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On Siegel–Shidlovskii's theory for q-difference equations

by

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Dedicated to Professor Yasuo Morita for his 60th birthday

1. Introduction. In the theory of E-functions Shidlovskii [Shi] proved an important auxiliary result, known as Shidlovskii's lemma. The lemma, in the abstract setting, concerns a solution $\overline{y} = {}^t(f_1, \ldots, f_m) \in K[[z]]^m$ for the system of linear homogeneous differential equations

$$\frac{d\overline{y}}{dz} = \mathcal{P}\overline{y},$$

where K is an arbitrary field of characteristic zero, $\mathcal{P}$ is an m-dimensional square matrix with entries from $K(z)$, and $f_1, \ldots, f_m$ are linearly independent over $K(z)$. Then, starting from a linear form $L_1 = \sum A_{1j} f_j$, $A_{1j} \in K[z]$, we construct linear forms $L_{i+1} = \sum A_{i+1,j} f_j$, $A_{i+1,j} \in K(z)$, by taking the ith derivative of L_1, where we use the differential system above. For this construction (or its slightly modified version taking the denominator of A_{ij} away) Shidlovskii's lemma asserts the nonvanishing of the m-dimensional determinant $\det(A_{ij})$ whenever the order of the zero of L_1 at $z = 0$ is greater than $(m-1)\max\deg A_{1j} + c_0$, where c_0 is a positive constant depending only on the differential system. The nonvanishing of this determinant is crucial in proving algebraic independence of the values of E-functions. On the other hand, the Chudnovsky brothers [CC] proved a result in some sense dual to Shidlovskii's by replacing a single linear form L_1 by simultaneous linear forms $L_{1j} = A_{10} f_j + A_{1j}$ $(1 \le j \le m)$. The result plays an essential role in the theory of G-functions.

The purpose of the present paper is firstly to prove analogues of both Shidlovskii's and Chudnovskys' results in the context of q-difference equa-

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tions, and secondly, to apply these to obtain a linear independence measure for the values of a fairly wide class of q-series.

For the first aim we take (and fix) a nonzero element q of K which is not a root of unity, and denote by J the q-difference operator on the rational function field $K(z)$, or more generally on the Laurent series field $K((z))$, that is, $J(a) := a(qz)$, $a \in K((z))$. Then we replace the differentiation operator above by the q-difference operator. It is worth noting that we may take a field K with positive characteristic provided that it contains a nonzero element q different from a root of unity, and that for the result analogous to Chudnovskys' we may add an inhomogeneous term to our q-difference system.

The first part, Sections 1–4, of the paper is organized as follows. First we state the main results of this part, Theorems 2.1 and 2.2. Since our proof formally follows the proof for the differential case, it is reasonable to isolate some arguments which are independent of such differential or difference system. That is done in Section 3. Then we give the crucial Lemma 4.1 by introducing an abstract difference system (F, ϕ), extending the difference field $(K(z), J)$, and deduce our main results from it together with the results formulated in the third section. The use of the abstract difference system (F, ϕ) is necessary because we have no a priori knowledge of the solution space, in contrast to the classical differential case. Also, in the proof of Lemma 4.1 we use a simple fact from the fundamentals of algebraic theory of linear difference equations, proved in the appendix. Further, we note that our considerations work also in the Mahler function case (see Remark 4.2).

In the second part, Sections 5–10, we give an application of the first part by exhibiting a linear independence measure for the values of the solutions, analytic at $z = 0$, of the q-difference equation

$$z^s \overline{y}(qz) = a(z)\mathcal{C}\overline{y}(z) + \overline{b}(z),$$

where $s \in \mathbb{Z}^+$, $\mathcal{C} \in \mathrm{GL}(m, \mathbb{K})$, and $a(z)$ and the components of $\overline{b}(z)$ belong to $\mathbb{K}[z]$. This result generalizes and improves the main result of [Vää] and implies a general quantitative linear independence result e.g. for the values of the q-series

$$\varphi_{\mu\nu}(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} n^{\nu}}{a(q)\cdots a(q^n)} (q^{\mu} z)^n,$$

$\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots, l-1$; $l \in \mathbb{Z}^+$. For detailed references on results concerning arithmetic properties of the values of q-series we refer to [AV3]. We begin the second part by introducing the notations and main results: Theorem 5.1, Corollaries 5.1 and 5.2. Then we prove Theorem 6.2 of general nature, which together with some Padé type approximation constructions implies the truth of our main results. In Section 9 we finally prove linear independence of the functions under consideration, which is needed in our results.

2. q-analogues of Shidlovskii's and Chudnovskys' lemmas. In Sections 2–4 we denote by K a field of arbitrary characteristic which has elements different from roots of unity, and fix such an element q. As in the introduction we denote by J the q-difference operator on the rational function field $K(z)$, or more generally on the Laurent series field $K((z))$, that is,

$$J(a) := a(qz), \qquad a \in K((z)).$$

Note that J is an automorphism of $K(z)$ fixing an element $a \in K(z)$ if and only if $a \in K$, and the same holds with $K(z)$ replaced by $K((z))$. In what follows, for $f = \sum_{\mu=M}^{\infty} a_\mu z^\mu \in K((z))$, $a_M \neq 0$, we define $\operatorname{ord} f$ to be M.

We first consider a formal power series solution $\overline{y} = {}^t(f_1, \ldots, f_m) \in K[[z]]^m$ of a linear homogeneous q-difference equation

$$J(\overline{y}) = \mathcal{P}\overline{y}, \tag{2.1}$$

where $\mathcal{P} \in \mathrm{GL}(m, K(z))$. For $\overline{A} = {}^t(A_1, \ldots, A_m) \in K(z)^m$ we define a linear form

$$L = \langle \overline{A}, \overline{y} \rangle := \sum_{j=1}^{m} A_j f_j.$$

Note that

$$J(L) = \langle J(\overline{A}), J(\overline{y}) \rangle = \langle J(\overline{A}), \mathcal{P}\overline{y} \rangle = \langle {}^t\mathcal{P} J(\overline{A}), \overline{y} \rangle.$$

Starting from $L_1 = \langle \overline{A}_1, \overline{y} \rangle$ with $\overline{A}_1 \in K(z)^m$, we define

$$\overline{A}_{i+1} := {}^t\mathcal{P} J(\overline{A}_i) \in K(z)^m, \qquad L_{i+1} := J(L_i) = \langle \overline{A}_{i+1}, \overline{y} \rangle \tag{2.2}$$

inductively for any $i \in \mathbb{Z}^+$. The following result, which generalizes particular cases obtained in [AV1], is an analogue of Lemma 8 in Chapter III of [Shi].

THEOREM 2.1. *Suppose that* (2.1) *has a solution $\overline{y} = {}^t(f_1, \ldots, f_m) \in K[[z]]^m$ such that $f_1, \ldots, f_m$ are linearly independent over $K(z)$. Let $L_1 = \langle \overline{A}_1, \overline{y} \rangle$, where $\overline{A}_1 = {}^t(A_{11}, \ldots, A_{1m}) \in K[z]^m \setminus \{\overline{0}\}$ with $\deg A_{1j} \le n$ for some positive integer n, and let $\overline{A}_{i+1} \in K(z)^m$ for $i \in \mathbb{Z}^+$ be defined inductively by* (2.2). *If the $K(z)$-vector space generated by $\overline{A}_i$ ($i \in \mathbb{Z}^+$) has dimension r, then*

$$\operatorname{ord} L_1 \le rn + O(1), \tag{2.3}$$

where the constant implied in the O-symbol depends on the system (2.1), *but not on n.*

COROLLARY 2.1. *Let the notations be as in Theorem* 2.1. *If* (2.3) *does not hold for any $r < m$, then* $\det(A_{ij})_{1 \le i,j \le m}$ *does not vanish.*

REMARK 2.1. As is shown in Theorem 3 of [AM], for certain cases of (2.1), the determinant $\det(A_{ij})_{1\le i,j\le m}$ does not vanish without any additional assumption except the necessary condition $\overline{A}_1 \neq \overline{0}$.

We next consider a formal power series solution $\overline{y} = {}^t(f_1,\dots,f_m) \in K[[z]]^m$ of a linear inhomogeneous q-difference equation

$$J(\overline{y}) = \mathcal{P}\overline{y} + \overline{b}, \tag{2.4}$$

where $\mathcal{P} \in \mathrm{GL}(m, K(z))$ and $\overline{b} \in K(z)^m$. In order to consider $\overline{y}$ we shall utilize $\widetilde{y} = {}^t(1, f_1,\dots,f_m)$, which is a solution of the linear homogeneous q-difference equation

$$J(\widetilde{y}) = \widetilde{\mathcal{P}}\widetilde{y}, \qquad \widetilde{\mathcal{P}} = \begin{pmatrix} 1 & {}^t\overline{0} \\ \overline{b} & \mathcal{P} \end{pmatrix}.$$

For $B \in K(z)$ and $\widetilde{A} \in K(z)^{m+1}$ having the first component $-B$, we define

$$\widetilde{L} = B\widetilde{y} + \widetilde{A}.$$

Note that

$$J(\widetilde{L}) = J(B)J(\widetilde{y}) + J(\widetilde{A}) = J(B)\widetilde{\mathcal{P}}\widetilde{y} + J(\widetilde{A}).$$

Then, multiplying by the inverse matrix $\widetilde{\mathcal{P}}^{-1}$ of $\widetilde{\mathcal{P}}$, we have

$$\widetilde{\mathcal{P}}^{-1}J(\widetilde{L}) = J(B)\widetilde{y} + \widetilde{\mathcal{P}}^{-1}J(\widetilde{A}).$$

Starting from $\widetilde{L}_1 = A_{1,0}\widetilde{y} + \widetilde{A}_1$ with $A_{1,0} \in K(z)$ and $\widetilde{A}_1 = {}^t(-A_{1,0}, A_{1,1}, \dots, A_{1,m}) \in K(z)^{m+1}$, we define

$$A_{i+1,0} := J(A_{i,0}) \in K(z), \qquad \widetilde{A}_{i+1} := \widetilde{\mathcal{P}}^{-1}J(\widetilde{A}_i) \in K(z)^{m+1}, \tag{2.5}$$

and

$$\widetilde{L}_{i+1} := \widetilde{\mathcal{P}}^{-1}J(\widetilde{L}_i) = A_{i+1,0}\widetilde{y} + \widetilde{A}_{i+1} \tag{2.6}$$

inductively for any $i \in \mathbb{Z}^+$. By definition the first component of $\widetilde{A}_i$ is $-A_{i,0}$ and that of $\widetilde{L}_i$ is 0.

