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Quintuple products and Ramanujan's partition congruence $p(11n+6) \equiv 0 \pmod{11}$

by

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Dedicated to my teacher Prof. L. C. Hsu
on the occasion of his 85th birthday

1. A difference equation for quintuple products. For two indeterminates q and z with $|q| < 1$, the q-shifted factorial of z of infinite order is defined by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n).$$

We abbreviate

$$[\alpha,\beta,\ldots,\gamma;q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty.$$

The object of this paper is to establish the following algebraic identity for infinite factorial products and show that it leads to a new proof of the Ramanujan congruence for the partition function modulo 11.

Theorem 1 (q-difference equation for a symmetric difference). *Let $f(x,y)$ be a bivariate function defined by two quintuple products*

$$f(x,y) = y[q^2, qx^2, q/x^2; q^2]_\infty[x^4, q^4/x^4; q^4]_\infty \times [q^2, y^2, q^2/y^2; q^2]_\infty[q^2y^4, q^2/y^4; q^4]_\infty. \tag{1}$$

Then the following algebraic identity holds on the symmetric difference:

$$f(x,y) - f(y,x) = y[x^2, q/x^2; q]_\infty[q, x/y, qy/x; q]_\infty \times [y^2, q/y^2; q]_\infty[q, xy, q/xy; q]_\infty. \tag{2}$$

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Proof. Extracting the common shifted factorials, we find that

$$f(x,y)-f(y,x)=y[x^2,y^2,q/x^2,q/y^2;q]_\infty\Omega(x,y),$$

where

$$\begin{aligned}\Omega(x,y)=&[q^2,-x^2,-q^2/x^2;q^2]_\infty[q^2,-qy^2,-q/y^2;q^2]_\infty\\&-\frac{x}{y}[q^2,-y^2,-q^2/y^2;q^2]_\infty[q^2,-qx^2,-q/x^2;q^2]_\infty.\end{aligned}$$

By means of the Jacobi triple product identity (cf. [1, p. 12])

$$[q,z,q/z;q]_\infty=\sum_{k=-\infty}^{\infty}q^{\binom{k}{2}}z^k \tag{3}$$

we can expand $\Omega(x,y)$ as follows:

$$\Omega(x,y)=\sum_{i,j}q^{i^2+j^2-i}\{x^{2i}y^{2j}-y^{2i-1}x^{1+2j}\}$$

where the double sum runs over $-\infty<i,j<\infty$. Changing the summation indices according to

$$\left.\begin{aligned}i+j=m\\ i-j=n\end{aligned}\right\}\quad\rightleftharpoons\quad\left\{\begin{aligned}i=(m+n)/2\\ j=(m-n)/2\end{aligned}\right.$$

and then letting $m\equiv_2 n$ indicate that the summation indices m and n are of the same parity, we may rewrite the double sum as

$$\begin{aligned}\Omega(x,y)&=\sum_{m\equiv_2 n}q^{\binom{m}{2}+\binom{n}{2}}(xy)^m\{(x/y)^n-(x/y)^{1-n}\}\\&=\sum_{m\equiv_2 n}q^{\binom{m}{2}+\binom{n}{2}}(xy)^m\{(x/y)^n-q^n(x/y)^{n+1}\}\end{aligned}$$

where the last passage is justified by replacement $n\mapsto -n$ in the penultimate line for the summand corresponding to the second term in the braces.

When both m and n are even, we perform the substitutions $m\mapsto 2m$ and $n\mapsto 2n$ and otherwise $m\mapsto 1+2m$ and $n\mapsto 1+2n$. Then $\Omega(x,y)$ can be rewritten as follows:

$$\begin{aligned}&\Omega(x,y)\\&=\sum_m q^{\binom{2m}{2}}(xy)^{2m}\sum_n q^{\binom{2n}{2}}\{(x/y)^{2n}-q^{2n}(x/y)^{1+2n}\}\\&\quad+\sum_m q^{\binom{1+2m}{2}}(xy)^{1+2m}\sum_n q^{\binom{1+2n}{2}}\{(x/y)^{1+2n}-q^{1+2n}(x/y)^{2+2n}\}.\end{aligned} \tag{4}$$

