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On Fleck quotients

by

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1. Introduction and main results. Let $m \in \mathbb{Z}^+ = \{1, 2, \ldots\}$, $n \in \mathbb{N} = \{0, 1, \ldots\}$ and $r \in \mathbb{Z}$, and define

$$C_m(n,r) = \sum_{k \equiv r \pmod{m}} \binom{n}{k} (-1)^k. \tag{1.0}$$

This sum has been studied by various authors and many applications have been found (cf. [S02] and the references therein). The following well-known observation is fundamental:

$$mC_m(n,r) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{\gamma^m=1} \gamma^{k-r} = \sum_{\gamma^m=1} \gamma^{-r}(1-\gamma)^n.$$

Note that

$$C_m(n+1,r) = C_m(n,r) - C_m(n,r-1)$$

since $x^{-r}(1-x)^{n+1} = x^{-r}(1-x)^n - x^{-r+1}(1-x)^n$.

Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. In 1913 A. Fleck (cf. [D, p. 274]) showed that

$$\mathrm{ord}_p(C_p(n,r)) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor,$$

where $\mathrm{ord}_p(\alpha)$ denotes the p-adic order of a p-adic number α, and $\lfloor\cdot\rfloor$ is the well-known floor function. Fleck's result is fundamental in the recent investigation of the ψ-operator related to Fontaine's theory, Iwasawa's theory, and p-adic Langlands correspondence (cf. [Co], [SW] and [W]); it also plays an indispensable role in Davis and Sun's study of homotopy exponents of special unitary groups (cf. [DS] and [SD]). In this paper we are interested

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in the *Fleck quotient*

$$F_p(n,r) := (-p)^{-\lfloor (n-1)/(p-1)\rfloor} C_p(n,r) + [\![n=0]\!]. \tag{1.1}$$

(Throughout this paper, for an assertion A we let $[\![A]\!]$ take the value 1 or 0 according as A holds or not.)

For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we use $\{a\}_m$ to denote the least nonnegative residue of a mod m (thus $\{a\}_m/m$ is the fractional part $\{a/m\}$ of a/m). For a prime p and an integer a, we define $q_p(a) = (a^{p-1}-1)/p$, which is an integer if $a \not\equiv 0 \pmod p$.

By a number-theoretic approach related to Gauss sums, we establish the following explicit result.

Theorem 1.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n_0 = \{n\}_p$ and $n_1 = \{n_0 - n\}_{p-1} = \{-\lfloor n/p \rfloor\}_{p-1}$. If $n_0 \le n_1$, then*

$$F_p(n,r) \equiv \frac{(-1)^{n_1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} \pmod p. \tag{1.2}$$

If $n_0 > n_1 = 0$, then

$$F_p(n,r) \equiv (-1)^{\{r\}_p} \binom{n_0}{\{r\}_p} \pmod p. \tag{1.3}$$

If $n_0 > n_1 > 0$, then

$$F_p(n,r) \equiv \frac{(-1)^{n_1-1}}{(n_1-1)!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} q_p(k-r) \pmod p. \tag{1.4}$$

Corollary 1.1. *Let p be a prime and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$F_p(pn,r) \equiv \frac{r^{n^*}}{n^*!} \pmod p \tag{1.5}$$

where $n^* = \{-n\}_{p-1}$. Consequently,

$$F_p\left(p\,\frac{p-1}{2}, r\right) \equiv \begin{cases} (-1)^{(h(-p)+1)/2}\left(\frac{r}{p}\right) \pmod p & \text{if } p \ne 3 \text{ and } 4 \mid p+1, \\ (-1)^{(h(p)-1)/2}\left(\frac{r}{p}\right)\frac{v}{2} \pmod p & \text{if } 4 \mid p-1, \end{cases} \tag{1.6}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, and $h(-p)$ and $h(p)$ are the class numbers of the quadratic fields $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{p})$ respectively, and for $p \equiv 1 \pmod 4$ we write the fundamental unit of $\mathbb{Q}(\sqrt{p})$ in the form $(v + u\sqrt{p})/2$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod 2$.

Proof. Note that $\{pn\}_p = 0$. By Theorem 1.1,

$$F_p(pn,r) \equiv \frac{(-1)^{n^*}}{n^*!}\sum_{k=0}^{0}\binom{0}{k}(-1)^k(k-r)^{n^*} = \frac{r^{n^*}}{n^*!} \pmod{p}.$$

When $p \neq 2$ and $n = (p-1)/2$, we have $n^* = (p-1)/2$ and hence

$$\begin{aligned} F_p\left(p\,\frac{p-1}{2}, r\right) &\equiv r^{(p-1)/2}(-1)^{(p-1)/2}\,\frac{((p-1)/2)!}{\prod_{k=1}^{(p-1)/2} k(p-k)} \\ &\equiv \left(\frac{r}{p}\right)(-1)^{(p-1)/2}\,\frac{((p-1)/2)!}{(p-1)!} \qquad \text{(by Euler's criterion)} \\ &\equiv (-1)^{(p+1)/2}\left(\frac{r}{p}\right)\frac{p-1}{2}! \pmod{p} \quad \text{(by Wilson's theorem).} \end{aligned}$$

If $p > 3$ and $p \equiv 3 \pmod 4$, then

$$\frac{p-1}{2}! \equiv (-1)^{(h(-p)+1)/2} \pmod{p}$$

by a result of L. J. Mordell [M]. If $p \equiv 1 \pmod 4$ and $\varepsilon_p = (v + u\sqrt{p})/2 > 1$ is the fundamental unit of $\mathbb{Q}(\sqrt{p})$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod 2$, then by S. Chowla [C] we have

$$\frac{p-1}{2}! \equiv (-1)^{(h(p)+1)/2}\,\frac{v}{2} \pmod{p}.$$

Combining the above we immediately obtain (1.6). ∎

Remark. Let n be a positive integer and $p > 2n+1$ be a prime. By the first part of Corollary 1.1 in the case $r = 0$, we have

$$\binom{2pn}{pn}(-1)^n + 2\sum_{k=0}^{n-1}\binom{2pn}{pk}(-1)^k = \sum_{k=0}^{2n}\binom{2pn}{pk}(-1)^{pk} \equiv 0 \pmod{p^{2n+1}}$$

and hence

$$\binom{2pn-1}{pn-1} = \frac{1}{2}\binom{2pn}{pn} \equiv \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2pn}{pk} \pmod{p^{2n+1}}. \tag{1.7}$$

When $n = 1$ and $p > 3$, this gives the Wolstenholme congruence

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

When $n = 2$ and $p > 5$, (1.7) yields the following new congruence:

$$\binom{4p-1}{2p-1} = \frac{1}{2}\binom{4p}{2p} \equiv \binom{4p}{p} - 1 \pmod{p^5}.$$

Our second approach to Fleck quotients is of combinatorial nature. It involves Stirling numbers of the second kind as well as higher-order Bernoulli polynomials.

Let $n \in \mathbb{N}$. The Stirling numbers $S(n,k)$ ($k \in \mathbb{N}$) of the second kind are given by

$$x^n = \sum_{k\in\mathbb{N}} S(n,k)(x)_k,$$

where

$$(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x-1)\cdots(x-k+1) \quad \text{for } k = 1,2,\ldots.$$

Clearly, $S(n,n) = 1$, and $S(n,k) = 0$ if $k > n$. When $n+k > 0$, $S(n,k)$ is actually the number of ways to partition a set of cardinality n into k nonempty subsets. Here is an explicit formula (cf. [LW, p. 126]) for Stirling numbers of the second kind:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n.$$

As $S(i,k) = 0$ for all those $i \in \mathbb{N}$ with $i < k$, we have *Euler's identity*

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j P(j) = 0,$$

where $P(x)$ is any polynomial with $\deg P < k$ having complex number coefficients. It is known (cf. [LW, p. 126]) that

$$\sum_{n=k}^{\infty} S(n,k)\,\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!};$$

in other words,

$$(e^x-1)^k = \sum_{n=k}^{\infty}\overline{S}(n,k)x^n \quad \text{with} \quad \overline{S}(n,k) = \frac{k!}{n!}\,S(n,k).$$

For $m = 0,1,\ldots$, the *mth order Bernoulli polynomials* $B_n^{(m)}(t)$ ($n \in \mathbb{N}$) are defined by

$$\frac{x^m e^{tx}}{(e^x-1)^m} = \sum_{n=0}^{\infty} B_n^{(m)}(t)\,\frac{x^n}{n!}, \tag{1.8}$$

and those $B_n^{(m)} = B_n^{(m)}(0)$ are called the *mth order Bernoulli numbers*. The usual Bernoulli polynomials and numbers are $B_n(t) = B_n^{(1)}(t)$ and $B_n = B_n(0) = B_n^{(1)}$ respectively. (It is well known that $B_0 = 1$, $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k = 1,2,\ldots$; the reader may consult [IR, pp. 228–248] for the basic properties of Bernoulli numbers.) For a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we use $[x^n]f(x)$ to denote the coefficient a_n of the monomial x^n in $f(x)$. Thus

$$\begin{aligned}B_n^{(m)}(t) &= [x^n]n!\left(\frac{x}{e^x-1}\right)^m e^{tx}\\ &= [x^n]n!\sum_{k=0}^{\infty}B_k^{(m)}\frac{x^k}{k!}\sum_{j=0}^{\infty}\frac{(tx)^j}{j!} = \sum_{k=0}^{n}\binom{n}{k}B_k^{(m)}t^{n-k}.\end{aligned}$$