The following result, which generalizes particular cases obtained in [Vää] and [AV2], is an analogue of Theorem 3.1 of [CC].

THEOREM 2.2. *Suppose that* (2.4) *has a solution* $\overline{y} = {}^t(f_1,\dots,f_m) \in K[[z]]^m$ *such that* 1 *and* $f_1,\dots,f_m$ *are linearly independent over* $K(z)$. *Let* $\widetilde{L}_1 = A_{1,0}\widetilde{y} + \widetilde{A}_1$ *with* $\widetilde{y} = {}^t(1, f_1,\dots,f_m)$, *where* $A_{1,0} \in K[z] \setminus \{0\}$ *and* $\widetilde{A}_1 = {}^t(-A_{1,0}, A_{1,1},\dots,A_{1,m}) \in K[z]^{m+1} \setminus \{\overline{0}\}$ *satisfy* $\max\deg A_{1,j} \le n$ *with a positive integer* n, *and let* $\widetilde{A}_i := {}^t(-A_{i,0}, A_{i,1},\dots,A_{i,m}) \in K(z)^{m+1}$ *for* $i \in \mathbb{Z}^+$ *be defined inductively by* (2.5). *If the* $K(z)$-*vector space generated by* $\widetilde{A}_i$ $(i \in \mathbb{Z}^+)$ *has dimension* $r \le m$, *then*

$$\min_j \operatorname{ord} L_{1,j} \le n + O(1), \tag{2.7}$$

where $L_{1,j} = A_{1,0}f_j + A_{1,j} \in K[[z]]$ $(1 \le j \le m)$ *and the implied constant depends on the system* (2.4), *but not on* n.

COROLLARY 2.2. *Let the notations be as in Theorem* 2.2. *If* (2.7) *does not hold*, *then* $\det(A_{ij})_{1\le i\le m+1,\,0\le j\le m}$ *does not vanish.*

REMARK 2.2. The assertion of Theorem 2.2 also holds for certain Mahler functions (see Remark 4.2 at the end of Section 4).

3. General formalism. In this section we formalize certain arguments used in the proof of Shidlovskii's and Chudnovskys' results, which are independent of any differential system. More precisely, we state and prove three lemmas among which Lemmas 3.1 and 3.3 are essentially contained in the proof of Lemma 8 in Chapter III, §5 of [Shi] (see Chapter VII, §3 of [Lan]), and Lemma 3.2 in the proof of Theorem 3.1 of [CC]. To this end we first give a fundamental notation.

Let K be a field of arbitrary characteristic. Let d and r be positive integers with $r \le d$. For any $r \times d$ matrix $A = (\overline{\alpha}_1, \ldots, \overline{\alpha}_d)$ with $\overline{\alpha}_i \in K(z)^r$ having rank r we define $\delta(A)$ by $\delta(A) := 0$ if $r = d$, and

$$\delta(A) := \max_I \max_{(i,j)\in I\times \bar{I}} \Big\{ \deg d_{ij} \;\Big|\; \overline{\alpha}_j = \sum_{i\in I} d_{ij}\overline{\alpha}_i,\ d_{ij} \in K(z)\Big\}$$

if $r < d$, where I in the first maximum runs over all subsets of $\{1, \ldots, d\}$ with r elements for which $\overline{\alpha}_i$ $(i \in I)$ are linearly independent over $K(z)$, and $\bar{I}$ is the complement of I in $\{1, \ldots, d\}$. Here by the degree of a rational function we mean the maximum of the degrees of its numerator and denominator, which are assumed to be relatively prime.

LEMMA 3.1 (Shidlovskii). *Let m and r be positive integers with $r \le m$, $A = (a_{ij})$ be an $r \times m$ matrix with $a_{ij} \in K[z]$ having rank r, and $f_1, \ldots, f_m$ be elements of $K[[z]]$ which are linearly independent over $K(z)$. Then, for $l_i = \sum_{j=1}^m a_{ij}f_j$ $(1 \le i \le r)$, we have*

$$\min_i \operatorname{ord} l_i \le r \max_{i,j} \deg a_{ij} + c_1, \tag{3.1}$$

where c_1 is a positive constant depending only on $f_1, \ldots, f_m$ and $\delta(A)$.

Proof. Renumbering the f_i if necessary, we may assume that the first r columns of A are linearly independent over $K(z)$. Then we have a decomposition $A = (B, BB') = B(E_r, B')$, where B is an element of $\mathrm{GL}(r, K[z])$, $B' = (b_{ij})$ is an $r \times (m - r)$ matrix with entries from $K(z)$, and E_r is the r-dimensional unit matrix. Then, for $\overline{f} = {}^t(f_1, \ldots, f_m)$ and $\overline{L} = {}^t(l_1, \ldots, l_r)$, we have $\overline{L} = A\overline{f} = B(E_r, B')\overline{f}$. Hence, multiplying by (Δ_{ij}), where $B^{-1} = \Delta^{-1}(\Delta_{ij})$ with $\Delta = \det B$, we get $(\Delta_{ij})\overline{L} = \Delta(E_r, B')\overline{f}$. Equating the first components on both sides, we have

$$\text{(3.2)} \qquad \sum_{j=1}^{r} \Delta_{1j} l_j = \Delta g, \qquad g = f_1 + \sum_{j=1}^{m-r} b_{1j} f_{r+j}.$$

Let $D \in K[z]$ be a common denominator of b_{1j}'s with $\max(\deg(Db_{1j}), \deg D) \le \delta(A)m$. Since $f_1, \ldots, f_m$ are linearly independent over $K(z)$, by Lemma 2.1 in Chapter VII, §3 of [Lan], we have a bound for $\operatorname{ord}(Dg)$ yielding $\operatorname{ord} g \le c_1$. Hence the assertion of Lemma 3.1 follows directly from (3.2).

LEMMA 3.2 (Chudnovskys). *Let m and r be positive integers with $r \le m$, $A = (a_{ij})$ be an $r \times (m+1)$ matrix with $a_{ij} \in K[z]$ $(1 \le i \le r,\ 0 \le j \le m)$ having rank r, and $f_1, \ldots, f_m$ be elements of $K[[z]]$ which together with 1 are linearly independent over $K(z)$. Then, for $l_{ij} = a_{i0} f_j + a_{ij}$ $(1 \le i \le r$, $1 \le j \le m)$, $(a_{10}, \ldots, a_{r0}) \ne {}^t\overline{0}$ we have*

$$\text{(3.3)} \qquad \sum_i \min_j \operatorname{ord} l_{ij} \le \sum_i \max_j \deg a_{ij} + c_2,$$

where c_2 is a positive constant depending only on $f_1, \ldots, f_m$ and $\delta(A)$. In particular,

$$\min_{i,j} \operatorname{ord} l_{ij} \le \max_{i,j} \deg a_{ij} + c_3 \qquad (c_3 = r^{-1} c_2).$$

Proof. As in the proof of Lemma 3.1 we obtain a decomposition $A = (B, BB') = B(E_r, B')$, where B is an element of $\mathrm{GL}(r, K[z])$ and $B' = (b_{ij})$ is an $r \times (m+1-r)$ matrix with entries from $K(z)$. Define an $(m+1) \times r$ matrix G by

$${}^tG = \begin{pmatrix} f_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \cdots & 0 \\ f_r & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = ({}^tG_I, {}^tG_{II}),$$

where G_I and G_{II} are an r-dimensional square matrix and an $(m+1-r) \times r$ matrix, respectively. Then we have $AG = (a_{i0} f_j + a_{ij})$. Hence, multiplying by (Δ_{ij}), where $B^{-1} = \Delta^{-1}(\Delta_{ij})$ with $\Delta = \det B$, we get $(\Delta_{ij})(a_{i0} f_j + a_{ij}) = \Delta(G_I + B' G_{II})$. Taking the determinants of both sides, we have

$$\text{(3.4)} \qquad \Delta^{r-1} \det (a_{i0} f_j + a_{ij}) = (-1)^{r+1} \Delta^r \Big((b_{11} + f_r) - \sum_{j=1}^{r-1} b_{j+1,1} f_j \Big).$$

As in the final part of the proof of Lemma 3.1, the assertion of the lemma follows directly from (3.4).

The above lemmas show the importance of estimating $\delta(A)$ in the case $r < d$. To state the next lemma we use the following notations. Let d and r be again positive integers with $r \le d$. Let C be a field extension of K and F be a field extension of $K(z)$ satisfying $K(z) \subset C(z) \subset F$. For any

$U \in \mathrm{GL}(d, F)$ we denote by $\mathcal{U}_C$ the C-vector space generated by the column vectors of U. Then, for any $r \times d$ matrix A with entries from $K(z)$, we denote by $\mathcal{U}_C(A)$ the subspace of $\mathcal{U}_C$ consisting of the elements which are orthogonal to all row vectors of A.

LEMMA 3.3 (Shidlovskii). *Let F, $C(z)$, U, and $\mathcal{U}_C$ be as above, and let $\mathcal{U}_C^{(d)}$ be the C-vector space generated by all monomials of the entries of U with degrees at most d. Then, for any $r \times d$ matrix A with entries from $K(z)$ having rank r such that $\dim_C \mathcal{U}_C(A) \geq d-r$, $\delta(A)$ is uniformly bounded from above by a positive constant depending only on $\mathcal{U}_C^{(d)}$.*

Proof. We may assume $r < d$. Let U' be a $d \times (d-r)$ matrix whose column vectors belong to $\mathcal{U}_C(A)$ and are linearly independent over C. Since $U \in \mathrm{GL}(d, F)$, the rank of U' is $d - r$. By the definition of $\mathcal{U}_C(A)$ we have $AU' = O$. Let us rearrange the column vectors of A and the row vectors of U' so that the resulting matrices A' and U'' also satisfy $A'U'' = O$, and A' decomposes as (B, BB'), where B is a nonsingular $r \times r$ matrix and $B' = (b_{ij})$ is an $r \times (d-r)$ matrix with $\max \deg b_{ij} = \delta(A)$. Then, denoting by U_I and U_{II} the matrices consisting of the first r and the last $d - r$ rows of U'', we have

$$A'U'' = (B, BB') \begin{pmatrix} U_I \\ U_{II} \end{pmatrix} = B(U_I + B'U_{II}) = O.$$

Since B is nonsingular, we obtain $U_I + B'U_{II} = O$. This means that the rank of U_{II} is at least the rank of U'', thus, U_{II} is nonsingular. Hence we have $B' = -U_I U_{II}^{-1}$, which implies that each b_{ij} can be expressed as a quotient of elements from $\mathcal{U}_C^{(d)}$. We can now conclude the proof by applying Lemma 2.2 in Chapter VIII of Dwork *et al.* [DGS] (which is essentially due to Shidlovskii).