The first sum with respect to n in (4) can be reduced, through the Jacobi triple product identity, to the following product:

$$\begin{aligned}\sum_n q^{\binom{2n}{2}}\{(x/y)^{2n} - q^{2n}(x/y)^{1+2n}\} \\ &= \sum_n q^{\binom{2n}{2}}(x/y)^{2n} - \sum_n q^{\binom{1+2n}{2}}(x/y)^{1+2n} \\ &= \sum_k (-1)^k q^{\binom{k}{2}}(x/y)^k = [q, x/y, qy/x; q]_\infty.\end{aligned}$$

The second sum with respect to n in (4) may be factorized similarly:

$$\begin{aligned}\sum_n q^{\binom{1+2n}{2}}\{(x/y)^{1+2n} - q^{1+2n}(x/y)^{2+2n}\} \\ &= \sum_n q^{\binom{1+2n}{2}}(x/y)^{1+2n} - \sum_n q^{\binom{2+2n}{2}}(x/y)^{2+2n} \\ &= \sum_k (-1)^{k+1} q^{\binom{k}{2}}(x/y)^k = -[q, x/y, qy/x; q]_\infty.\end{aligned}$$

Finally, substituting these triple products into (4), we can further simplify it to

$$\begin{aligned}\frac{\Omega(x,y)}{[q, x/y, qy/x; q]_\infty} &= \sum_m q^{\binom{2m}{2}}(xy)^{2m} - \sum_m q^{\binom{1+2m}{2}}(xy)^{1+2m} \\ &= \sum_k (-1)^k q^{\binom{k}{2}}(xy)^k = [q, xy, q/xy; q]_\infty.\end{aligned}$$

This proves the identity stated in the theorem. ∎

2. Ramanujan's partition congruence $p(11n+6) \equiv 0 \pmod{11}$. After some routine modification, the quintuple product identity (cf. [1, p. 134]) may be stated as

$$\begin{aligned}(5)\quad [q, z, q/z; q]_\infty [qz^2, q/z^2; q^2]_\infty &= \sum_{n=-\infty}^{\infty} \{1 - z^{1+6n}\} q^{3\binom{n}{2}}(q^2/z^3)^n \\ &= \sum_{n=-\infty}^{\infty} \{1 - (q/z^2)^{1+3n}\} q^{3\binom{n}{2}}(qz^3)^n.\end{aligned}$$

Expanding the two products displayed in (1) respectively by using the last two expressions, we get the following double formal power series expression:

$$\begin{aligned}\frac{f(x,y)}{x^2y^2} &= \Big\{ \sum_{m=-\infty}^{\infty} (x^{2+6m} - x^{-2-6m})\, q^{6\binom{m}{2}+5m} \Big\} \\ &\quad \times \Big\{ \sum_{n=-\infty}^{\infty} (y^{1+6n} - y^{-1-6n})\, q^{6\binom{n}{2}+4n} \Big\}.\end{aligned}$$

The q-difference equation in Theorem 1 can accordingly be reformulated as follows:

$$\begin{aligned}(6)\quad &\frac{f(x,y)-f(y,x)}{x^2y^2}\\ &= \frac{(q;q)_\infty^2}{x^2y}\,[x^2,y^2,xy,x/y,q/x^2,q/y^2,q/xy,qy/x;q]_\infty\\ &= \sum_m (x^{2+6m}-x^{-2-6m})q^{6\binom{m}{2}+5m}\sum_n (y^{1+6n}-y^{-1-6n})q^{6\binom{n}{2}+4n}\\ &\quad -\sum_m (y^{2+6m}-y^{-2-6m})q^{6\binom{m}{2}+5m}\sum_n (x^{1+6n}-x^{-1-6n})q^{6\binom{n}{2}+4n}.\end{aligned}$$

Applying throughout first $y\partial/\partial y$ and then three times $x\partial/\partial x$ at $x=y=1$, we derive the following q-series identity.

Proposition 2.

$$\begin{aligned}3(q;q)_\infty^{10} = 4&\Big\{\sum_{m=-\infty}^{\infty}(1+3m)^3q^{3m^2+2m}\Big\}\times\Big\{\sum_{n=-\infty}^{\infty}(1+6n)q^{3n^2+n}\Big\}\\ &-\Big\{\sum_{m=-\infty}^{\infty}(1+3m)q^{3m^2+2m}\Big\}\times\Big\{\sum_{n=-\infty}^{\infty}(1+6n)^3q^{3n^2+n}\Big\}.\end{aligned}$$

This identity leads us to a new proof of the Ramanujan congruence for the partition function modulo 11.