It is also easy to verify that $B_n^{(m)}(m-t)=(-1)^nB_n^{(m)}(t)$, and

$$\frac{B_n^{(m)}(t)}{n!} = \sum_{k_0+\dots+k_{m-1}=n}\frac{B_{k_0}(t)}{k_0!}\prod_{0<i<m}\frac{B_{k_i}}{k_i!}\quad \text{provided } m>0.$$

If $0\le n<p-1$, then $B_0,\dots,B_n$ are p-adic integers by the von Staudt–Clausen theorem (cf. [IR, p. 233]) or the recurrence $\sum_{k=0}^{l}\binom{l+1}{k}B_k=0$ $(l=1,2,\dots)$, therefore $B_n^{(m)}(t)\in\mathbb{Z}_p[t]$ where $\mathbb{Z}_p$ is the ring of p-adic integers.

Our discovery of the next theorem was actually motivated by Theorem 1.1.

THEOREM 1.2. *Let p be a prime, and let $n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Set $n^*=\{-n\}_{p-1}$. For any integer $m\equiv n \pmod p$, if $m\ge 0$ then $(-1)^nF_p(n,r)$ is congruent to*

$$\begin{aligned}(1.9)\qquad \sum_{k=0}^{n^*}\overline{S}(n^*-k+m,m)\frac{(-r)^k}{k!} &= \sum_{k=0}^{n^*}\overline{S}(m+n^*,m+k)\binom{-r}{k}\\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{(k-r)^{m+n^*}}{(m+n^*)!}\end{aligned}$$

modulo p; if $m\le 0$ then we have

$$\begin{aligned}(1.10)\qquad & F_p(n,r)\\ &\equiv \frac{(-1)^{n^*}}{n^*!}B_{n^*}^{(-m)}(-r)\equiv -(p-1-n^*)!B_{n^*}^{(-m)}(-r) \pmod p.\end{aligned}$$

The following consequence determines $B_n^{(m)}(a)$ modulo a prime p for $m\in\{1,\dots,p\}$, $n\in\{0,\dots,p-2\}$ and $a\in\mathbb{Z}$.

COROLLARY 1.2. *Let p be a prime and $r\in\mathbb{Z}$. Let $n_0\in\{0,\dots,p-1\}$ and $n_1\in\{0,\dots,p-2\}$. If $n_0\le n_1$, then*

$$(1.11)\qquad B_{n_1-n_0}^{(p-n_0)}(-r)\equiv\frac{1}{(n_1)_{n_0}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^{n_0-k}(k-r)^{n_1} \pmod p.$$

If $n_0>n_1=0$, then

$$(1.12)\qquad B_{p-n_0+n_1-1}^{(p-n_0)}(-r)\equiv\frac{(-1)^{\{r\}_p-1}}{n_0!}\binom{n_0}{\{r\}_p} \pmod p.$$

If $n_0 > n_1 > 0$, *then*

$$(1.13)\quad B^{(p-n_0)}_{p-n_0+n_1-1}(-r) \equiv \frac{(-1)^{n_1}}{(n_0-n_1)!(n_1-1)!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{n_1}q_p(k-r) \pmod p.$$

Proof. Let n be a nonnegative integer with $n \equiv n_0 - pn_1 \pmod{p(p-1)}$. Applying (1.10) with $m = n_0 - p$ we obtain

$$F_p(n,r) \equiv \frac{(-1)^{n^*}}{n^*!}B^{(p-n_0)}_{n^*}(-r) \equiv -(p-1-n^*)!B^{(p-n_0)}_{n^*}(-r) \pmod p,$$

where $n^* = \{-n\}_{p-1}$.

If $n_0 \le n_1$, then $n^* = n_1 - n_0$ and hence

$$B^{(p-n_0)}_{n_1-n_0}(-r) \equiv (-1)^{n_1-n_0}(n_1-n_0)!F_p(n,r) \pmod p,$$

which implies (1.11) with the help of (1.2).

Now we consider the case $n_0 > n_1$. Clearly $n^* = n_1 - n_0 + p - 1$ and $p - 1 - n^* = n_0 - n_1$. Therefore

$$F_p(n,r) \equiv -(n_0-n_1)!B^{(p-n_0)}_{n_1-n_0+p-1}(-r) \pmod p.$$

The case $n_1 = 0$ of this, together with (1.3), yields (1.12). When $n_1 > 0$, combining the last congruence with (1.4) we obtain (1.13). ∎

Corollary 1.3. *Let p be a prime and let $n \in \mathbb{Z}^+$. Then* $\mathrm{ord}_p(C_p(n,r)) = \lfloor (n-1)/(p-1)\rfloor$ *for at least $p - n^* \ge 2$ values of $r \in \{0,\dots,p-1\}$, where $n^* = \{-n\}_{p-1}$.*

Proof. For any $r \in \mathbb{Z}$, $\mathrm{ord}_p(C_p(n,r)) = \lfloor (n-1)/(p-1)\rfloor$ if and only if $F_p(n,r) \not\equiv 0 \pmod p$. By Theorem 1.2,

$$F_p(n,r) \equiv \frac{(-1)^{n^*}}{n^*!}B^{(p-\{n\}_p)}_{n^*}(-r) \pmod p \quad \text{for all } r = 0,\dots,p-1.$$

Recall that $B^{(p-\{n\}_p)}_{n^*}(x) \in \mathbb{Z}_p[x]$ is monic and of degree n^*. Also, a polynomial of degree n^* over the field $\mathbb{Z}/p\mathbb{Z}$ cannot have more than n^* distinct zeroes in the field (cf. [IR, p. 39]). So the congruence equation $F_p(n,r) \equiv 0 \pmod p$ has at most n^* solutions with $r \in \{0,\dots,p-1\}$. This yields the desired result. ∎

Corollary 1.4. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Then*

$$(1.14)\quad (-1)^n F_p(n,0) \equiv \overline{S}(n^* + \{n\}_p, \{n\}_p) \equiv \frac{B^{(m)}_{n^*}}{n^*!} \pmod p,$$

where m is any nonnegative integer with $m + n \equiv 0 \pmod p$. Also, (1.15) $$(-1)^n F_p(pn+p-1,r) \equiv \frac{B_{n^*}(-r)}{n^*!} \equiv -(p-1-n^*)!B_{n^*}(r+1) \pmod p$$

for all $r \in \mathbb{Z}$, and in particular

(1.16) $$\binom{2p-1}{p+r} + (-1)^p \binom{2p-1}{r} \equiv (-1)^r p^2 B_{p-2}(-r) \pmod{p^3}$$

for every $r = 0, \ldots, p-1$.

Proof. Applying Theorem 1.2 with $r = 0$ we immediately get (1.14).

As $pn+p-1 \equiv -1 \pmod p$ and $n^* = \{-(pn+p-1)\}_{p-1}$, by the second part of Theorem 1.2 and the identity $(-1)^{n^*} B_{n^*}(x) = B_{n^*}(1-x)$, whenever $r \in \mathbb{Z}$ we have

$$\begin{aligned}(-1)^{n^*} F_p(pn+p-1,r) \equiv \frac{B_{n^*}(-r)}{n^*!} &\equiv (-1)^{n^*+1}(p-1-n^*)!B_{n^*}(-r) \\ &\equiv -(p-1-n^*)!B_{n^*}(r+1) \pmod p\end{aligned}$$

and hence (1.15) holds.

Now let $r \in \{0, \ldots, p-1\}$. By (1.15) in the case $n = 1$,

$$-F_p(2p-1,r) \equiv -(p-1-(p-2))!B_{p-2}(r+1) \pmod p$$

and hence

$$F_p(2p-1,r) \equiv B_{p-2}(1-(-r)) = (-1)^{p-2} B_{p-2}(-r) \pmod p,$$

which is equivalent to (1.16). ∎

Let p be an odd prime, and let h_p and h_p^+ denote the class numbers of the cyclotomic field $\mathbb{Q}(\zeta_p)$ and its maximal real subfield $\mathbb{Q}(\zeta_p+\zeta_p^{-1})$ respectively, where ζ_p is a primitive pth root of unity in the complex field $\mathbb{C}$. It is well known that $h_p^- = h_p/h_p^+$ is an integer. If p divides none of the numerators of the Bernoulli numbers $B_0, B_2, \ldots, B_{p-3} \in \mathbb{Z}_p$, then p is said to be a *regular* prime. In 1850 E. Kummer proved that

$$\begin{aligned} p \nmid h_p \ &\Leftrightarrow \ p \nmid h_p^- \ \Leftrightarrow \ p \text{ is regular} \\ &\Rightarrow \ x^p + y^p = z^p \text{ has no integer solution with } xyz \neq 0. \end{aligned}$$

Furthermore,

$$h_p^- \equiv \prod_{0<n\le(p-3)/2} \left(-\frac{B_{2n}}{4n}\right) \pmod p$$

by the proof of Theorem 5.16 in [Wa, p. 62].