4. Proof of the main results. In general, for a field F and an automorphism ϕ on F we call a pair (F, ϕ) a *difference field*. It is easily seen that a subset C of F consisting of the elements $a \in F$ with $\phi(a) = a$ is a field, which is called the *constant field* of (F, ϕ). In this section we introduce a difference field $(K(z), \phi)$ with constant field K including $\phi = J$ as a particular case, and prove the crucial Lemma 4.1 with the aid of Lemma 3.3. Then we deduce Theorems 2.1 and 2.2 from Lemma 4.1 together with Lemma 3.1 and Lemma 3.2, respectively.

LEMMA 4.1. *Let $(K(z), \phi)$ be a difference field with constant field K. Assume that $\phi^i(z) \neq \phi^j(z)$ for any $i, j \in \mathbb{Z}^+$ with $i \neq j$, where ϕ^i is the ith iteration of ϕ. Let $P \in \mathrm{GL}(d, K(z))$ with a positive integer d, and let $\overline{A}_1$ be a nonzero vector in $K(z)^d$. Starting from $\overline{A}_1$ we define for $i \in \mathbb{Z}^+$*

inductively

$$\overline{A}_{i+1} := {}^t P\phi(\overline{A}_i) \in K(z)^d, \tag{4.1}$$

and denote by r the dimension of the $K(z)$-vector space generated by $\overline{A}_i$ $(i \in \mathbb{Z}^+)$. Then the first r vectors $\overline{A}_1, \dots, \overline{A}_r$ are linearly independent, and for the $r \times d$ matrix A having the transpose of $\overline{A}_i$ in the ith row, $\delta(A)$ is bounded from above by a positive constant which depends on the linear homogeneous difference system

$$\phi(\overline{y}) = P\overline{y}, \qquad \overline{y} = {}^t(y_1, \dots, y_d), \tag{4.2}$$

but not on the choice of any particular initial vector $\overline{A}_1$.

Proof. Let k be the positive integer such that the first k vectors of $\overline{A}_i$ are linearly independent, but the first $k+1$ vectors are not. Using the expression

$$\overline{A}_{k+1} = \sum_{i=1}^{k} g_i \overline{A}_i, \qquad g_i \in K(z), \tag{4.3}$$

we have

$$\overline{A}_{k+2} = {}^t P\phi(\overline{A}_{k+1}) = \sum_{i=1}^{k} \phi(g_i)\, {}^t P\phi(\overline{A}_i) = \sum_{i=1}^{k} \phi(g_i)\overline{A}_{i+1}.$$

This implies that $k = r$, which proves the first assertion of the lemma.

To estimate $\delta(A)$ we construct a difference field extending $(K(z), \phi)$ as follows. Let u_{ij} $(1 \le i, j \le d)$ be indeterminates which are algebraically independent over $K(z)$, and denote by F the rational function field $K(z, u_{ij})$. Then we extend ϕ to an automorphism of F, also called ϕ, by

$$\phi(U) = PU, \qquad U = (u_{ij}) \in \mathrm{GL}(d, F), \tag{4.4}$$

which is well defined by our assumption $P \in \mathrm{GL}(d, K(z))$. Let C be the constant field of (F, ϕ). We note an important fact that z is still transcendental over C. To show this assume that

$$\sum_{j=0}^{k} C_j z^j = 0, \qquad C_j \in C. \tag{4.5}$$

Making ϕ act repeatedly on equality (4.5), we get the equalities $\sum C_j \phi^i(z)^j = 0$ $(0 \le i \le k)$. Since $\det(\phi^i(z)^j)_{0 \le i,j \le k}$ is a Vandermonde determinant, it does not vanish because of our assumption on ϕ. This implies that $C_j = 0$ for all j, as desired.

To estimate $\delta(A)$ we introduce $\mathcal{U}_C$ and $\mathcal{U}_C(A)$ for U, C, and A as in the previous section. Let $\mathcal{W}$ be the C-vector space consisting of $\sigma(\overline{u}) := \langle \overline{A}_1, \overline{u} \rangle$ for $\overline{u} \in \mathcal{U}_C$. Note that the map σ from $\mathcal{U}_C$ to $\mathcal{W}$ is C-linear. We show that the kernel of σ is $\mathcal{U}_C(A)$. In fact, the inclusion $\mathcal{U}_C(A) \subseteq \operatorname{Ker} \sigma$ holds trivially. To see the opposite inclusion, setting $P_i = \phi^i(P)\phi^{i-1}(P) \cdots P$, we note that

$\overline{A}_{i+1} = {}^tP_i\phi^i(\overline{A}_1)$ and $\phi^i(\overline{u}) = P_i\overline{u}$ for $\overline{u} \in \mathcal{U}_C$. It follows that, for $u = \sigma(\overline{u})$ with $\overline{u} \in \mathcal{U}_C$ and for $i \in \mathbb{Z}^+$,

$$\phi^i(u) = \langle \phi^i(\overline{A}_1), P_i\overline{u}\rangle = \langle {}^tP_i\phi^i(\overline{A}_1), \overline{u}\rangle = \langle \overline{A}_{i+1}, \overline{u}\rangle. \tag{4.6}$$

Hence, for $u = \sigma(\overline{u})$ $(= \langle \overline{A}_1, \overline{u}\rangle)$, $u = 0$ implies that $\phi^i(u) = \langle \overline{A}_{i+1}, \overline{u}\rangle = 0$, which shows the inclusion $\operatorname{Ker}\sigma \subseteq \mathcal{U}_C(A)$, as desired. Moreover, assuming the expression (4.3) with k replaced by r, we deduce from (4.6) that, for $u = \sigma(\overline{u})$ with $\overline{u} \in \mathcal{U}_C$,

$$\phi^r(u) = \Big\langle \sum_{i=1}^{r} g_i\overline{A}_i, \overline{u}\Big\rangle = \sum_{i=1}^{r} g_i\langle \overline{A}_i, \overline{u}\rangle = \sum_{i=1}^{r} g_i\phi^{i-1}(u).$$

This means that every $u = \sigma(\overline{u})$ satisfies the same linear homogeneous difference equation of order r. Hence, as in the case of linear homogeneous differential equations, the dimension of $\mathcal{W}$ is at most r by Corollary 11.1 in the Appendix. We now combine the above observations to get

$$\dim_C \mathcal{U}_C(A) = \dim_C \operatorname{Ker}\sigma = d - \dim_C \operatorname{Im}\sigma \geq d - r. \tag{4.7}$$

Thus, by Lemma 3.3, we obtain a bound for $\delta(A)$ depending on the system (4.2), but not on A. This completes the proof of the lemma.

REMARK 4.1. The idea of constructing (F, ϕ) comes from [PS], where van der Put and Singer construct a *Picard–Vessiot ring* for the system (4.2), which corresponds to a Picard–Vessiot extension in the case of differential equations. For the details, see [PS].

Proof of Theorem 2.1 and Corollary 2.1. Let $\overline{A}_i = {}^t(A_{i1}, \ldots, A_{im})$ $(i \in \mathbb{Z}^+)$ and r be as in Theorem 2.1. Since (4.1) with $d = m$, $\phi = J$, $P = \mathcal{P}$, and $\overline{A}_1 = \overline{A}_1$ corresponds to the definition of $\overline{A}_{i+1}$ given in (2.2), we see by the first assertion of Lemma 4.1 that the first r vectors $\overline{A}_1, \ldots, \overline{A}_r$ are linearly independent over $K(z)$. Then, by the second assertion of Lemma 4.1, we have $\delta(\mathcal{A}) = O(1)$ for the $r \times m$ matrix $\mathcal{A} = (A_{ij})$, where the implied constant depends only on the system (2.1). We now apply Lemma 3.1 by taking a common denominator D of all entries of $\mathcal{A}$ into account, where $\deg D = O(1)$ and $\operatorname{ord} D = O(1)$. This means that we set $a_{ij} = DA_{ij}$ and $l_i = DL_i$ in Lemma 3.1. Since, by the definition (2.2), $\deg a_{ij} \leq n + O(1)$ and $\operatorname{ord} l_i = \operatorname{ord} l_1$, and since $\delta(D\mathcal{A}) = \delta(\mathcal{A}) = O(1)$, we obtain the desired bound for $\operatorname{ord} L_1$ given in (2.3). This completes the proof of Theorem 2.1.

Under the assumption of Corollary 2.1, it follows directly from Theorem 2.1 that the dimension of the $K(z)$-vector space generated by $\overline{A}_i$ $(i \in \mathbb{Z}^+)$ is m. Then, as noted above, the first m vectors $\overline{A}_1, \ldots, \overline{A}_m$ are linearly independent over $K(z)$. Thus Corollary 2.1 is proved.

Proof of Theorem 2.2 and Corollary 2.2. Let $\widetilde{A}_i = {}^t(-A_{i,0}, A_{i,1}, \ldots, A_{i,m})$ $(i \in \mathbb{Z}^+)$ and r be as in Theorem 2.2. Then (4.1) with $d = m+1$, $\phi = J$,

$P = {}^t(\widetilde{\mathcal{P}}^{-1})$, and $\overline{A}_1 = \widetilde{A}_1$ corresponds to (2.6). The remaining part of the proof, using Lemma 3.2 instead of Lemma 3.1, is completely similar to the corresponding part of the proof of Theorem 2.1. The deduction of Corollary 2.2 from Theorem 2.2 is also similar to that of Corollary 2.1 from Theorem 2.1. This completes the proof of Theorem 2.2 and its corollary.

REMARK 4.2 (A variant for Mahler functions). We here explain briefly a variant of Theorem 2.2 for certain Mahler functions (see [Nis]). Let $\widehat{J}$ be an endomorphism of $K(z)$, or more generally of $K((z))$, defined by

$$\widehat{J}(a) := a(z^h), \qquad a \in K((z)),$$

where $h \in \mathbb{Z}^+$, $h \geq 2$. In the theory of Mahler functions (of one variable) we consider solutions of certain (systems of) functional equations involving the transformation $\widehat{J}$. In particular, we can consider the system (2.4) with $\widehat{J}$ instead of J, namely,

$$\widehat{J}\overline{y} = \mathcal{P}\overline{y} + \overline{b}.$$

In this setting the statement of Theorem 2.2 with $\widehat{J}$ instead of J remains valid. To show this we first note that Lemma 4.1 still holds upon replacing an automorphism ϕ of $K(z)$ with an endomorphism ϕ of $K(z)$. In fact, by the remark at the end of the Appendix, we do not need any essential change in the proof. Hence, by the inequality (3.3) in Lemma 3.2, we obtain

$$\sum_{i=0}^{r-1} h^i \min_j \operatorname{ord} L_{1j} \leq \sum_{i=0}^{r-1} h^i \max_j \deg A_{1j} + O(1),$$

which implies the desired assertion.

