} = a_{52} = 1 \) without loss of generality. So, we have \( a_{23} = 0 \) and \( a_{43} = a_{53} = 1 \) by our assumption and

\[
R_K = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & *
\end{pmatrix},
\]

where the asterisks “*” mean 0 or 1. We put \( F = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_4p_5}) \). Since \( \left( \frac{p_2}{p_j} \right) = 1 \) and \( \left( \frac{p_4p_5}{p_j} \right) = (-1)(-1) = 1 \) for \( j \in \{1, 3\} \), the two rational primes \( p_1 \) and \( p_3 \) split completely in \( F \) and ramify in \( K \). Therefore, the 2-class field
tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of $K$ is also infinite.

(ii) $a_{21} = a_{31} = a_{41} = a_{51} = 1$: If there exists a column vector $\mathbf{a}_j = (a_{ij})$ (2 $\leq j$ $\leq 5$) of $R_K$ satisfying $a_{ij} = 1$ for all $i$ (2 $\leq i$ $\leq 5$, $i \neq j$), then we put

$$F = \mathbb{Q}(\sqrt{p_{k}p_{l}}, \sqrt{p_{k}p_{m}}) \quad \{(j, k, l, m) = (2, 3, 4, 5)\}.$$  

In this case, as in the first half of the case (i), we see that the two rational primes $p_1$ and $p_j$ split completely in $F$ and ramify in $K$. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of $K$ is also infinite. However, if there exists no such column vector $\mathbf{a}_j$ (2 $\leq j$ $\leq 5$) of $R_K$, then we cannot find an appropriate field $F$ which satisfies the condition of Martinet’s inequality. In this case, we have $a_{23} = a_{32} = a_{45} = a_{54} = 0$, by changing the order of $p_i$’s. So, $R_K$ is as described in the assertion of our Theorem. This completes the proof of the Theorem.

**Remark 1.** In Theorem 1 of [1], Benjamin classified the case with only one negative prime discriminant ($\neq -4$) and 2-rank $C_K = 4$ into 32 types, by using “Kronecker symbol configurations”. Among them, the infinitude of the 2-class field tower remained unsettled for 5 types. Actually, there are two more Kronecker symbol configurations

$$\begin{pmatrix} p_1 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \end{pmatrix} = 1$$

with 4-rank $C_K = 0$, and

$$\begin{pmatrix} p_1 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \end{pmatrix} = -1$$

with 4-rank $C_K = 2$. The numbers of Rédei matrices (= the numbers of Kronecker symbol configurations) with given 4-rank are as follows:

<table>
<thead>
<tr>
<th>4-rank $C_K$</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Rédei matrices</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>10</td>
<td>13</td>
<td>34</td>
</tr>
</tbody>
</table>

In our Theorem, we showed the infinitude of the 2-class field tower for 33 types, except the third case of Theorem 1(C) in [1] where $p_1^*$ is negative ($\neq -4$) and

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_5 \end{pmatrix} = 1.$$  

**Examples.** The following are some examples of imaginary quadratic number fields with only one negative prime discriminant ($\neq -4$), 2-rank $C_K$
and the Rédei matrix of exceptional type:

\[ \mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 29 \cdot 8 \cdot 17}), \quad \mathbb{Q}(\sqrt{-7 \cdot 5 \cdot 41 \cdot 13 \cdot 17}), \]
\[ \mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 41 \cdot 17 \cdot 53}), \quad \mathbb{Q}(\sqrt{-8 \cdot 5 \cdot 61 \cdot 37 \cdot 53}). \]

**Remark 2.** In Theorems 1 and 2 of [2], Benjamin proved the infinitude of the 2-class field tower of an imaginary quadratic number field \( K \) with 2-rank \( C_K = 4 \) and 4-rank \( C_K = 2 \), in the case where \( R_K \) is not of the type

\[
\begin{pmatrix}
* & 1 & 1 & 0 & 0 \\
* & 1 & 1 & 1 & 1 \\
* & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & *
\end{pmatrix}
\]

with

\[ p_1^* = -4, \quad p_2^* < 0, \quad p_3^* < 0, \quad p_4^* > 0, \quad p_5^* > 0. \]

With the methods above one can prove the theorems of Benjamin and Koch–Hajir as well.

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**References**

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