COROLLARY 1.5. *Let p be a prime.*

(i) *For every $n = 2, \ldots, p$ we have*

$$\sum_{k=1}^{n}(-1)^{pk-1}\binom{pn-1}{pk-1} \equiv (n-1)!B_{p-n}p^n \pmod{p^{n+1}}. \tag{1.17}$$

(ii) *Suppose that $p > 3$. Then p does not divide the class number h_p of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ if and only if*

$$\mathrm{ord}_p\bigg(\sum_{k=1}^{n}(-1)^k\binom{pn-1}{pk-1}\bigg) = n \quad \textit{for all } n = 3, 5, \ldots, p-2.$$

Also,

$$\sum_{k=1}^{(p-1)/2}(-1)^{k-1}\binom{p(p-1)/2-1}{pk-1} \equiv [\![4 \mid p+1]\!](-1)^{(h(-p)+1)/2}h(-p)p^{(p-1)/2} \pmod{p^{(p+1)/2}}, \tag{1.18}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Proof. (i) Let $n \in \{2, \ldots, p\}$. Then $\lfloor (pn-1-1)/(p-1) \rfloor = n$ and hence

$$F_p(pn-1,-1) = (-p)^{-n}C_p(pn-1,-1) = (-p)^{-n}\sum_{k=1}^{n}\binom{pn-1}{pk-1}(-1)^{pk-1}.$$

By Corollary 1.4, $(-1)^nF_p(pn-1,-1)$ is congruent to

$$(p-1-\{-(n-1)\}_{p-1})!B_{\{-(n-1)\}_{p-1}}(-1+1) = (n-1)!B_{p-n}$$

modulo p. Therefore (1.17) holds.

(ii) In view of part (i),

$$\begin{aligned}\mathrm{ord}_p\bigg(\sum_{k=1}^{n}(-1)^k\binom{pn-1}{pk-1}\bigg) = n \quad &\text{for } n = 3, 5, \ldots, p-2\\ \Leftrightarrow\ B_{p-n} \not\equiv 0 \pmod{p} \quad &\text{for } n = 3, 5, \ldots, p-2\\ \Leftrightarrow\ p \text{ is regular } \Leftrightarrow\ h_p \not\equiv 0 &\pmod{p}.\end{aligned}$$

Taking $n = (p-1)/2$ in (1.17) we get

$$\sum_{k=1}^{(p-1)/2}(-1)^{k-1}\binom{p(p-1)/2-1}{pk-1} \equiv \frac{((p-1)/2)!}{(p-1)/2}\,p^{(p-1)/2}B_{(p+1)/2} \pmod{p^{(p+1)/2}}.$$

If $p \equiv 1 \pmod 4$, then $B_{(p+1)/2} = 0$ since $(p+1)/2 \in \{3, 5, \ldots\}$. If $p \equiv 3 \pmod 4$, then we have $h(-p) \equiv -2B_{(p+1)/2} \pmod{p}$ (cf. [IR, p. 238]), and

$((p-1)/2)! \equiv (-1)^{(h(-p)+1)/2} \pmod p$ by Mordell [M]. So (1.18) follows from the above. ∎

REMARK. Let p be an odd prime. If $p \ge 5$, then (1.17) in the case $n=2$ reduces to Wolstenholme's congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ since $B_{p-2}=0$. Taking $n=3$ in (1.17) we get

$$\binom{3p-1}{p-1} - \binom{3p-1}{2p-1} + \binom{3p-1}{3p-1} \equiv 2B_{p-3}p^3 \pmod{p^4};$$

as $\binom{3p-1}{2p-1} = 2\binom{3p-1}{p-1}$ this yields the congruence

$$\binom{3p-1}{p-1} \equiv 1 - 2p^3 B_{p-3} \pmod{p^4}.$$

This was first obtained by J. W. L. Glaisher (cf. [G1, p. 21] and [G2, p. 323]) who showed that

$$\binom{pn-1}{p-1} \equiv 1 - \frac{n(n-1)}{3}\, p^3 B_{p-3} \pmod{p^4} \quad \text{for } n=1,2,\ldots.$$

COROLLARY 1.6. *Let p be an odd prime, and let $n \in \{3,\ldots,p\}$ and $r \in \mathbb{Z}$. Then*

$$F_p(pn-2,r) \equiv -n!\left(\frac{B_{p-n+1}(-r)}{n-1} + (r+1)\,\frac{B_{p-n}(-r)}{n}\right) \pmod p. \tag{1.19}$$

Proof. Clearly $\{-(pn-2)\}_{p-1} = p-n+1$. By Theorem 1.2, $F_p(pn-2,r)$ is congruent to

$$-(p-1-(p-n+1))!B^{(2)}_{p-n+1}(-r) = -(n-2)!B^{(2)}_{p-n+1}(-r)$$

modulo p.

Let $m=p-n+1$. By [PS, (2.14)] or [SP, (1.12)],

$$\begin{aligned}
&\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_kB_{m-k}(x) - \frac{B_m(1-x)}{m}\,B_0\\
&\quad= -\sum_{k=0}^{1}\binom{1}{k}B_{1-k}(x)B_{m-1+k}(1-x) - B_1B_{m-1}(1-x)\\
&\quad= -B_1(x)B_{m-1}(1-x) - B_0(x)B_m(1-x) - B_1B_{m-1}(1-x)\\
&\quad= (-1)^m((B_1(x)+B_1)B_{m-1}(x) - B_m(x))\\
&\quad= (-1)^m((x-1)B_{m-1}(x) - B_m(x)).
\end{aligned}$$

It follows that

$$\begin{aligned}B_m^{(2)}(-r) &= \sum_{k=0}^{m}\binom{m}{k}B_kB_{m-k}(-r)\\ &= (1-m)B_m(-r)+m(-r-1)B_{m-1}(-r)\\ &\equiv (1+n-1)B_{p-n+1}(-r)-(r+1)(-n+1)B_{p-n}(-r)\\ &\equiv n(n-1)\left(\frac{B_{p-n+1}(-r)}{n-1}+(r+1)\,\frac{B_{p-n}(-r)}{n}\right)\ (\operatorname{mod} p).\end{aligned}$$

Combining the above we immediately obtain (1.19). ■

By Theorem 1.1 or 1.2, for any prime p the Fleck quotient $F_p(n,r)$ (with $n\in\mathbb{N}$ and $r\in\mathbb{Z}$) modulo p only depends on p and r and the remainder of n modulo $p(p-1)$. This observation can be further extended as follows.

THEOREM 1.3. *Let p be a prime, and let $a,l,n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kF_p(kp^a(p-1)+l,r)\equiv 0\ (\operatorname{mod} p^{an+\lceil (n-l^*)/(p-1)\rceil}), \tag{1.20}$$

where $l^*=\{-l\}_{p-1}$ and $\lceil\cdot\rceil$ is the ceiling function.

The following consequence is somewhat similar to Kummer's congruence for Bernoulli numbers (cf. [IR, pp. 238–241]).

COROLLARY 1.7. *Let p be a prime, and let $a,l\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then*

$$\begin{aligned}F_p(p^a(p-1)+l,r)&\equiv F_p(l,r)\ (\operatorname{mod} p^a),\\ F_p(2p^a(p-1)+l,r)&\equiv 2F_p(p^a(p-1)+l,r)-F_p(l,r)\ (\operatorname{mod} p^{2a}),\\ F_p(3p^a(p-1)+l,r)&\equiv 3F_p(2p^a(p-1)+l,r)-3F_p(p^a(p-1)+l,r)\\ &\qquad +F_p(l,r)\ (\operatorname{mod} p^{3a}).\end{aligned}$$

Proof. Simply apply (1.20) with $n=1,2,3$. ■

Let p be a prime, and let $a\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$. In 1977 C. S. Weisman [We] extended Fleck's result by showing that if $n\geq p^{a-1}$ then

$$C_{p^a}(n,r)\equiv 0\ (\operatorname{mod} p^{\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor}),$$

where φ is Euler's totient function. In view of this, we define the *generalized Fleck quotient*

$$F_{p^a}(n,r)=(-p)^{-\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor}C_{p^a}(n,r)+[\![n<p^{a-1}]\!]\in\mathbb{Z}.$$

Note that $F_{p^a}(n,r)\equiv 1\ (\operatorname{mod} p)$ for $n=0,\dots,p^{a-1}-1$.