5. Linear independence results for function values. Let $\mathbb{K}$ be an algebraic number field of degree κ over $\mathbb{Q}$. In the following we shall consider arithmetic properties of certain analytic solutions of the q-difference equation

$$z^s\overline{y}(qz) = a(z)\mathcal{C}\overline{y}(z) + \overline{b}(z), \tag{5.1}$$

where $s \in \mathbb{Z}^+$, $\mathcal{C} \in \mathrm{GL}(m, \mathbb{K})$, $a(z) \in \mathbb{K}[z]$ satisfies $a(0) \neq 0$ and $t := \deg a(z) \leq s$. Further, let $u = \deg \overline{b}(z) \leq s$ denote the maximum of the degrees of the components of $\overline{b}(z) \in \mathbb{K}[z]^m$.

We shall now introduce the notations to be used in our arithmetic results. If the finite place v of $\mathbb{K}$ lies over the prime p, we write $v \mid p$, and for an infinite place v of $\mathbb{K}$ we write $v \mid \infty$. We normalize the absolute value $|\ |_v$ of $\mathbb{K}$ so that

$$|p|_v = p^{-1} \quad \text{if } v \mid p,$$
$$|x|_v = |x| \quad \text{if } v \mid \infty,$$

where $|\ |$ denotes the ordinary absolute value in $\mathbb{Q}$. By using the notation

$$\|\alpha\|_v = |\alpha|_v^{\kappa_v/\kappa}, \qquad \kappa_v = [\mathbb{K}_v : \mathbb{Q}_v],$$

the product formula has the form

$$\prod_v \|\alpha\|_v = 1 \qquad \forall \alpha \in \mathbb{K}^*.$$

The height $H(\alpha)$ of α is defined by the formula

$$H(\alpha) = \prod_v \|\alpha\|_v^*, \qquad \|\alpha\|_v^* = \max\{1, \|\alpha\|_v\},$$

and the height $H(\overline{\alpha})$ of the vector $\overline{\alpha} = {}^t(\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$ is given by

$$H(\overline{\alpha}) = \prod_v \|\overline{\alpha}\|_v^*, \qquad \|\overline{\alpha}\|_v^* = \max_{i=1,\ldots,m} \{1, \|\alpha_i\|_v\}.$$

Further, for any place v of $\mathbb{K}$, and $q \in \mathbb{K}^*$ with $\|q\|_v \neq 1$, we define the number

$$l = l_q = \frac{\log H(q)}{\log \|q\|_v}.$$

From now on we assume that v is a place of $\mathbb{K}$ and $q \in \mathbb{K}^*$ is such that $\|q\|_v < 1$ (implying $\lambda \leq -1$). Then, as we prove in Section 7, the equation (5.1) has a unique solution $\overline{f}(z)$ with components $f_i(z)$ converging in some neighbourhood of the origin in $\mathbb{K}_v$, and using (5.1) we define $f_i(z)$ for all $z \in \mathbb{K}_v$ satisfying $a(q^k z) \neq 0$ for all $k = 0, 1, \ldots$. We are interested in linear independence of the values of these functions.

To state our result we choose $0 < \delta < 1$, and define

$$A(\tau) = s\tau^2/2 + m\tau + K, \qquad B(\tau) = s\tau^2/2 + (m + 1 - \delta)\tau,$$

where

$$K = m^2/2 + ((m + 1 - \delta)^3 - m^3)/(6\delta).$$

Let

$$\mu = \mu(\varrho_0) = \frac{B(\varrho_0)}{B(\varrho_0) + \lambda A(\varrho_0)}, \tag{5.2}$$

where ϱ_0 is the positive solution of

$$s(1-\delta)\varrho^2 + (s(1-\delta)\delta m - 2sK)\varrho - (s\delta m + 2(m + 1 - \delta))K = 0. \tag{5.3}$$

Note that $B(\varrho_0) > A(\varrho_0)$. Then we have

THEOREM 5.1. *Assume that the functions* $1, f_1(z), \ldots, f_m(z)$ *are linearly independent over* $\mathbb{K}(z)$, *and let* $\alpha \in \mathbb{K}^*$ *satisfy* $a(\alpha q^k) \neq 0$, $k = 0, 1, \ldots$. *If*

$$-\frac{B(\varrho_0)}{A(\varrho_0)} < \lambda \leq -1, \tag{5.4}$$

then the numbers $1, f_1(\alpha), \ldots, f_m(\alpha)$ *belonging to* $\mathbb{K}_v$ *are linearly independent over* $\mathbb{K}$. *Further*, *there exist positive constants* C, D, H_0 *depending on* (5.1), α *and* δ *such that for all* $\bar{l} = {}^t(l_0, l_1, \ldots, l_m) \in \mathbb{K}^{m+1} \setminus \{\bar{0}\}$, *we have*

$$(5.5) \qquad |l_0 + l_1 f_1(\alpha) + \cdots + l_m f_m(\alpha)|_v > \frac{C}{H^{\mu\kappa/\kappa_v} H^{D(\log H)^{-1/2}}},$$

where $H = \max(H(\bar{l}), H_0)$ *and* μ *is given in* (5.2).

Clearly the condition (5.4) restricts the choice of q, but already $\lambda = -1$ contains several interesting cases, for example the following:

1. $\mathbb{K} = \mathbb{Q}$ or an imaginary quadratic field, v is the infinite place of $\mathbb{K}$, and $q = 1/r$, $\|r\|_v > 1$, $r \in \mathbb{Z}_{\mathbb{K}}$, the ring of integers in $\mathbb{K}$;
2. $\mathbb{K} = \mathbb{Q}$, $v = p$, and $q = p^l$, $l \in \mathbb{Z}^+$;
3. q is a negative power of a PV-number, for example in $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, $q_1 = (1+\sqrt{5})/2$, $q = q_1^l$, $l \in \mathbb{Z}^-$.

To give a more precise idea of μ we consider the values $\lambda = -1$ and $\delta = 1/2$. Then we have (see Section 9)

$$(5.6) \qquad \mu \le \frac{8m}{8m-1}\,(8sm^2 + (s+4)m + s/3 + 2).$$

Theorem 5.1 generalizes and improves Theorem 1 of [Vää] (see also [AV3] for other earlier results generalized by our theorem).

Let q, s and $a(z)$ be as above, and let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of $\mathbb{K}$ satisfying

$$(5.7) \qquad \alpha_i/\alpha_j \ne q^n \quad \text{for all } i \ne j,\ n \in \mathbb{Z}.$$

Let

$$a(z) = \sum_{\nu=0}^{t} a_\nu z^\nu, \qquad a_0 a_t \ne 0.$$

If $\deg a(z) = s$, assume further that for all $i = 1, \ldots, m$ and $n = s, s+1, \ldots,$

$$(5.8) \qquad \alpha_i q^n \ne a_s.$$

Further, we use the notations $[a(z)]_0 = 1$ and $[a(z)]_n = a(z)a(qz)\cdots a(q^{n-1}z)$ for all $n \in \mathbb{Z}^+$. In Corollary 5.1 we consider the functions

$$f_{j\mu\nu}(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{[a(qz)]_n}\, n^\nu (q^\mu \alpha_j)^n,$$

where $j = 1, \ldots, m$; $\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots, l-1$.

Corollary 5.1. *Let* $\alpha_1, \ldots, \alpha_m$ *be as above*, *and let* $\alpha \in \mathbb{K}^*$ *satisfy* $a(\alpha q^k) \ne 0$, $k = 1, 2, \ldots$. *Assume further that the condition* (5.4) *of Theorem* 5.1 *is satisfied with* m *replaced by* $M = msl$. *Then the* $M+1$ *num-*

bers $1, f_{j\mu\nu}(\alpha)$ *belonging to* $\mathbb{K}_v$ *are linearly independent over* $\mathbb{K}$ *and have a linear independence measure given in Theorem* 5.1.

If we define

$$\varphi_\mu(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}}}{[a(q)]_n} (q^\mu z)^n, \qquad \mu = 0, 1, \ldots, s-1,$$

and for $\nu = 0, 1, \ldots,$

$$\varphi_{\mu\nu}(z) = \left(z\frac{d}{dz}\right)^\nu \varphi_\mu(z) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} n^\nu}{[a(q)]_n} (q^\mu z)^n,$$

then Corollary 5.1 implies

Corollary 5.2. *Let* $\alpha_1, \ldots, \alpha_m$ *be as above and assume that* $a(q^k) \neq 0$, $k = 1, 2, \ldots.$ *Assume further that the condition* (5.4) *of Theorem* 5.1 *is satisfied with* m *replaced by* $M = msl$. *Then the* $M+1$ *numbers* $1, \varphi_{\mu\nu}(\alpha_j)$, $j = 1, \ldots, m$; $\mu = 0, 1, \ldots, s-1$; $\nu = 0, 1, \ldots, l-1$, *belonging to* $\mathbb{K}_v$ *are linearly independent over* $\mathbb{K}$ *and have a linear independence measure given in Theorem* 5.1.

In the special case $a(z) = (1 - b_1 z) \cdots (1 - b_t z)$, $t < s$, with $b_i \in \mathbb{K}^*$ this corollary follows from Theorem 1 of [SV] which generalizes earlier results of Stihl [Sti] and Katsurada [Kat]. There the explicit Padé approximation construction is used and the dependence on H in the measure is better than in our more general result.

6. A theorem on linear independence measure. In this section our aim is to formulate and prove a general theorem to be used in the proof of Theorem 5.1. Assume that we have a sequence of linear forms

$$\overline{L}_{n,T} = B_{n,T}\overline{\Theta} + \overline{A}_{n,T} \tag{6.1}$$

of $\overline{\Theta} = {}^t(\Theta_1, \ldots, \Theta_m) \in \mathbb{K}_v^m$, where $B_{n,T} \in \mathbb{K}$, $\overline{A}_{n,T} = {}^t(A_{n,T,1}, \ldots, A_{n,T,m}) \in \mathbb{K}^m$ and $\overline{L}_{n,T} = {}^t(L_{n,T,1}, \ldots, L_{n,T,m})$. Let

$$\max\{\|B_{n,T}\|_w^*, \|\overline{A}_{n,T}\|_w^*\} \leq P_w(n,T) \quad \forall w, \tag{6.2}$$

$$\|\overline{L}_{n,T}\|_v \leq R_v(n,T), \tag{6.3}$$

and let ϱ_1, ϱ_2 with $\varrho_1 < \varrho_2$ and c_4 be positive constants independent of n such that

$$\Delta_{n,T} = \begin{vmatrix} -B_{n,T} & -B_{n,T+1} & \ldots & -B_{n,T+m} \\ A_{n,T,1} & A_{n,T+1,1} & \ldots & A_{n,T+m,1} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n,T,m} & A_{n,T+1,m} & \ldots & A_{n,T+m,m} \end{vmatrix} \neq 0 \tag{6.4}$$

with some integer $T \in [\varrho_1 n, \varrho_2 n - m]$ for all $n \geq c_4$. (Here and later on, c_i's are positive constants independent of n.)

