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Iwasawa λ-invariants and Mordell–Weil ranks of abelian varieties with complex multiplication

by

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To our teacher, Professor Tsuneo Kanno, on his 77th birthday

Let A be an abelian variety of dimension d defined over a Galois CM number field K of degree $2d$, with full complex multiplication by the ring of integers $\mathcal{O}_K$ of K. Let $\{K; \sigma_1, \ldots, \sigma_d\}$ be a CM-type of A, and p an odd prime number of good reduction for A which splits completely in K with $p\mathcal{O}_K = \prod_{\sigma \in G(K/\mathbb{Q})} \pi^\sigma \mathcal{O}_K$, where π is a prime element of K and $G(K/\mathbb{Q})$ is the Galois group of K over $\mathbb{Q}$. In this paper we prove the following theorem which is a generalization of [1, p. 365]:

THEOREM. *Let A, K, p, π and d be as above, K_0 the Galois extension obtained by adjoining the π-torsion points of A, and K_∞ the $\mathbb{Z}_p$-extension of K_0 obtained by adjoining the π-power torsion points of A. Then K_∞ has λ-invariant greater than or equal to $r - d$, where r is the $\mathcal{O}_K$-rank of the K-rational points $A(K)$.*

Proof. We call the above r the $\mathcal{O}_K$-*Mordell–Weil rank* of A. We denote by $\mathrm{Tor}(A(K))$ the torsion parts of $A(K)$. Then there exists an isomorphism φ of the factor group $A(K)/\mathrm{Tor}(A(K))$ onto the direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are ideals of K.

Since $p\mathcal{O}_K = \prod_{\sigma \in G(K/\mathbb{Q})} \pi^\sigma \mathcal{O}_K$, we may assume that $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are prime to π. Let $\omega_{i1}, \omega_{i2}, \ldots, \omega_{i,2d}$ be a basis of $\mathfrak{a}_i$ over $\mathbb{Z}$. We may assume that ω_{i1} is prime to π. Then there exists an element P_i of $A(K)$ with $\varphi(P_i \bmod \mathrm{Tor}(A(K))) = \omega_{i1}$. We denote by K_n the nth layer of the $\mathbb{Z}_p$-extension K_∞/K_0. Let Q be an element of $A(K_{n-1})$ with $\pi^n Q = \sum_{i=1}^r \xi_i P_i$ for some $\xi_1, \ldots, \xi_r \in \mathcal{O}_K$. Let t_n be a generator of the π^n-torsion points A_{π^n}, σ_0 a generator of $G(K_{n-1}/K)$ and s an integer with $t_n^{\sigma_0} = st_n$ (cf. [4, Propo-

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sition 3.1]). Then $s \bmod p^n$ is a generator of $(\mathbb{Z}/p^n\mathbb{Z})^\times$, which means that $s-1$ is prime to p. Since $\pi^n(Q^{\sigma_0}-Q)=0$, there exists an element t of A_{π^n} with $Q^{\sigma_0}-Q=t^{\sigma_0}-t$. Hence we have $Q-t\in A(K)$, which means that there exists an element α_i of $\mathfrak{a}_i$ with $\varphi((Q-t) \bmod \mathrm{Tor}(A(K)))=\sum_{i=1}^r \alpha_i$, which shows $\xi_i\omega_{i1}=\pi^n\alpha_i$ for $i=1,\dots,r$. This implies $\xi_i\equiv 0 \pmod{\pi^n}$ because ω_{i1} is prime to π. Hence

$$G(K_{n-1}(\pi^{-n}P_1,\dots,\pi^{-n}P_r)/K_{n-1})\cong(\mathbb{Z}/p^n\mathbb{Z})^r$$

by Kummer theory (cf. [3]) and $K_{n-1}(\pi^{-n}P_1,\dots,\pi^{-n}P_r)/K_{n-1}$ is unramified outside $\{\pi^{\sigma_j};\, j=1,\dots,d\}$ by [4, Lemma 5.1].