Theorem 1.4. *Let p be a prime, and let $a, n \in \mathbb{Z}^+$ with $n \ge p^{a-1}$.*

(i) *For any $r \in \mathbb{Z}$ we have*

$$F_{p^a}(n, r) \equiv \sum_{k=0}^{d} \binom{r+k-1}{k} F_{p^a}(n+k, 0) \pmod{p}, \tag{1.21}$$

where $d = \{p^{a-1} - 1 - n\}_{\varphi(p^a)}$ is the least nonnegative integer with $n + d \equiv p^{a-1} - 1 \pmod{\varphi(p^a)}$.

(ii) *We have*

$$\operatorname{ord}_p\left(C_{p^a}(n, r)\right) = \left\lfloor \frac{n - p^{a-1}}{\varphi(p^a)} \right\rfloor \quad (\text{i.e., } p \nmid F_{p^a}(n, r)) \text{ for some } r \in \mathbb{Z}. \tag{1.22}$$

If $n \ge 2p^{a-1}$, then

$$F_{p^a}(n + p^a(p-1), r) \equiv F_{p^a}(n, r) \pmod{p} \quad \text{for all } r \in \mathbb{Z}. \tag{1.23}$$

In view of the first congruence in Corollary 1.7 and the last congruence in Theorem 1.4, we propose the following conjecture.

Conjecture 1.1. *Let p be a prime, and let $a, b, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. If $n \ge 2p^{a+b-2}$, then*

$$F_{p^a}(n + \varphi(p^{a+b}), r) \equiv F_{p^a}(n, r) \pmod{p^b}.$$

Theorems 1.1, 1.2 and 1.3 will be proved in Sections 2, 3 and 4 respectively. In Section 5 we will first give a new proof of Weisman's congruence via roots of unity, and then establish Theorem 1.4.

2. Proof of Theorem 1.1

Lemma 2.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Define $G(n) = \sum_{a=1}^{p-1} a^n \zeta_p^a$ and $\pi = 1 - \zeta_p$, where ζ_p is a primitive pth root of unity in the complex field $\mathbb{C}$. Then*

$$G(n) \equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}, \tag{2.1}$$

where $s(m, 0), \ldots, s(m, m)$ are Stirling numbers of the first kind defined by $(x)_m = \sum_{k=0}^{m} (-1)^{m-k} s(m, k) x^k$.

Proof. Clearly,

$$\begin{aligned} G(n) &= \sum_{a=1}^{p-1} a^n (1-\pi)^a = \sum_{a=1}^{p-1} a^n \sum_{m=0}^{a} \binom{a}{m} (-\pi)^m = \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n (a)_m \\ &= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n \sum_{k=0}^{m} (-1)^{m-k} s(m,k) a^k \\ &= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{k=0}^{m} (-1)^{m-k} s(m,k) \sum_{a=1}^{p-1} a^{n+k}. \end{aligned}$$

Since

$$1 + x + \cdots + x^{p-1} = \frac{x^p - 1}{x - 1} = \prod_{a=1}^{p-1} (x - \zeta_p^a),$$

we have

$$\frac{p}{\pi^{p-1}} = \prod_{a=1}^{p-1} \frac{1 - \zeta_p^a}{\pi} = \prod_{a=1}^{p-1} \frac{1 - (1-\pi)^a}{\pi} \equiv \prod_{a=1}^{p-1} a \equiv -1 \pmod{\pi}$$

with the help of Wilson's theorem. Note also that

$$\sum_{a=1}^{p-1} a^{n+k} \equiv -[\![p-1 \mid n+k]\!] \pmod{p}$$

by elementary number theory (see, e.g., [IR, pp. 235–236]). Therefore

$$\begin{aligned} G(n) &\equiv \sum_{m=0}^{p-2} \frac{\pi^m}{m!} \sum_{k=0}^{m} (-1)^k s(m,k) (-[\![k = n^*]\!]) \\ &\equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \, \frac{\pi^m}{m!} \pmod{p}. \ \blacksquare \end{aligned}$$

REMARK. Let p be an odd prime. For each $a \in \mathbb{Z}$ let $\overline{a} = a + p\mathbb{Z} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let ω be the Teichmüller character of the multiplicative group $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{\overline{0}\}$. For $\overline{a} \in \mathbb{F}_p^*$, $\omega(\overline{a})$ is just the $(p-1)$th root of unity in the unique unramified extension of the p-adic field $\mathbb{Q}_p$ with $\omega(\overline{a}) \equiv a \pmod{p}$. (See, e.g., [Wa, p. 51].) If ζ_p is a primitive pth root of unity in the algebraic closure of $\mathbb{Q}_p$, then for $n \in \mathbb{N}$ and $\pi = 1 - \zeta_p$ we have

$$\sum_{a=1}^{p-1} a^n \zeta_p^a \equiv \sum_{a=1}^{p-1} \omega^n(\overline{a}) \zeta_p^a \equiv -\frac{(-\pi)^{n^*}}{n^*!} \pmod{\pi^{n^*+1}}$$

with $n^* = \{-n\}_{p-1}$, by Stickelberger's congruence for Gauss' sums (cf. [BEW, pp. 344–345]).

LEMMA 2.2. *Let p be a prime, and let ζ_p be a primitive pth root of unity in $\mathbb{C}$. Let $n = p^a m + n_0 > 0$ with $a \in \mathbb{Z}^+$ and $m, n_0 \in \mathbb{N}$. Then for any $r \in \mathbb{Z}$ we have*

$$\begin{aligned}&\pi^{-p^a m} C_p(n,r) - [\![p-1 \mid m]\!] C_p(n_0,r)\\ &\quad\equiv \frac{G(p^a m)}{p} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{p^a m^*} \pmod{p^{a-1}\pi^{\min\{n_0+1,\, p-1\}}},\end{aligned}$$

where $\pi = 1 - \zeta_p$ and $m^* = \{-m\}_{p-1}$.

Proof. Let $j \in \{1, \ldots, p-1\}$. Then

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^m = \left(\frac{1-(1-\pi)^j}{\pi}\right)^m = \left(\sum_{i=1}^{j} \binom{j}{i} (-\pi)^{i-1}\right)^m = j^m + \beta_j \pi,$$

where β_j is a suitable element in the ring $\overline{\mathbb{Z}}$ of algebraic integers. For $i = 0, 1, \ldots,$ if

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^i m} = j^{p^i m} + p^i \pi \beta_j^{(i)}$$

for some $\beta_j^{(i)} \in \overline{\mathbb{Z}}$, then

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^{i+1} m} = (j^{p^i m} + p^i \pi \beta_j^{(i)})^p = j^{p^{i+1} m} + p^{i+1} \pi \beta_j^{(i+1)}$$

for some $\beta_j^{(i+1)} \in \overline{\mathbb{Z}}$. So

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^a m} \equiv j^{p^a m} \pmod{p^a \pi}.$$

Observe that

$$pC_p(n,r) = \sum_{j=0}^{p-1} \zeta_p^{-jr} (1-\zeta_p^j)^n = \pi^{p^a m} \sum_{j=1}^{p-1} \zeta_p^{-jr} \left(\frac{1-\zeta_p^j}{\pi}\right)^{p^a m} (1-\zeta_p^j)^{n_0}.$$

As π^{n_0} divides $(1-\zeta_p^j)^{n_0}$ in the ring $\overline{\mathbb{Z}}$, by the above $\pi^{-p^a m} pC_p(n,r)$ is congruent to

$$\sum_{j=1}^{p-1} \zeta_p^{-jr} j^{p^a m} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \zeta_p^{jk} = \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k S_{k-r}$$

modulo $p^a \pi^{n_0+1}$, where

$$S_{k-r} = \sum_{j=1}^{p-1} j^{p^a m} \zeta_p^{j(k-r)}.$$

If $k \not\equiv r \pmod p$, then

$$\begin{aligned}S_{k-r} &= (k-r)^{-p^a m}\sum_{j=1}^{p-1}(j(k-r))^{p^a m}\zeta_p^{j(k-r)}\\ &\equiv (k-r)^{p^a m^*}\sum_{t=1}^{p-1}t^{p^a m}\zeta_p^t = (k-r)^{p^a m^*}G(p^a m) \pmod{p^{a+1}}.\end{aligned}$$

(Note that if $j(k-r)\equiv t \pmod p$ then $(j(k-r))^{p^a}\equiv t^{p^a} \pmod{p^{a+1}}$.)