Now we suppose that the assumptions (6.2)–(6.4) are valid with

$$\prod_w P_w(n,\tau n) \le c_5^n H(q)^{A(\tau)n^2}, \quad c_5 \ge 1, \tag{6.5}$$

$$R_v(n,\tau n) \le c_6^n \|q\|_v^{B(\tau)n^2}, \quad c_6 \ge 1, \tag{6.6}$$

for all $\varrho_1 \le \tau \le \varrho_2$. In (6.5) and (6.6) we also suppose that $A(\tau)$ and $B(\tau)$ are bounded positive functions on the interval $\varrho_1 \le \tau \le \varrho_2$ satisfying

$$B(\tau) + \lambda A(\tau) \ge c_7 \tag{6.7}$$

with some $c_7 > 0$. Further we define

$$\mu(\tau) = \frac{B(\tau)}{B(\tau)+\lambda A(\tau)}, \quad \mu = \sup_{\varrho_1 \le \tau \le \varrho_2} \mu(\tau).$$

THEOREM 6.1. *If the above assumptions* (6.4)–(6.7) *are valid, then there exist positive constants C, D and H_0 depending on the numbers $\Theta_1,\dots,\Theta_m$ and ϱ_1, ϱ_2, c_4, c_5, c_6 and c_7 such that*

$$|\beta_0 + \beta_1\Theta_1 + \dots + \beta_m\Theta_m|_v > \frac{C}{H^{\mu\kappa/\kappa_v} H^{D(\log H)^{-1/2}}} \tag{6.8}$$

for all $\overline{\beta} = {}^t(\beta_0,\beta_1,\dots,\beta_m) \in \mathbb{K}^{m+1} \setminus \{\overline{0}\}$ with $H = \max\{H(\overline{\beta}), H_0\}$.

Proof. Let

$$\Lambda := \beta_0 + \beta_1\Theta_1 + \dots + \beta_m\Theta_m.$$

Using (6.1) we have

$$B_{n,T}\Lambda = B_{n,T}\beta_0 - \langle\widehat{\beta}, \overline{A}_{n,T}\rangle + \langle\widehat{\beta}, \overline{L}_{n,T}\rangle = G_{n,T} + \langle\widehat{\beta}, \overline{L}_{n,T}\rangle, \tag{6.9}$$

where $\widehat{\beta} = {}^t(\beta_1,\dots,\beta_m)$ and $G_{n,T} = B_{n,T}\beta_0 - \langle\widehat{\beta}, \overline{A}_{n,T}\rangle$.

From the determinant condition (6.4) it follows that $G_{n,T'} \ne 0$ with some integer $T' \in [\varrho_1 n, \varrho_2 n]$. Thus we may use the product formula and (6.9) to get

$$\begin{aligned} 1 &= \prod_w \|G_{n,T'}\|_w = \|B_{n,T'}\Lambda - \langle\widehat{\beta}, \overline{L}_{n,T'}\rangle\|_v \prod_{w\ne v} \|G_{n,T'}\|_w \\ &\le (\|\widehat{\beta}\|_v R_v(n,T') + \|\Lambda\|_v P_v(n,T'))2^\kappa \prod_{w\ne v} P_w(n,T') \prod_{w\ne v} \|\overline{\beta}\|_w^*. \end{aligned} \tag{6.10}$$

By choosing $T' = \tau n$, where $\tau \in [\varrho_1,\varrho_2]$, we have

$$\begin{aligned} 1 &\le c_6^n \|q\|_v^{B(\tau)n^2} H(\overline{\beta}) c_5^n H(q)^{A(\tau)n^2} + c_5^n \|\Lambda\|_v H(\overline{\beta}) H(q)^{A(\tau)n^2} \\ &\le S(n,\tau) + W(n,\tau), \end{aligned} \tag{6.11}$$

where

$$S(n,\tau) = 2^{-1} c_8^n H(\overline{\beta}) \|q\|_v^{(B(\tau)+\lambda A(\tau))n^2},$$
$$W(n,\tau) = c_5^n \|\Lambda\|_v H(\overline{\beta}) H(q)^{A(\tau)n^2}.$$

Now we proceed in the usual way (see e.g. [Mat, formulae (3.22)–(3.28)]). Choosing H large enough, say $H \ge H_0$, and using (6.7) we can find a largest $n_1 \ge c_4$ such that

$$S(n_1,\tau) \ge 1/2. \tag{6.12}$$

Consequently, (6.12) implies

$$W(n_1+1,\tau) > 1/2. \tag{6.13}$$

First we use the inequality (6.12) to get

$$(B(\tau)+\lambda A(\tau))n_1^2 + \frac{\log c_8}{\log \|q\|_v}\, n_1 + \frac{\log H(\overline{\beta})}{\log \|q\|_v} \le 0,$$

which implies the bounds

$$n_1 \le c_9 + \sqrt{\frac{-\log H(\overline{\beta})}{(B(\tau)+\lambda A(\tau))\log \|q\|_v}} \tag{6.14}$$

and

$$n_1^2 \le c_9^2 + 2c_9\sqrt{\frac{-\log H(\overline{\beta})}{(B(\tau)+\lambda A(\tau))\log \|q\|_v}} - \frac{\log H(\overline{\beta})}{(B(\tau)+\lambda A(\tau))\log \|q\|_v}. \tag{6.15}$$

Then using (6.13) and the bounds (6.14), (6.15) we get

$$\begin{aligned} 1/2 &< \|\varLambda\|_v H(\overline{\beta}) c_{10}^{n_1} H(q)^{A(\tau)n_1^2} \\ &< \|\varLambda\|_v H(\overline{\beta}) c_{11} H(\overline{\beta})^{-\lambda A(\tau)/(B(\tau)+\lambda A(\tau)) + c_{12}/\sqrt{\log H(\overline{\beta})}}, \end{aligned} \tag{6.16}$$

where $c_{11} = c_{11}(\tau)$ and $c_{12} = c_{12}(\tau)$ are bounded by the assumption (6.7). The estimate (6.16) is valid for some $\varrho_1 \le \tau \le \varrho_2$ and hence the choice

$$\mu = \sup_{\varrho_1 \le \tau \le \varrho_2} \frac{B(\tau)}{B(\tau)+\lambda A(\tau)}$$

proves (6.8).

7. A construction by Siegel's lemma. By setting

$$a(z) = \sum_{\nu=0}^{t} a_\nu z^\nu, \qquad a_0 a_t \ne 0,$$
$$\overline{b}(z) = \sum_{\nu=0}^{u} \overline{b}_\nu z^\nu, \qquad \overline{f}(z) = \sum_{\nu=0}^{\infty} \overline{f}_\nu z^\nu,$$

we see that $\overline{y} = \overline{f}$ satisfies (5.1) if and only if

$$\overline{f}_\nu = -a_0^{-1}\mathcal{C}^{-1}\overline{b}_\nu - a_0^{-1}\sum_{j=1}^{\min(t,\nu)} a_j \overline{f}_{\nu-j}, \qquad 0 \le \nu < s, \tag{7.1}$$

$$\overline{f}_\nu = -a_0^{-1}\mathcal{C}^{-1}(\overline{b}_\nu - q^{\nu-s}\overline{f}_{\nu-s}) - a_0^{-1}\sum_{j=1}^{t} a_j \overline{f}_{\nu-j}, \qquad \nu \ge s, \tag{7.2}$$

where $\overline{b}_\nu = \overline{0}$ for all $\nu > u$. If v is a place of $\mathbb{K}$ such that $|q|_v < 1$, it follows that

$$\|\overline{f}_\nu\|_v \le c_{13}^{\nu+1} \tag{7.3}$$

for some constant $c_{13} \ge 1$ depending on v and (5.1). Furthermore, the recursion (7.2) implies that for any place w,

$$\|\overline{f}_\nu\|_w \le (2m+t)^{\delta(w)\nu} \|\overline{\zeta}_1\|_w^{\nu+1} \|q\|_w^{*\binom{\nu}{2}}, \tag{7.4}$$

where $\delta(w) = 1$ if $w \mid \infty$, $\delta(w) = 0$ if w is finite, and $\overline{\zeta}_1$ is a constant vector with coordinates in $\mathbb{K}$ depending on (5.1) (as are also other $\overline{\zeta}_i$ later).

Thus it follows that (5.1) has a unique analytic solution $\overline{f}(z)$ with components $f_i(z)$ converging in a neighbourhood of the origin in $\mathbb{K}_v$. By using (5.1) these can be continued and we have $f_i(z) \in \mathbb{K}_v$ for all $z \in \mathbb{K}$ satisfying $a(q^k z) \neq 0$, $k = 0, 1, \ldots$. Furthermore, for all those z,

$$\overline{f}(z) = -\sum_{n=0}^{\infty} \frac{q^{s\binom{n}{2}} z^{sn}}{[a(z)]_{n+1}} (\mathcal{C}^{-1})^{n+1} \overline{b}(q^n z).$$

We now construct Padé type approximations of the second kind for the components of $\overline{f}(z)$. These are given in the following

Lemma 7.1. *Suppose that $0 < \delta < 1$, and let n denote a positive integer. There exist polynomials $B(z)$ and $A_i(z)$, $i = 1, \ldots, m$, in $\mathbb{K}[z]$ of degree $\le mn$, not all identically zero, such that the components $L_i(z)$ of*

$$\overline{L}(z) = B(z)\overline{f}(z) + \overline{A}(z), \qquad \overline{A}(z) = {}^t(A_1(z), \ldots, A_m(z)),$$

satisfy

$$\operatorname{ord} L_i(z) \ge \sigma := (m+1)n - [\delta n]. \tag{7.5}$$

Furthermore, if $\eta = ((m+1-\delta)^3 - m^3)/(6\delta)$ and a place v_0 is given, the following estimates are valid (in $H(B)$ and $|B|_v$ we denote by B the vector having the coefficients of $B(z)$ as components):

$$H(B) \le 2^{O(n)} H(q)^{\eta n^2}, \qquad \|B\|_w \le 1 \quad \textit{for all } w \neq v_0, \tag{7.6}$$

and

$$\|\overline{A}\|_w^* \le c_{14}^{\delta(w)n} \|\overline{\zeta}_2\|_w^n \|q\|_w^{*\, m^2n^2/2} H(q)^{\epsilon(w)\eta n^2}, \tag{7.7}$$

where $\epsilon(v_0) = 1$ and $\epsilon(w) = 0$ if $w \neq v_0$. If $v \neq v_0$, then

$$\|\overline{L}(z)\|_v \le 2^{\delta(v)O(n)} c_{13}^{\sigma+1} \|z\|_v^\sigma \quad \textit{for all } \|z\|_v < 1/(2c_{13}). \tag{7.8}$$

The constants in $O(n)$ depend on the system (5.1).