Let $\mathfrak{p}$ be a prime ideal of K which is one of $\{\pi^{\sigma_j}\mathcal{O}_K;\, j=1,\dots,d\}$. Let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$, and $\mathcal{O}_{K_\mathfrak{p}}$ be the ring of integers of $K_\mathfrak{p}$. Let $\mathfrak{p}_{n-1}$ be the unique prime ideal of K_{n-1} lying above $\mathfrak{p}$, and $D_{\mathfrak{p}_{n-1}}$ the decomposition group of $\mathfrak{p}_{n-1}$ in the extension $K_{n-1}(\pi^{-n}P_1,\dots,\pi^{-n}P_r)$ over K_{n-1}.

Let $\mathcal{F}_{A,\mathfrak{p}}$ be the formal group law over $\mathbb{Z}_p=\mathcal{O}_{K_\mathfrak{p}}$ on the kernel of reduction of $A \bmod \mathfrak{p}$. Let N be the order of $\widetilde{A}(\mathbb{Z}/p\mathbb{Z})$, where $\widetilde{A}$ is the reduction of A at $\mathfrak{p}$. By [4, Proposition 2.9], $\mathcal{F}_{A,\mathfrak{p}}$ is strictly isomorphic to a product $\mathcal{G}=\bigoplus_{j=1}^d\mathcal{G}_j$ of d one-dimensional formal groups $\mathcal{G}_j$ of Lubin–Tate type over $\mathcal{O}_{K_\mathfrak{p}}$. We denote by ϱ the isomorphism of $\mathcal{F}_{A,\mathfrak{p}}$ onto $\mathcal{G}$ and define the endomorphism ι_j of $\mathcal{G}$ by

$$\iota_j:(t_1,\dots,t_j,\dots,t_d)\to(0,\dots,t_j,\dots,0)$$

for $j=1,\dots,d$. We put $f_j=\varrho^{-1}\circ\iota_j\circ\varrho$ for $j=1,\dots,d$. The f_j's are endomorphisms of $\mathcal{F}_{A,\mathfrak{p}}$ over $\mathcal{O}_{K_\mathfrak{p}}$. Then there exists an element P in $\mathcal{F}_{A,\mathfrak{p}}(p\mathcal{O}_{K_\mathfrak{p}})$ such that $f_1(P),\dots,f_d(P)$ are free over $\mathbb{Z}_p$ (e.g. $P=\varrho^{-1}(p,\dots,p)$). We put

$$m=(\mathcal{F}_{A,\mathfrak{p}}(p\mathcal{O}_{K_\mathfrak{p}}):\langle f_1(P),\dots,f_d(P)\rangle_{\mathbb{Z}_p}).$$

Then $mNP_i\in\langle f_1(P),\dots,f_d(P)\rangle_{\mathbb{Z}_p}$ for $i=1,\dots,r$.

Hence $K_{n-1}K_\mathfrak{p}(\pi^{-n}P_1,\dots,\pi^{-n}P_r)\subset K_{n-1}K_\mathfrak{p}(m^{-1}N^{-1}\pi^{-n}P)$. This shows that there exists an integer c, independent of n, such that the order of $D_{\mathfrak{p}_n}$ is less than p^{n+c}. Hence there exists an unramified extension of K_{n-1} whose Galois group is isomorphic to $(\mathbb{Z}/p^{n-c'}\mathbb{Z})^{r-d}$, where c' is a non-negative integer independent of n. This means $\lambda\ge r-d$ by Iwasawa theory (cf. [2, p. 249]). ∎

Example. Let $k=\mathbb{Q}(e^{2\pi i/5})$, C the curve defined by the equation $y^2=x^5+13$, and J the Jacobian variety of C. Then the $\mathcal{O}_k$-rank of $J(k)$ is 3. Since $d=2$, the λ-invariant in the Theorem is positive in this case. The above computation is due to Dr. K. Matsuno. The authors would like to express their hearty thanks for him.

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