Choose a primitive root g modulo p. Since

$$(g^{p^a m}-1)\sum_{j=1}^{p-1}j^{p^a m} = \sum_{j=1}^{p-1}(gj)^{p^a m} - \sum_{t=1}^{p-1}t^{p^a m} \equiv 0 \pmod{p^{a+1}},$$

if $p-1\nmid m$ then $g^{p^a m}-1 \not\equiv 0 \pmod p$ and so $\sum_{j=1}^{p-1} j^{p^a m} \equiv 0 \pmod{p^{a+1}}$. Thus, when $k\equiv r \pmod p$ we have

$$S_{k-r} = \sum_{j=1}^{p-1}j^{p^a m} \equiv (p-1)[\![p-1\mid m]\!] \pmod{p^{a+1}}.$$

Recall that $p/\pi^{p-1}\equiv -1 \pmod{\pi}$. In view of the above,

$$\begin{aligned}&\pi^{-p^a m}pC_p(n,r) - \sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{p^a m^*}G(p^a m)\\ &\quad\equiv \sum_{\substack{k=0\\ p\mid k-r}}^{n_0}\binom{n_0}{k}(-1)^k([\![p-1\mid m]\!](p-1)-(k-r)^{p^a m^*}G(p^a m))\\ &\quad\equiv C_p(n_0,r)[\![p-1\mid m]\!]p \pmod{p^a\pi^{\min\{n_0+1,\,p-1\}}},\end{aligned}$$

where we have noted that if $p-1\mid m$ (i.e., $m^*=0$) then

$$p-1-G(p^a m) \equiv p - \sum_{t=0}^{p-1}\zeta_p^t = p - \frac{1-\zeta_p^p}{1-\zeta_p} = p \pmod{p^{a+1}}.$$

Therefore the desired congruence follows. ∎

Proof of Theorem 1.1. In the case $n=0$, (1.2) holds since $n_1=n_0=0$ and $F_p(n,r) = -pC_p(0,r)+1$. Below we assume $n>0$.

Let ζ_p be a primitive pth root of unity in $\mathbb{C}$, and set $\pi = 1-\zeta_p$. By Lemma 2.2 in the case $a=1$,

$$\begin{aligned}&\pi^{-p\lfloor n/p\rfloor}C_p(n,r) - [\![n_1=0]\!]C_p(n_0,r)\\ &\quad\equiv \frac{G(p\lfloor n/p\rfloor)}{p}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1} \pmod{\pi^{\min\{n_0+1,\,p-1\}}}.\end{aligned}$$

In view of Lemma 2.1,

$$G\left(p\left\lfloor\frac{n}{p}\right\rfloor\right)\equiv G\left(\left\lfloor\frac{n}{p}\right\rfloor\right)\equiv(-1)^{n_1-1}\sum_{m=n_1}^{p-2}s(m,n_1)\,\frac{\pi^m}{m!}\pmod{p}.$$

If $n_0>n_1$, then

$$\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\equiv\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{n_1}=0\pmod{p},$$

where we have applied Fermat's little theorem and Euler's identity (mentioned in Section 1). Therefore

$$\begin{aligned}&\pi^{-p\lfloor n/p\rfloor}C_p(n,r)-[\![n_1=0]\!]C_p(n_0,r)\\&\qquad\equiv\frac{(-1)^{n_1-1}}{p}\sum_{m=n_1}^{p-2}s(m,n_1)\,\frac{\pi^m}{m!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\\&\qquad\qquad\pmod{\pi^{[\![n_0>n_1]\!]\min\{n_0+1,\,p-1\}}}.\end{aligned}$$

Recall that $-p/\pi^{p-1}\equiv 1\pmod{\pi}$. Since $s(n_1,n_1)=1$ and

$$\frac{p^{[\![n_0\le n_1]\!]}}{\pi^{n_1}}\,\pi^{[\![n_0>n_1]\!]\min\{n_0+1,\,p-1\}}\equiv 0\pmod{\pi},$$

by the above we have

$$\begin{aligned}&\frac{p^{[\![n_0\le n_1]\!]}C_p(n,r)}{\pi^{p\lfloor n/p\rfloor+n_1}}-p^{[\![n_0=0]\!]}[\![n_1=0]\!]C_p(n_0,r)\\&\qquad\equiv\frac{(-1)^{n_1-1}/n_1!}{p^{[\![n_0>n_1]\!]}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\pmod{\pi}.\end{aligned}$$

Note that

$$\left\lfloor\frac{n-1}{p-1}\right\rfloor=\left\lfloor\frac{p\lfloor n/p\rfloor+n_0-1}{p-1}\right\rfloor=\frac{p\lfloor n/p\rfloor+n_1}{p-1}-[\![n_0\le n_1]\!]$$

and hence

$$\begin{aligned}\frac{(-p)^{[\![n_0\le n_1]\!]}C_p(n,r)}{\pi^{p\lfloor n/p\rfloor+n_1}}&=\frac{C_p(n,r)}{(-p)^{\lfloor(n-1)/(p-1)\rfloor}}\left(\frac{-p}{\pi^{p-1}}\right)^{(p\lfloor n/p\rfloor+n_1)/(p-1)}\\&\equiv F_p(n,r)\pmod{\pi}.\end{aligned}$$

In view of the above,

$$\begin{aligned}&(-1)^{[n_0\le n_1]}F_p(n,r)-[\![n_0>n_1=0]\!]C_p(n_0,r)\\&\qquad\equiv\frac{(-1)^{n_1-1}/n_1!}{p^{[\![n_0>n_1]\!]}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\pmod{\pi}.\end{aligned}$$

As the rational p-adic integer

$$D = F_p(n,r) - [\![n_0 > n_1 = 0]\!]C_p(n_0,r) - \frac{(-1)^{n_1}}{(-p)^{[\![n_0>n_1]\!]}\cdot n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}$$

is divisible by π, we have $D^{p-1} \equiv 0 \pmod p$ and hence $D \equiv 0 \pmod p$. Thus

$$F_p(n,r) - [\![n_0 > n_1 = 0]\!]C_p(n_0,r) \equiv \frac{(-1)^{n_1}}{(-p)^{[\![n_0>n_1]\!]}\cdot n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1} \pmod p. \tag{2.2}$$

In the case $n_0 \le n_1$, (2.2) reduces to (1.2). When $n_0 > n_1 = 0$, (2.2) yields (1.3) since $C_p(n_0,r) = (-1)^{\{r\}_p}\binom{n_0}{\{r\}_p}$ and $\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k = (1-1)^{n_0} = 0$.

Now assume that $n_0 > n_1 > 0$. As $\sum_{k=0}^{n_0}\binom{n_0}{k}(k-r)^{n_1} = 0$ by Euler's identity, (2.2) implies that

$$F_p(n,r) \equiv \frac{(-1)^{n_1-1}}{n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k\,\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p} \pmod p.$$

If $n_1 = 1$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p} = (k-r)^{n_1}n_1q_p(k-r);$$

if $n_1 \ge 2$ and $k \equiv r \pmod p$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p} \equiv 0 \equiv (k-r)^{n_1}n_1q_p(k-r) \pmod p;$$

if $a = k-r \not\equiv 0 \pmod p$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p} = a^{n_1}\,\frac{(1+p\cdot q_p(a))^{n_1}-1}{p} \equiv a^{n_1}n_1q_p(a) \pmod p.$$

Therefore (1.4) follows. ∎

3. Proof of Theorem 1.2. The following lemma is a refinement of an induction technique used by Sun [S06].

LEMMA 3.1. *Let p be a prime, and let $n \in \mathbb{N}$ with $n \ge p$. Then*

$$F_p(n,r) \equiv -\sum_{j=1}^{p-1}\frac{1}{j}\sum_{i=0}^{j-1}F_p(n-p+1,r-i) \pmod p. \tag{3.1}$$

Proof. Set $n' = n - (p-1) > 0$. By the Chu–Vandermonde convolution identity (cf. [GKP, (5.27)]),

$$\begin{aligned} F_p(n,r) &= (-p)^{-\lfloor (n-1)/(p-1)\rfloor} \sum_{\substack{0\le k\le n\\ k\equiv r\,(\mathrm{mod}\,p)}} \sum_{j=0}^{k} \binom{p-1}{j}\binom{n'}{k-j}(-1)^k \\ &= -\frac{1}{p}\sum_{j=0}^{p-1}\binom{p-1}{j}(-p)^{-\lfloor (n'-1)/(p-1)\rfloor} \sum_{\substack{j\le k\le n\\ p\mid k-r}} \binom{n'}{k-j}(-1)^k \\ &= -\frac{1}{p}\sum_{j=0}^{p-1}\binom{p-1}{j}(-1)^j F_p(n', r-j). \end{aligned}$$

For any $j = 0,\dots,p-1$, clearly

$$\begin{aligned} \binom{p-1}{j}(-1)^j &= \prod_{0<i\le j}\left(1-\frac{p}{i}\right) \\ &\equiv 1 - \sum_{0<i\le j}\frac{p}{i} \equiv (-1)^{p-1} + p\sum_{j<k<p}\frac{1}{k} \pmod{p^2}. \end{aligned}$$