Proof. Write $B(z) = \sum_{j=0}^{mn} b_j z^j$ with unknown coefficients b_j. Then

$$B(z)\overline{f}(z) = \sum_{\nu=0}^{\infty} \Big(\sum_{j=0}^{\min(\nu,mn)} b_j \overline{f}_{\nu-j} \Big) z^\nu,$$

and

$$(7.9)\qquad \sum_{j=0}^{mn} b_j \bar{f}_{\nu-j} = 0, \qquad mn+1 \le \nu \le \sigma - 1,$$

is a set of $m(n-[\delta n]-1)$ linear homogeneous equations in $mn+1$ unknowns b_j. By (7.4),

$$\prod_w \max_{0\le j\le mn} \|\bar{f}_{\nu-j}\|_w \le \prod_w 2^{\delta(w)O(\nu)} \|\bar{\zeta}_1\|_w^{\nu+1} \|q\|_w^{*\binom{\nu}{2}} \le 2^{O(n)} H(q)^{\binom{\nu}{2}},$$

and therefore the use of Siegel's lemma in the form given by [Bom] implies the existence of a polynomial $B(z) \ne 0$ satisfying (7.9) and

$$H(B) \le 2^{O(n)} H(q)^{m\sum_{\nu=mn+1}^{\sigma-1}\binom{\nu}{2}/(mn\delta)} \le 2^{O(n)} H(q)^{\eta n^2}.$$

Furthermore, we may choose $B(z)$ in such a way that $\|B\|_w \le 1$ for all $w \ne v_0$.

By defining

$$\bar{A}(z) = -\sum_{\nu=0}^{mn} \Big(\sum_{j=0}^{\nu} b_j \bar{f}_{\nu-j} \Big) z^\nu,$$

we then obtain $\operatorname{ord} L_i(z) \ge \sigma$ for the components of

$$\bar{L}(z) = B(z)\bar{f}(z) + \bar{A}(z).$$

The bound for the components of $\bar{A}(z)$ follows from

$$\Big\| \sum_{j=0}^{\nu} b_j \bar{f}_{\nu-j} \Big\|_w \le (\nu+1)^{\delta(w)} \|B\|_w \max_{0\le j\le \nu} \|\bar{f}_{\nu-j}\|_w$$
$$\le c_{14}^{\delta(w)n} \|\bar{\zeta}_2\|_w^n \|q\|_w^{*\,m^2n^2/2} H(q)^{\epsilon(w)\eta n^2}, \qquad 0 \le \nu \le mn,$$

where we used (7.4) and (7.6). Finally, if $v \ne v_0$ and $\|z\|_v < 1/(2c_{13})$, then

$$\|\bar{L}(z)\|_v = \Big\| \sum_{\nu=\sigma}^{\infty} \Big(\sum_{j=0}^{mn} b_j \bar{f}_{\nu-j} \Big) z^\nu \Big\|_v$$
$$\le \sum_{\nu=\sigma}^{\infty} (mn+1)^{\delta(v)} c_{13}^{\nu+1} \|z\|_v^\nu \le 2^{\delta(v)O(n)} c_{13}^{\sigma+1} \|z\|_v^\sigma$$

by (7.3). This proves Lemma 7.1.

8. Nonvanishing lemma. Set $B_0(z) = B(z)$, $A_{0,i}(z) = A_i(z)$ and $L_{0,i}(z) = L_i(z)$ with $i = 1, \ldots, m$, where $B(z)$, $A_i(z)$ and $L_i(z)$ are given in Lemma 7.1, and define

$$\widetilde{L}_0(z) := {}^t(L_{0,0}(z), \ldots, L_{0,m}(z)), \qquad \widetilde{A}_0(z) = {}^t(A_{0,0}(z), \ldots, A_{0,m}(z)),$$

where $A_{0,0}(z) = -B_0(z)$ and $L_{0,0}(z) \equiv 0$. We now construct inductively $B_k(z)$, $\widetilde{A}_k(z)$ and $\widetilde{L}_k(z)$ with $k = 1, 2, \ldots$ as in (2.6) but with a slight modification. It follows from (5.1) that $\widetilde{y} = {}^t(1, f_1, \ldots, f_m)$ satisfies the homogeneous q-difference equation

$$z^s J(\widetilde{y}) = \widetilde{\mathcal{C}}\widetilde{y}, \qquad \widetilde{\mathcal{C}} = \begin{pmatrix} z^s & {}^t\overline{0} \\ \overline{b}(z) & a(z)\mathcal{C} \end{pmatrix}. \tag{8.1}$$

Hence $z^{-s}\widetilde{\mathcal{C}}$ plays the role of $\widetilde{\mathcal{P}}$ in Theorem 2.2. Since the inverse of $z^{-s}\widetilde{\mathcal{C}}$ contains rational components with the denominator $a(z)$, we define

$$\widetilde{\mathcal{Q}} := a(z) z^s \widetilde{\mathcal{C}}^{-1} = \begin{pmatrix} a(z) & {}^t\overline{0} \\ -\mathcal{C}^{-1}\overline{b}(z) & z^s\mathcal{C}^{-1} \end{pmatrix}$$

and use it instead of the inverse of $z^{-s}\widetilde{\mathcal{C}}$. Therefore, $\widetilde{L}_k(z)$ here is defined inductively by

$$\widetilde{L}_k := \widetilde{\mathcal{Q}} J(\widetilde{L}_{k-1}) = B_k \widetilde{y} + \widetilde{A}_k, \qquad k \in \mathbb{Z}^+, \tag{8.2}$$

where

$$\widetilde{A}_k = \widetilde{\mathcal{Q}} J(\widetilde{A}_{k-1}), \qquad B_k = a(z) J(B_{k-1}). \tag{8.3}$$

By definition the first component of $\widetilde{A}_k$ is $-B_k$ and that of $\widetilde{L}_k$ is 0. We then set

$$\varDelta_k(z) = \begin{vmatrix} -B_k(z) & -B_{k+1}(z) & \ldots & -B_{k+m}(z) \\ A_{k,1}(z) & A_{k+1,1}(z) & \ldots & A_{k+m,1}(z) \\ \ldots & \ldots & \ldots & \ldots \\ A_{k,m}(z) & A_{k+1,m}(z) & \ldots & A_{k+m,m}(z) \end{vmatrix}, \qquad k = 0, 1, \ldots,$$

for which the recursion $\varDelta_k(z) = (\det \widetilde{\mathcal{Q}}(z)) J \varDelta_{k-1}(z)$ holds. Thus we have

$$\varDelta_k(z) = [\det \widetilde{\mathcal{Q}}(z)]_k \varDelta_0(q^k z) = [z^s a(z) \mathcal{C}^{-1}]_k \varDelta_0(q^k z). \tag{8.4}$$

By Corollary 2.2 and the construction of Lemma 7.1 it follows that

$$\varDelta_0(z) \neq 0 \tag{8.5}$$

for all $n \geq c_{16}$, if the functions $1, f_1(z), \ldots, f_m(z)$ are linearly independent over $\mathbb{K}(z)$.

To estimate $\deg \varDelta_0(z)$ we note that using the recursions (8.3) we obtain $\deg B_i(z) \leq mn + it$ and $\deg \widetilde{A}_i(z) \leq mn + is$. Therefore

$$\deg \varDelta_0(z) \leq (m+1)mn + \binom{m+1}{2} s.$$

Since $\operatorname{ord} \widetilde{L}_i(z) \geq \sigma + is$, $\operatorname{ord} \varDelta_0(z) \geq m\sigma + \binom{m}{2}s$. These estimates give

$$\deg \varDelta_0(z) - \operatorname{ord} \varDelta_0(z) \leq \delta mn + c_{17} \qquad \text{if } n \geq c_{16}. \tag{8.6}$$

Thus, if $\alpha \neq 0$ and $k \geq 0$ is fixed,

$$\varDelta_0(q^{k+l}\alpha) \neq 0 \tag{8.7}$$

for some

$$0 \le l \le \delta mn + c_{17}. \tag{8.8}$$

Thus we obtain

LEMMA 8.1. *Let $a(\alpha q^k) \neq 0$ for all $k = 0, 1, \ldots,$ and let the functions $1, f_1(z), \ldots, f_m(z)$ be linearly independent over $\mathbb{K}(z)$. Then for any $\varrho > 0$ there exists an integer T satisfying*

$$\varrho n \le T \le (\varrho + \delta m)n + c_{17}, \tag{8.9}$$

$$\Delta_T(\alpha) \neq 0, \tag{8.10}$$

if $\alpha \neq 0$ and $n \ge c_{16}$.

We shall also need the following consequence of the recurrences (8.3) and Lemma 7.1.

LEMMA 8.2. *Let $\alpha \in \mathbb{K}^*$. For all w we have*

$$\|B_k(\alpha)\|_w \le 2^{\delta(w)O(n)} \|\overline{\zeta}_3\|_w^{O(n)} \|q\|_w^{*\,kmn+tk^2/2} H(q)^{\epsilon(w)\eta n^2}, \tag{8.11}$$

$$\max(\|A_{ki}(\alpha)\|_w) \le 2^{\delta(w)O(n)} \|\overline{\zeta}_4\|_w^{O(n)} \|q\|_w^{*\,kmn+m^2n^2/2+sk^2/2} H(q)^{\epsilon(w)\eta n^2}. \tag{8.12}$$

Further, if $v \neq v_0$ and $n \ge c_{19}$, then

$$\|L_{ki}(\alpha)\|_v \le 2^{O(n)} \|q\|_v^{k\sigma + sk^2/2}. \tag{8.13}$$

The constants in $O(n)$ here (and in the next section) depend only on the system (5.1), α and ϱ.

9. Proof of Theorem 5.1. We shall use Theorem 6.1 and Lemmas 8.1 and 8.2 in our proof. From Lemmas 8.1 and 8.2 it follows that, for any constant $\varrho > 0$, we may apply Theorem 6.1, where

$$[\varrho_1, \varrho_2] = [\varrho, \varrho + \delta m] \tag{9.1}$$

(taking n large enough, $\varrho + \delta m + (c_{17} - m)/n$ approaches $\varrho + \delta m$) and

$$P_w(T) = 2^{\delta(w)O(n)} \|\overline{\zeta}_5\|_w^{O(n)} \|q\|_w^{*\,Tmn+sT^2/2+m^2n^2/2+\epsilon(w)\eta n^2},$$
$$R_w(T) = 2^{O(n)} \|q\|_v^{T\sigma + sT^2/2}.$$

Therefore, by writing $T = \tau n$, we have

$$A(\tau) = s\tau^2/2 + m\tau + K, \quad K = m^2/2 + ((m+1-\delta)^3 - m^3)/(6\delta),$$
$$B(\tau) = s\tau^2/2 + L\tau, \qquad L = m + 1 - \delta.$$

The minimum value of $A(\tau)/B(\tau)$, $\tau > 0$, is reached at τ_0 satisfying

$$\frac{s}{2}(1-\delta)\tau_0^2 - sK\tau_0 - KL = 0. \tag{9.2}$$

Hence the optimal value of

$$\mu = \max_{\varrho\le\tau\le(\varrho+\delta m)} \frac{B(\tau)}{B(\tau)+\lambda A(\tau)} = \max\{\mu(\varrho), \mu(\varrho+\delta m)\}$$

will be attained when

$$(9.3) \qquad A(\varrho_0)/B(\varrho_0) = A(\varrho_0+\delta m)/B(\varrho_0+\delta m), \qquad \varrho_0 \le \tau_0 \le \varrho_0+\delta m,$$

where the unique positive solution of (9.3) satisfies the equation

$$s(1-\delta)\varrho_0^2 + (s(1-\delta)\delta m - 2sK)\varrho_0 - (s\delta m + 2(m+1-\delta))K = 0.$$

Here we note that, if $0 < B(\varrho_0)/A(\varrho_0)+\lambda$, then $0 < B(\tau)/A(\tau)+\lambda$ for all $\tau \in [\varrho_0, \varrho_0+\delta m]$. This completes the proof of Theorem 5.1.

Finally, we prove (5.6). Let $\delta = 1/2$ and $\lambda = -1$. Then

$$\tau_0 \ge \varrho_0 \ge 4K - m/2 =: \varrho^*, \qquad m < \varrho^*/(4m).$$

Because the function $A(\tau)/B(\tau)$ is decreasing on the interval $(0, \tau_0)$ we get

$$\begin{aligned}\mu = \mu(\varrho_0) \le \mu(\varrho^*) &= \frac{2(s\varrho^{*2} + (2m+1)\varrho^*)}{2\varrho^* - 4K} \\ &= \frac{2(s\varrho^{*2} + (2m+1)\varrho^*)}{\varrho^* - m/2} \le \frac{2(s\varrho^* + 2m+1)}{1-1/(8m)} \\ &= \frac{8m}{8m-1}(8sm^2 + (s+4)m + s/3 + 2).\end{aligned}$$

10. Proof of Corollary 5.1. We shall start from the q-functions

$$f(z) = f(z,\alpha) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{[a(qz)]_n}\,\alpha^n$$

satisfying the q-difference equation

$$(10.1) \qquad \alpha(qz)^s f(qz) = a(qz)f(z) - a(qz).$$

Further, set

$$f_\nu(z,\alpha) = \left(\alpha\frac{\partial}{\partial\alpha}\right)^{\nu} f(z,\alpha)$$

and

$$(10.2) \qquad f_{j\mu\nu}(z) = f_\nu(z, q^\mu\alpha_j) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{[a(qz)]_n}\, n^\nu (q^\mu \alpha_j)^n,$$

where $j = 1,\ldots,m$; $\mu = 0,1,\ldots,s-1$; $\nu = 0,1,\ldots,l-1$. We then have

Lemma 10.1. *The functions* (10.2) *satisfy a system of q-difference equations*

$$(10.3) \quad q^\mu \alpha_j (qz)^s f_{j\mu\nu}(qz) = a(qz)\left(\sum_{i=0}^{\nu} (-1)^{\nu-i}\binom{\nu}{i} f_{j\mu i}(z) + (-1)^{\nu+1}\right).$$

Proof. By writing $f_\nu(z) = f_\nu(z,\alpha)$ we have

$$\begin{aligned}
\alpha(qz)^s f_\nu(qz) &= a(qz)\sum_{n=0}^{\infty} \frac{q^{s\binom{n+2}{2}} z^{s(n+1)}}{a(qz)a(q^2z)\cdots a(q^{n+1}z)}\, n^\nu \alpha^{n+1} \\
&= a(qz)\sum_{n=1}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{a(qz)\cdots a(q^n z)}\, (n-1)^\nu \alpha^{n} \\
&= a(qz)\sum_{i=0}^{\nu}(-1)^{\nu-i}\binom{\nu}{i}\sum_{n=1}^{\infty} \frac{q^{s\binom{n+1}{2}} z^{sn}}{a(qz)\cdots a(q^n z)}\, n^i \alpha^{n} \\
&= a(qz)\left\{(-1)^\nu(f_0(z)-1) + \sum_{i=1}^{\nu}(-1)^{\nu-i}\binom{\nu}{i} f_i(z)\right\} \\
&= a(qz)\sum_{i=0}^{\nu}(-1)^{\nu-i}\binom{\nu}{i} f_i(z) + (-1)^{\nu+1}a(qz),
\end{aligned}$$

which implies (10.3).

Lemma 10.1 tells us that $\overline{f} = {}^t(f_0(z), f_1(z), \ldots, f_\nu(z))$ satisfies

$$\alpha(qz)^s J\overline{f} = a(qz)\mathcal{P}\overline{f} + a(qz)\overline{b}$$

where $\mathcal{P}$ is a nonsingular constant matrix and $\overline{b} = {}^t(-1, 1, -1, 1, \ldots, (-1)^{\nu+1})$. Also, we note that the system (10.3) is of the type (5.1). Therefore Corollary 5.1 follows from Theorem 5.1 immediately, once we prove the following

Lemma 10.2. *The $msl+1$ functions*

(10.4) $\quad 1,\ f_{j\mu\nu}(z), \quad j = 1,\ldots,m;\ \mu = 0,1,\ldots,s-1;\ \nu = 0,1,\ldots,l-1,$

are linearly independent over $\mathbb{K}(z)$.

Proof. Let us first show that the functions

(10.5) $\quad 1,\ f_{j\mu}(z) \quad (j = 1,\ldots,m;\ \mu = 0,1,\ldots,s-1)$

are linearly independent over $\mathbb{K}(z)$, where $f_{j\mu}(z) = f_{j\mu 0}(z)$. Assume, on the contrary, that these functions are linearly dependent over $\mathbb{K}(z)$. So suppose we have a nontrivial relation

$$F(z) = F_0(z) + \sum_{j=1}^{m} F_j(z) = 0, \qquad F_j(z) = \sum_{\mu=0}^{s-1} F_{j\mu}(z) f_{j\mu}(z), \tag{10.6}$$

with $F_0, F_{j\mu} \in \mathbb{K}[z]$ such that

$$N = \#\{(j,\mu) \mid F_{j\mu} \neq 0\}$$

is minimal. Further define $\Lambda_j = \{\mu \mid F_{j\mu} \neq 0\}$ and $\Lambda = \{j \mid \Lambda_j \neq \emptyset\}$. Under these notations we can rewrite the above linear dependence relation as

$$F(z) = F_0(z) + \sum_{j\in\Lambda} F_j(z) = 0, \qquad F_j(z) = \sum_{\mu\in\Lambda_j} F_{j\mu}(z) f_{j\mu}(z). \tag{10.7}$$

In what follows, for each $j \in \Lambda$, we denote by s_j the maximal element of Λ_j.

Claim 1. *For each $j \in \Lambda$ there exist $d_{j0}, \ldots, d_{js_j} \in \mathbb{K}$ with $d_{js_j} = 1$ such that*

$$z^{s_j} F_j(z) = F_{js_j}(z) \sum_{\mu=0}^{s_j} d_{j\mu} z^{\mu} f_{j\mu}(z). \tag{10.8}$$

Proof of Claim 1. If $s_j = 0$ or $d_{j\mu} = 0$ for all $\mu < s_j$, then the claim holds. Therefore, our task is to show that, for each nonnegative integer μ with $\mu < s_j$ and $F_{j\mu} \neq 0$, there exists $d_{j\mu} \in \mathbb{K}$ such that

$$z^{s_j} F_{j\mu}(z) = d_{j\mu} z^{\mu} F_{js_j}(z). \tag{10.9}$$

It follows from (10.3) that for each $k \in \Lambda$,

$$\begin{aligned} q^{s_k} \alpha_k (qz)^s F_k(qz) &= a(qz) \sum_{\mu=0}^{s_k} q^{s_k-\mu} F_{k\mu}(qz)(f_{k\mu}(z) - 1) \\ &\equiv a(qz) \sum_{\mu=0}^{s_k} q^{s_k-\mu} F_{k\mu}(qz) f_{k\mu}(z) \pmod{\mathbb{K}[z]}. \end{aligned}$$

Hence, by setting $S_j = S - s_j$ with the maximum S of s_j's and $\beta_j = (\alpha_1 \cdots \alpha_m)/\alpha_j$, the identity

$$q^{S_j} \beta_j a(qz) F_{js_j}(qz) \cdot F(z) - F_{js_j}(z) \cdot q^S \alpha_1 \cdots \alpha_m (qz)^s F(qz) = 0$$

(following from (10.7)) can be expressed as

$$G_0(z) + \sum_{k\in\Lambda} G_k(z) = 0, \qquad G_k(z) = \sum_{\mu\in\Lambda_k} G_{k\mu}(z) f_{k\mu}(z),$$

where $G_0, G_{k\mu} \in \mathbb{K}[z]$ with

$$G_{j\mu}(z) = q^{S_j} \beta_j a(qz) \{F_{js_j}(qz) F_{j\mu}(z) - q^{s_j-\mu} F_{j\mu}(qz) F_{js_j}(z)\}.$$

Since $G_{js_j} = 0$, by the minimality of N, all the $G_{k\mu}$'s are 0. In particular,

$$F_{js_j}(qz) F_{j\mu}(z) - q^{s_j-\mu} F_{j\mu}(qz) F_{js_j}(z) = 0 \qquad (\mu \in \Lambda_j,\ \mu \neq s_j),$$

or equivalently

$$\frac{F_{j\mu}(z)}{F_{js_j}(z)} = q^{s_j-\mu} \frac{F_{j\mu}(qz)}{F_{js_j}(qz)} \qquad (\mu \in \Lambda_j,\ \mu \neq s_j).$$

Since $F_{j\mu}(z)/F_{js_j}(z)$ has an isolated singularity at $z = 0$, its Laurent series expansion about this point implies that $F_{j\mu}(z)/F_{js_j}(z) = d_{j\mu} z^{\mu-s_j}$ with some $d_{j\mu} \in \mathbb{K}$, as desired. Hence Claim 1 is proved.