(Note that $2\sum_{k=1}^{p-1} 1/k = \sum_{k=1}^{p-1}(1/k + 1/(p-k)) \equiv 0 \pmod{p}$.) Also,

$$\sum_{j=0}^{p-1} F_p(n', r-j) = (-p)^{-\lfloor (n'-1)/(p-1)\rfloor}\sum_{k=0}^{n'}\binom{n'}{k}(-1)^k = 0.$$

Therefore

$$F_p(n,r) \equiv -\sum_{j=0}^{p-1}\sum_{j<k<p}\frac{F_p(n', r-j)}{k} = -\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=0}^{k-1}F_p(n', r-j) \pmod{p}.$$

This proves (3.1). ∎

Proof of Theorem 1.2. (i) Suppose $m \ge 0$. Then

$$\begin{aligned} &\sum_{k=0}^{n^*}\overline{S}(m+n^*-k, m)\,\frac{(-r)^k}{k!} \\ &\quad = [x^{m+n^*}]\sum_{l=m}^{\infty}\overline{S}(l,m)x^l\sum_{k=0}^{\infty}\frac{(-rx)^k}{k!} = [x^{m+n^*}](e^x-1)^m e^{-rx} \\ &\quad = [x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{-rx} = [x^{m+n^*}]\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}e^{(k-r)x} \\ &\quad = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,\frac{(k-r)^{m+n^*}}{(m+n^*)!}. \end{aligned}$$

By the identity (2.4) of Sun [S03], for any $l = 0, 1, \ldots$ we have

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+l)^{m+n^*} &= \sum_{j=0}^{l}\binom{l}{j}(m+j)!S(m+n^*, m+j)\\ &= \sum_{j=0}^{n^*}\binom{l}{j}(m+j)!S(m+n^*, m+j).\end{aligned}$$

Thus

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+x)^{m+n^*} = \sum_{j=0}^{n^*}\binom{x}{j}(m+j)!S(m+n^*, m+j)$$

and hence

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{(k-r)^{m+n^*}}{(m+n^*)!} = \sum_{j=0}^{n^*}\binom{-r}{j}\overline{S}(m+n^*, m+j).$$

If $m \le 0$, then

$$\frac{B_{n^*}^{(-m)}(-r)}{n^*!} = [x^{n^*}]\left(\frac{x}{e^x-1}\right)^{-m}e^{-rx} = [x^{n^*}]\left(\frac{e^x-1}{x}\right)^{m}e^{-rx}.$$

Note also that

$$\frac{1}{n^*!} = \frac{\prod_{j=1}^{p-1-n^*}(p-j)}{(p-1)!} \equiv (-1)^{n^*+1}(p-1-n^*)! \pmod{p}$$

by Wilson's theorem.

In view of the above, whether $m \ge 0$ or $m \le 0$, we only need to show that

$$(-1)^n F_p(n, r) \equiv [x^{n^*}]\left(\frac{e^x-1}{x}\right)^{m}e^{-rx} \pmod{p}.$$

(ii) All those formal power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with $a_k \in \mathbb{Q}$ and $a_0, \ldots, a_{n^*} \in \mathbb{Z}_p$ form a ring R_{n^*} under the usual addition and multiplication. In particular, this ring contains

$$e^{-rx} = \sum_{k=0}^{\infty}(-r)^k\frac{x^k}{k!}, \qquad \frac{e^x-1}{x} = \sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}, \qquad \frac{x}{e^x-1} = \sum_{k=0}^{\infty}B_k\frac{x^k}{k!}.$$

(Recall that $n^* < p-1$ and $B_0, \ldots, B_{n^*} \in \mathbb{Z}_p$.) If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and

$g(x)=\sum_{k=0}^\infty b_k x^k$ belong to R_{n^*}, then

$$\begin{aligned}[x^{n^*}]f(x)g(x)^p &= [x^{n^*}]\sum_{j=0}^{n^*} a_j x^j\Big(\sum_{k=0}^{n^*} b_k x^k\Big)^p \\ &\equiv [x^{n^*}]\sum_{j=0}^{n^*} a_j x^j \sum_{k=0}^{n^*} b_k^p x^{pk} = a_{n^*}b_0^p \\ &\equiv [x^{n^*}]f(x)[x^0]g(x) \pmod p.\end{aligned}$$

Consequently, for any $a\in\mathbb{Z}$ we have

$$[x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{ax}\equiv [x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{ax} \pmod p)$$

since $m\equiv n \pmod p$. From this and part (i), it suffices to use induction on n to show that

$$(-1)^n F_p(n,r)\equiv [x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{-rx} \pmod p. \tag{3.2}$$

(iii) Obviously

$$(-1)^0F_p(0,r)=-pC_p(0,r)+1\equiv 1=[x^0]\left(\frac{e^x-1}{x}\right)^0 e^{-rx} \pmod p.$$

So (3.2) holds for $n=0$.

Suppose that $0<n\le p-1$. Then $n^*=p-1-n$ and

$$\begin{aligned}&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{-rx}=[x^{p-1}](e^x-1)^n e^{-rx} \\ &\quad=\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}[x^{p-1}]e^{(k-r)x}=\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}\frac{(k-r)^{p-1}}{(p-1)!} \\ &\quad\equiv(-1)^{n-1}\sum_{k\not\equiv r \,(\mathrm{mod}\, p)}\binom{n}{k}(-1)^k \pmod p.\end{aligned}$$

(To get the last congruence we have applied Wilson's theorem and Fermat's little theorem.) Since

$$-\sum_{k\not\equiv r\,(\mathrm{mod}\,p)}\binom{n}{k}(-1)^k=\sum_{k\equiv r\,(\mathrm{mod}\,p)}\binom{n}{k}(-1)^k=F_p(n,r),$$

the desired (3.2) follows.

Now fix $n \geq p$ and assume that (3.2) holds for smaller values of n. Clearly $n' = n-(p-1) > 0$ and $\{-n'\}_{p-1} = n^*$. In light of Lemma 3.1,

$$F_p(n,r) \equiv -\sum_{j=1}^{p-1}\frac{1}{j}\sum_{k=0}^{j-1}F_p(n',r-k) \pmod{p}.$$

By the induction hypothesis and part (ii),

$$\begin{aligned}(-1)^{n'}F_p(n',r-k) &\equiv [x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n'}e^{-(r-k)x}\\ &\equiv [x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n+1}e^{(k-r)x} \pmod{p}.\end{aligned}$$

Thus $(-1)^{n-1}F_p(n,r)$ is congruent to

$$\begin{aligned}&\sum_{j=1}^{p-1}\frac{1}{j}\sum_{k=0}^{j-1}\bigg([x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n+1}e^{(k-r)x}\bigg)\\ &\qquad = [x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n+1}e^{-rx}\sum_{j=1}^{p-1}\Big(\frac{1}{j}\cdot\frac{e^{jx}-1}{e^x-1}\Big)\\ &\qquad = [x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n}e^{-rx}\sum_{j=1}^{p-1}\frac{e^{jx}-1}{jx}\end{aligned}$$

modulo p. This yields

$$\begin{aligned}(-1)^nF_p(n,r) &\equiv\ -[x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n}e^{-rx}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\frac{(jx)^{k-1}}{k!}\\ &\equiv [x^{n^*}]\Big(\frac{e^x-1}{x}\Big)^{n}e^{-rx} \pmod{p},\end{aligned}$$

since $n^* < p-1$ and $\sum_{j=1}^{p-1} j^{k-1} \equiv -[\![p-1 \mid k-1]\!] \pmod{p}$.

In view of the above, we have completed the proof. ■

4. Proof of Theorem 1.3. Let ζ_p be a primitive pth root of unity in $\mathbb{C}$, and set $\pi = 1-\zeta_p$. For any $k = 0,\ldots,n$, we have

$$\begin{aligned}pC_p(kp^a(p-1)+l,r) &= \sum_{j=0}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l}\\ &= \sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l} + [\![k=l=0]\!]\end{aligned}$$

and thus

$$\begin{aligned}&F_p(kp^a(p-1)+l,r)\\&\quad=(-p)^{-\lfloor (kp^a(p-1)+l-1)/(p-1)\rfloor}C_p(kp^a(p-1)+l,r)+[\![k=l=0]\!]\\&\quad=-(-p)^{-kp^a-\lfloor (l-1)/(p-1)\rfloor-1}\sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l}.\end{aligned}$$

Therefore, for $S_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^kF_p(kp^a(p-1)+l,r)$ we have

$$S_n=-\sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^l(-p)^{-\lfloor (l-1)/(p-1)\rfloor-1}c_{n,j},\tag{4.1}$$

where

$$\begin{aligned}c_{n,j}&=\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-p)^{-kp^a}(1-\zeta_p^j)^{kp^a(p-1)}\\&=(1-(-p)^{-p^a}(1-\zeta_p^j)^{p^a(p-1)})^n.\end{aligned}$$

Let $j\in\{1,\dots,p-1\}$. Clearly

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p-1}=\left(\frac{1-(1-\pi)^j}{\pi}\right)^{p-1}\equiv j^{p-1}\equiv 1\ (\mathrm{mod}\,\pi)$$

and hence

$$b_j:=\frac{(1-\zeta_p^j)^{p-1}}{-p}=\left(\frac{1-\zeta_p^j}{\pi}\right)^{p-1}\frac{\pi^{p-1}}{-p}\equiv 1\ (\mathrm{mod}\,\pi).$$

(Recall the congruence $p/\pi^{p-1}\equiv -1\ (\mathrm{mod}\,\pi)$.) It follows that $b_j^{p^a}\equiv 1\ (\mathrm{mod}\,p^a\pi)$ and

$$c_{n,j}=(1-b_j^{p^a})^n\equiv 0\ (\mathrm{mod}\,p^{an}\pi^n).\tag{4.2}$$

Since $(1-\zeta_p^j)^l\equiv 0\ (\mathrm{mod}\,\pi^l)$ and $\mathrm{ord}_p(\pi)=1/(p-1)$, in view of (4.1) and (4.2) we have

$$\mathrm{ord}_p(S_n)\ge\frac{l+n}{p-1}+an-\left\lfloor\frac{l-1}{p-1}\right\rfloor-1=an+\frac{l+n}{p-1}-\frac{l+l^*}{p-1}=an+\frac{n-l^*}{p-1}$$

and hence $\mathrm{ord}_p(S_n)\ge an+\lceil (n-l^*)/(p-1)\rceil$. This proves (1.20). ■

5. On generalized Fleck quotients

LEMMA 5.1. *Let $d,q\in\mathbb{Z}^+$, $n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Let ζ_{dq} be a primitive dqth root of unity in $\mathbb{C}$. Then*

$$C_{dq}(n,r)=\frac{1}{d}\sum_{k=0}^{n}\binom{n}{k}C_q(k,r)\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}(1-\zeta_{dq}^j)^{n-k}.\tag{5.1}$$

Proof. Note that $\zeta = \zeta_{dq}^d$ is a primitive qth root of unity. Thus

$$\begin{aligned}
q\sum_{k=0}^{n}\binom{n}{k}C_q(k,r)\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}(1-\zeta_{dq}^{j})^{n-k}&\\
=\sum_{k=0}^{n}\binom{n}{k}\sum_{s=0}^{q-1}\zeta^{-sr}(1-\zeta^{s})^{k}\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}(1-\zeta_{dq}^{j})^{n-k}&\\
=\sum_{s=0}^{q-1}\sum_{j=0}^{d-1}\zeta_{dq}^{-(ds+j)r}\sum_{k=0}^{n}\binom{n}{k}(\zeta_{dq}^{j}(1-\zeta_{dq}^{ds}))^{k}(1-\zeta_{dq}^{j})^{n-k}&\\
=\sum_{s=0}^{q-1}\sum_{j=0}^{d-1}\zeta_{dq}^{-(ds+j)r}(1-\zeta_{dq}^{ds+j})^{n}=\sum_{t=0}^{dq-1}\zeta_{dq}^{-tr}(1-\zeta_{dq}^{t})^{n}=dqC_{dq}(n,r).&
\end{aligned}$$

So we have (5.1). ∎

With the help of Lemma 5.1 we can prove the following result via roots of unity.

THEOREM 5.1 (Weisman, 1977). *Let p be a prime, and let $a \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then $F_{p^a}(n,r) \in \mathbb{Z}$.*

Proof. We use induction on a.

The case $a = 1$ reduces to Fleck's result. A proof of Fleck's result via roots of unity was given by A. Granville [Gr].

Now let $a \geq 2$ and assume that $F_{p^{a-1}}(n', r') \in \mathbb{Z}$ for all $n' \in \mathbb{N}$ and $r' \in \mathbb{Z}$. If $n < p^a$, then $\lfloor (n - p^{a-1})/\varphi(p^a) \rfloor \leq 0$ and hence $F_{p^a}(n,r) \in \mathbb{Z}$. Below we suppose $n \geq p^a$ and let ζ_{p^a} be a primitive p^ath root of unity in $\mathbb{C}$.

By Lemma 5.1,

$$C_{p^a}(n,r) = \frac{1}{p}\sum_{k=0}^{n}\binom{n}{k}C_{p^{a-1}}(k,r)\sum_{j=0}^{p-1}\zeta_{p^a}^{j(k-r)}(1-\zeta_{p^a}^{j})^{n-k}. \tag{5.2}$$

Observe that

$$\prod_{\substack{j=1\\ p\nmid j}}^{p^a-1}(1-\zeta_{p^a}^{j}) = \prod_{\substack{\gamma^{p^a}=1\\ \gamma^{p^{a-1}}\neq 1}}(1-\gamma) = \lim_{x\to 1}\frac{x^{p^a}-1}{x^{p^{a-1}}-1} = \frac{p^a}{p^{a-1}} = p.$$

If $p \nmid j$, then $(1-\zeta_{p^a}^{j})/(1-\zeta_{p^a})$ is a unit in the ring $\mathbb{Z}[\zeta_{p^a}]$ and thus

$$\mathrm{ord}_p(1-\zeta_{p^a}^{j}) = \mathrm{ord}_p(1-\zeta_{p^a}) = \frac{1}{\varphi(p^a)}.$$

From this and the induction hypothesis, for any $k = 0, \ldots, n$ we have

$$\begin{aligned}
&\mathrm{ord}_p\Big(C_{p^{a-1}}(k,r)\sum_{j=0}^{p-1}\zeta_{p^a}^{j(k-r)}(1-\zeta_{p^a}^j)^{n-k}\Big)\\
&\quad\ge \max\left\{0, \left\lfloor\frac{k-p^{a-2}}{\varphi(p^{a-1})}\right\rfloor\right\} + \frac{n-k}{\varphi(p^a)}\\
&\quad= \max\left\{0, \frac{pk-p^{a-1}}{\varphi(p^a)} - \left\{\frac{k-p^{a-2}}{\varphi(p^{a-1})}\right\}\right\} + \frac{n-k}{\varphi(p^a)}\\
&\quad= \max\left\{\frac{n-k}{\varphi(p^a)}, \frac{n-p^{a-1}}{\varphi(p^a)} + \frac{k}{p^{a-1}} - \left\{\frac{k-p^{a-2}}{\varphi(p^{a-1})}\right\}\right\} > \frac{n-p^{a-1}}{\varphi(p^a)}.
\end{aligned}$$

(Note that if $k \ge p^{a-1}$ then $k/p^{a-1} \ge 1 > \{(k-p^{a-2})/\varphi(p^{a-1})\}$.) Therefore, from (5.2) we get

$$\mathrm{ord}_p(C_{p^a}(n,r)) > \frac{n-p^{a-1}}{\varphi(p^a)} - 1 \ge \left\lfloor\frac{n-p^{a-1}}{\varphi(p^a)}\right\rfloor - 1.$$

So $F_{p^a}(n,r) = (-p)^{-\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor}C_{p^a}(n,r) \in \mathbb{Z}$ as desired. ∎

Proof of Theorem 1.4. (i) Write $n+d = p^{a-1}-1+m\varphi(p^a)$ with $m \in \mathbb{N}$. Then, for any $k = 0, \ldots, d$ we have

$$\left\lfloor\frac{n+k-p^{a-1}}{\varphi(p^a)}\right\rfloor = \left\lfloor m - \frac{d-k+1}{\varphi(p^a)}\right\rfloor = m-1.$$

Below we use induction on d to show the desired congruence (1.21).

In the case $d = 0$ (i.e., $n - p^{a-1} \equiv -1 \pmod{\varphi(p^a)}$), we have $F_{p^a}(n,r) \equiv F_{p^a}(n,0) \pmod p$ because

$$F_{p^a}(n,i) - F_{p^a}(n,i-1) = (-p)^{-m+1}C_{p^a}(n+1,i) = -pF_{p^a}(n+1,i)$$

for all $i \in \mathbb{Z}$. Furthermore, by a result of Weisman [We] (see also [SW, Theorem 1.5]), $F_{p^a}(n,r) \equiv 1 \pmod p$ if $d = 0$.