Set

$$g_j(z) = \sum_{\mu=0}^{s_j} d_{j\mu} z^{\mu} f_{j\mu}(z) \qquad (j \in \Lambda).$$

Then, by (10.3), we have the following q-difference equation for $g_j(z)$:

$$\alpha_j(qz)^s g_j(qz) = a(qz)(g_j(z) - Q_j(z)), \qquad Q_j(z) = \sum_{\mu=0}^{s_j} d_{j\mu} z^{\mu}. \tag{10.10}$$

CLAIM 2. *None of the functions $g_j(z)$ is a polynomial.*

Proof of Claim 2. Assume, on the contrary, that $g_j(z)$ is a polynomial of degree n. Since $a(0) \neq 0$, $g_j(z) - Q_j(z)$ is divisible by z^s in $\mathbb{K}[z]$. By comparing both sides of (10.10) this implies that either $g_j(z) = Q_j(z)$ or $n \geq s$. The former gives $g_j(qz) = 0$ by (10.10), which is inconsistent with $g_j(z) = Q_j(z) \neq 0$. The latter gives, by comparing the degrees of both sides of (10.10), $\deg a = s$ and $\alpha_j q^n = a_s$, which contradicts (5.8). This proves Claim 2.

We can now conclude the proof that the functions (10.5) are linearly independent over $\mathbb{K}(z)$. In fact, by Claim 1, we have

$$z^S F(z) = z^S F_0(z) + \sum_{j \in \Lambda} z^{S_j} F_{js_j}(z) g_j(z) = 0.$$

However, by Claim 2, it follows completely analogously to the proof of Lemma 1 of [AV1] that this is not the case.

We next show our lemma in full generality. Assume, on the contrary, that the functions (10.4) are linearly dependent over $\mathbb{K}(z)$. Take a nontrivial linear dependence relation

$$G(z) = F_0(z) + \sum_{j=1}^{m} \sum_{\mu=0}^{s-1} \sum_{\nu=0}^{l-1} F_{j\mu\nu}(z) f_{j\mu\nu}(z) = 0 \tag{10.11}$$

so that the number L defined by

$$L = \sum_{j=1}^{m} \sum_{\mu=0}^{s-1} l_{j\mu}$$

is minimum, where $l_{j\mu}$ for each (j, μ) is defined to be the maximal element of the set $\{\nu \mid F_{j\mu\nu} \neq 0\}$, or zero if this set is empty. Since $L \geq 1$, we can take j, s_j and l_j with $1 \leq j \leq m, 0 \leq s_j \leq s-1$ and $1 \leq l_j \leq l-1$ such that

$$F_{js_jl_j} \neq 0, \qquad F_{js_j\nu} = 0 \qquad (\nu = l_j + 1, \ldots, l-1).$$

For simplicity, we write $f_{js_j\nu} = f_{\nu}$ and $F_{js_j\nu} = F_{\nu}$ for $0 \leq \nu \leq l_j$. Then, by (10.3) and (10.11),

$$a(qz) F_{l_j}(qz) \cdot G(z) - F_{l_j}(z) \cdot q^{s_j} \alpha_j(qz)^s G(qz) = 0$$

can be expressed in the form similar to (10.11), with a nonnegative integer L' corresponding to L in this expression. Since the coefficient of $f_{l_j}(z)$ in this expression is zero, L' is smaller than L. Let us show that the coefficient of $f_{l_j-1}(z)$, say $c(z)$, is not zero. This will contradict the minimality of L and prove Lemma 10.2. In fact, by setting $h(z) = F_{l_j-1}(z)/F_{l_j}(z)$, we have

$$\begin{aligned} c(z) &= a(qz)(F_{l_j}(qz)F_{l_j-1}(z) + l_jF_{l_j}(qz)F_{l_j}(z) - F_{l_j-1}(qz)F_{l_j}(z)) \\ &= a(qz)F_{l_j}(z)F_{l_j}(qz)(h(z) + l_j - h(qz)), \end{aligned}$$

where $h(z) + l_j - h(qz) \neq 0$ since $l_j \neq 0$. This completes the proof of Lemma 10.2.

REMARK 10.1. Let $A(x)$ and $B(x)$ be any polynomials. Then the q-series

$$G(y) = \sum_{n=0}^{\infty} y^n \prod_{k=0}^{n-1} \frac{B(q^k)}{A(q^k)}$$

satisfies the q-difference equation

$$\{A(J/q) - yB(J)\}G(y) = A(1/q), \tag{10.12}$$

where $J = J_y$. Put now

$$B(x) = (qzx)^s, \qquad A(x) = a_0 + a_1(qzx) + \cdots + a_t(qzx)^t,$$

where $s \in \mathbb{Z}^+$, $t \leq s$, and $a_0a_t \neq 0$. Then

$$G(y) = \sum_{n=0}^{\infty} y^n \prod_{k=0}^{n-1} \frac{(zq^{k+1})^s}{a(zq^{k+1})}$$

satisfies, by (10.12),

$$a_0G(y) + a_1zG(qy) + \cdots + a_tz^tG(q^ty) - y(qz)^sG(q^sy) = a(z), \tag{10.13}$$

where $a(z) = a_0 + a_1z + \cdots + a_tz^t$. If for any $\alpha \in \mathbb{K}^*$ we define

$$g_i(z) = G(q^i\alpha) = \sum_{n=0}^{\infty} \frac{q^{s\binom{n+1}{2}}z^{sn}}{a(qz)\cdots a(q^nz)}\,(q^i\alpha)^n, \qquad i = 0, 1, \ldots, s, \tag{10.14}$$

then (10.13) implies

$$\sum_{i=0}^{t} a_iz^ig_i(z) - \alpha(qz)^sg_s(z) = a(z).$$

Hence the functions 1 and (10.14) are linearly dependent over $\mathbb{K}(z)$. In particular, if $t = s$ and $\alpha q^s = a_s$, then $1, g_0, g_1, \ldots, g_{s-1}$ are linearly dependent over $\mathbb{K}(z)$ and satisfy

$$\sum_{i=0}^{s-1} a_iz^ig_i(z) = a(z).$$

11. Appendix. In this appendix, for the convenience of the readers, we supply the proof of the following fundamental Lemma 11.1 on a linear homogeneous difference system and its Corollary 11.1 for a linear homogeneous difference equation; the latter has been used in the proof of Lemma 4.1.

LEMMA 11.1. *Let (F, ϕ) be a difference field whose constant field is C, and let $\phi(\overline{y}) = P\overline{y}$ be a linear homogeneous difference system, where $P \in \mathrm{GL}(m, F)$. Then a set $\overline{y}_1, \ldots, \overline{y}_r \in F^m$ of solutions of the system is linearly dependent over F if and only if it is linearly dependent over C.*

Proof. Assume that the solutions $\overline{y}_1, \ldots, \overline{y}_r \in F^m$ of the system are linearly dependent over F. We shall show that they are linearly dependent over C. We may assume that any $r-1$ vectors among $\overline{y}_i$'s are linearly independent over F. Let $\sum d_i \overline{y}_i = \overline{0}$ be a linear dependence relation over F, where we may assume that $d_1 = 1$. Making ϕ act on both sides of this relation, we have by the difference system $P \sum \phi(d_i)\overline{y}_i = \overline{0}$. Since P is nonsingular, we obtain $\sum \phi(d_i)\overline{y}_i = \overline{0}$. We then subtract the resulting relation from the original one to get $\sum_{i=2}^{r}(d_i - \phi(d_i))\overline{y}_i = \overline{0}$, which means $\phi(d_i) = d_i$ by the assumption on $\overline{y}_i$'s. This shows our desired assertion. Thus the lemma is proved.

The following corollary to Lemma 11.1 on a linear homogeneous difference equation is an analogue of Lemma 1.2 in Chapter III, §1 of [DGS] on a linear homogeneous differential equation.

COROLLARY 11.1. *Let (F, ϕ) and C be as in Lemma 11.1, and let $L = \sum_{k=0}^{r} d_k \phi^k$ be a linear homogeneous difference operator over F, where $d_i \in F$ with $d_r \neq 0$. Then the dimension of the set of solutions $y \in F$ of $Ly = 0$ as a C-vector space is at most r.*

Proof. Take the maximum integer s for which $d_{r-s} \neq 0$, and define a linear homogeneous difference operator $\widetilde{L}$ over F by

$$\widetilde{L} = \sum_{k=0}^{s} d_{r-k}\phi^{s-k}.$$

Then the set of solutions of $Ly = 0$ is exactly that of $\widetilde{L}y = 0$. We may assume $d_r = 1$; then for any solution $y \in F$ of $\widetilde{L}y = 0$, the vector $\overline{y}$ defined by $\overline{y} = {}^t(y, \phi(y), \ldots, \phi^{s-1}(y))$ satisfies $\phi(\overline{y}) = P\overline{y}$, where

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_{r-s} & -d_{r-s+1} & \cdots & -d_{r-1} \end{pmatrix}.$$

Since $P \in \mathrm{GL}(s, F)$, and since y is the first component of $\overline{y}$, the assertion of the corollary follows directly from Lemma 11.1.

REMARK 11.1. Both Lemma 11.1 and its corollary remain valid for a pair (F, ϕ) with F a field and ϕ an endomorphism of F, under the notations used in difference fields. Though there is no change in the proofs, we note a point in the proof of Corollary 11.1. Namely, to ensure that the set of solutions of $Ly = 0$ is exactly that of $\widetilde{L}y = 0$, we use the inverse of the isomorphism ϕ^{r-s} from F to $\phi^{r-s}(F)$.

Note added in proof. Daniel Bertrand has informed us about his recent work [Be], where he obtains independently a result (Corollary of Theorem 1) analogous to our Theorem 2.1.

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