Now let $d > 0$ and assume that the desired result holds for smaller values of d. Clearly, $(n+1)+(d-1) = p^{a-1}-1+m\varphi(p^a)$ and

$$\left\lfloor\frac{n+1+k-p^{a-1}}{\varphi(p^a)}\right\rfloor = m-1 \quad \text{for } k = 0, \ldots, d-1.$$

If $r \ge 0$ then

$$C_{p^a}(n,r) - C_{p^a}(n,0) = \sum_{0<i\le r}(C_{p^a}(n,i) - C_{p^a}(n,i-1)) = \sum_{0<i\le r}C_{p^a}(n+1,i);$$

if $r<0$ then

$$C_{p^a}(n,r)-C_{p^a}(n,0)=\sum_{r<i\le 0}(C_{p^a}(n,i-1)-C_{p^a}(n,i))$$
$$=-\sum_{r<i\le 0}C_{p^a}(n+1,i).$$

Therefore

$$F_{p^a}(n,r)-F_{p^a}(n,0)=\begin{cases}\sum_{0<i\le r}F_{p^a}(n+1,i) & \text{if } r\ge 0,\\ -\sum_{r<i\le 0}F_{p^a}(n+1,i) & \text{if } r<0.\end{cases}$$

By the induction hypothesis, whenever $i\in\mathbb{Z}$ we have

$$F_{p^a}(n+1,i)\equiv\sum_{k=0}^{d-1}\binom{i+k-1}{k}F_{p^a}(n+1+k,0)\ (\operatorname{mod} p).$$

For any $k=0,\dots,d-1$, if $r\ge 0$ then

$$\sum_{0<i\le r}\binom{i+k-1}{k}=\sum_{j=0}^{r+k-1}\binom{j}{k}=\binom{r+k}{k+1}$$

by an identity of S.-C. Chu (cf. [GKP, (5.10)]); if $r<0$ then

$$-\sum_{r<i\le 0}\binom{i+k-1}{k}=(-1)^{k+1}\sum_{r<i\le 0}\binom{-i}{k}=(-1)^{k+1}\sum_{j=0}^{-r-1}\binom{j}{k}$$
$$=(-1)^{k+1}\binom{-r}{k+1}=\binom{r+k}{k+1}.$$

Thus, by the above, $F_{p^a}(n,r)$ is congruent to

$$F_{p^a}(n,0)+\sum_{k=0}^{d-1}\binom{r+k}{k+1}F_{p^a}(n+1+k,0)=\sum_{k=0}^{d}\binom{r+k-1}{k}F_{p^a}(n+k,0)$$

modulo p. This concludes the induction proof of (1.21).

(ii) In the case $a=1$, the desired results in Theorem 1.4(ii) follow from Corollaries 1.3 and 1.7.

Now we let $a\ge 2$ and $r\in\mathbb{Z}$. Write $n=p^{a-2}(pn_1+n_0)+s$ and $r=p^{a-2}(pr_1+r_0)+t$, where $s,t\in\{0,\dots,p^{a-2}-1\}$, $n_0,r_0\in\{0,\dots,p-1\}$ and $n_1\in\mathbb{N}$ and $r_1\in\mathbb{Z}$.

If $p^{a-1}\le n<p^a$, then

$$F_{p^a}(n,r)=C_{p^a}(n,r)=\binom{n}{\{r\}_{p^a}}(-1)^{\{r\}_{p^a}},$$

and in particular $\operatorname{ord}_p(C_{p^a}(n,0))=0=\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor$.

Below we assume that $n \ge 2p^{a-1}$ (i.e., $n_1 \ge 2$). By [SD, Theorem 1.7],

$$F_{p^a}(n,r) \equiv (-1)^t \binom{s}{t} F_{p^2}(pn_1+n_0, pr_1+r_0) \pmod p.$$

If $p \mid n_1$, or $p-1 \nmid n_1-1$, or $n_0 = r_0 = p-1$, then by [SW, Theorem 1.2] in the case $l=0$, we have

$$F_{p^2}(pn_1+n_0, pr_1+r_0) \equiv (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1, r_1) \pmod p$$

and hence $F_{p^a}(n,r) \equiv b_{n,r} F_p(n_1,r_1) \pmod p$, where

$$\begin{aligned} b_{n,r} &:= (-1)^{\{r\}_{p^{a-1}}} \binom{\{n\}_{p^{a-1}}}{\{r\}_{p^{a-1}}} = (-1)^{p^{a-2}r_0+t} \binom{p^{a-2}n_0+s}{p^{a-2}r_0+t} \\ &\equiv (-1)^t \binom{s}{t} (-1)^{r_0} \binom{n_0}{r_0} \pmod p \quad \text{(by Lucas' theorem (cf. [HS]))}. \end{aligned}$$

By Corollary 1.3, there is an $r_1' \in \mathbb{Z}$ such that $F_p(n_1, r_1') \not\equiv 0 \pmod p$. Thus, if $p \mid n_1$ or $p-1 \nmid n_1 - 1$, then

$$F_{p^a}(n, p^{a-1}r_1') \equiv F_p(n_1, r_1') \not\equiv 0 \pmod p.$$

If $n_0 = p-1$, then

$$F_{p^a}(n, p^{a-2}(pr_1'+p-1)) \equiv (-1)^{p-1} \binom{p-1}{p-1} F_p(n_1, r_1') \not\equiv 0 \pmod p.$$

When $p \nmid n_1$, $p-1 \mid n_1 - 1$ and $n_0 < r_0$, by applying the second part of [SW, Theorem 1.2] in the case $l=0$, we have

$$F_{p^2}(pn_1+n_0, pr_1+r_0) \equiv [\![n_1 > 1]\!] \frac{(-1)^{n_0} n_1}{r_0\binom{r_0-1}{n_0}} = \frac{(-1)^{n_0} n_1}{r_0\binom{r_0-1}{n_0}} \pmod p$$

and hence

$$F_{p^a}(n,r) \equiv (-1)^{n_0+t} \frac{n_1\binom{s}{t}}{r_0\binom{r_0-1}{n_0}} \pmod p.$$

In particular, if $p \nmid n_1$, $p-1 \mid n_1-1$ and $n_0 < p-1$, then

$$F_{p^a}(n, p^{a-2}(n_0+1)) \equiv \frac{(-1)^{n_0} n_1}{n_0+1} \not\equiv 0 \pmod p.$$

In view of the above, we already have (1.22).

To prove the congruence in (1.23), we also have to consider the case $p \nmid n_1$, $p-1 \mid n_1-1$ and $n_0 \ge r_0$. By [SW, Lemmas 3.2 and 3.3],

$$
\begin{aligned}
&p^{-\lfloor (pn_1+n_0-p)/\varphi(p^2)\rfloor} C_{p^2}(pn_1+n_0, pr_1+r_0)\\
&\qquad -(-1)^{r_0}\binom{n_0}{r_0} p^{-\lfloor (n_1-1)/(p-1)\rfloor} C_p(n_1, r_1)\\
&\quad \equiv (-1)^{n_1-1} p^{-\lfloor (n_1-1-1)/(p-1)\rfloor} C_p(n_1-1, r_1)(-1)^{n_1+r_0} n_1 \binom{n_0}{r_0} \frac{\sigma_{n_0,r_0}(n_1)}{p}\\
&\quad \equiv -(-1)^{r_0}\binom{n_0}{r_0} p^{-(n_1-1)/(p-1)+1} C_p(n_1-1, r_1) n_1 \,\frac{\sigma_{n_0,r_0}(n_1)}{p} \pmod p,
\end{aligned}
$$

where

$$
\sigma_{n_0,r_0}(n_1) = 1 + (-1)^p \,\frac{\prod_{1\le i\le p,\, i\ne p-r_0}(p(n_1-1)+r_0+i)}{\prod_{1\le i\le p,\, i\ne p-(n_0-r_0)}(n_0-r_0+i)} \equiv 0 \pmod p.
$$

Therefore

$$
\begin{aligned}
&F_{p^2}(pn_1+n_0, pr_1+r_0) - (-1)^{r_0}\binom{n_0}{r_0} F_p(n_1, r_1)\\
&\qquad\qquad \equiv (-1)^{r_0}\binom{n_0}{r_0} F_p(n_1-1, r_1) n_1 \,\frac{\sigma_{n_0,r_0}(n_1)}{p} \pmod p
\end{aligned}
$$

and hence

$$
F_{p^a}(n, r) \equiv b_{n,r}\left(F_p(n_1, r_1) + F_p(n_1-1, r_1) n_1 \,\frac{\sigma_{n_0,r_0}(n_1)}{p}\right) \pmod p.
$$

Observe that $n+p^a(p-1) = p^{a-2}(pn_1'+n_0)+s$ with $n_1' = n_1+p(p-1)$. Clearly $F_p(n_1', r_1) \equiv F_p(n_1, r_1) \pmod p$ by Corollary 1.7, and $\sigma_{n_0,r_0}(n_1') \equiv \sigma_{n_0,r_0}(n_1) \pmod{p^2}$ if $n_0 \ge r_0$. Thus, by the above, $F_{p^a}(n+p^a(p-1), r) \equiv F_{p^a}(n, r) \pmod p$. ∎

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