Corollary 3 (Ramanujan [2]). *Let $p(n)$ denote the number of unrestricted partitions of the natural number n. Then*

$$p(11n+6)\equiv 0 \pmod{11}.$$

Proof. By means of the congruence

$$q^5\sum_{n=0}^{\infty}p(n)q^n = \frac{q^5}{(q;q)_\infty} = q^5\,\frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \equiv \frac{q^5(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} \pmod{11},$$

the assertion follows if we can show that whenever K is a multiple of 11, the coefficient of q^K in the formal power series expansion of $q^5(q;q)_\infty^{10}$ is divisible by 11.

In view of the linear factorization

$$\begin{aligned}&4(1+3m)^3(1+6n)-(1+3m)(1+6n)^3\\ &\qquad = 3(1+3m)(1+6n)(1+2m+2n)(1+6m-6n)\end{aligned}$$

we can rewrite the identity stated in Proposition 2 in the form

$$(7)\qquad q^5(q;q)_\infty^{10} = \sum_{m,n}(1+2m+2n)(1+6m-6n) \times (1+3m)(1+6n)q^{3m^2+3n^2+2m+n+5}.$$

Let K be a multiple of 11. The exponent of q^K in this double series expansion can be expressed as

$$0 \equiv K = 5+2m+n+3m^2+3n^2 \equiv (2+6m)^2+(1+6n)^2 \pmod{11}.$$

By examining the quadratic residues modulo 11, we conclude that this can happen only when

$$1+6n \equiv 2+6m \equiv 0 \pmod{11}.$$

With the help of the linear combinations

$$3(1+2m+2n) = (2+6m)+(1+6n),$$
$$1+6m-6n = (2+6m)-(1+6n),$$

we see that the coefficient of q^K in (7) is divisible by 11^4, which is stronger than what we really need. ■

3. Winquist's proof of $p(11n+6) \equiv 0 \pmod{11}$. In order to clarify the difference between the identity stated in Theorem 1 and Winquist's identity, we restate the latter with a trivial modification as the following theorem.

THEOREM 4 (Winquist [3, Theorem 1.1]).

$$(8)\qquad \frac{(q;q)_\infty^2}{x^2y}[x^2, y^2, x^2y^2, x^2/y^2, q/x^2, q/y^2, q/x^2y^2, qy^2/x^2; q]_\infty$$
$$= \Big\{\sum_{i=-\infty}^{\infty}(-1)^i q^{3\binom{i+1}{2}}x^{3+6i}\Big\} \times \Big\{\sum_{j=-\infty}^{\infty}(-1)^j(y^{1+6j}-y^{-1-6j})q^{3\binom{j}{2}+2j}\Big\}$$
$$- \Big\{\sum_{i=-\infty}^{\infty}(-1)^i q^{3\binom{i+1}{2}}y^{3+6i}\Big\} \times \Big\{\sum_{j=-\infty}^{\infty}(-1)^j(x^{1+6j}-x^{-1-6j})q^{3\binom{j}{2}+2j}\Big\}.$$

This has been used by Winquist to provide an elementary proof of Corollary 3. Following the argument of the last section, we can rewrite Winquist's proof in a more symmetric form. In fact, applying first $y\partial/\partial y$ and then three times $x\partial/\partial x$ at $x=y=1$ to both sides of the identity stated in Theorem 4, one gets the following identity, which should be attributed to Winquist even though it did not appear in [3] explicitly.

Proposition 5.

$$16(q;q)_\infty^{10}$$
$$= 9\Big\{\sum_{i=-\infty}^{\infty}(-1)^i(1+2i)^3q^{3\binom{i+1}{2}}\Big\}\times\Big\{\sum_{j=-\infty}^{\infty}(-1)^j(1+6j)q^{3\binom{j}{2}+2j}\Big\}$$
$$-\Big\{\sum_{i=-\infty}^{\infty}(-1)^i(1+2i)q^{3\binom{i+1}{2}}\Big\}\times\Big\{\sum_{j=-\infty}^{\infty}(-1)^j(1+6j)^3q^{3\binom{j}{2}+2j}\Big\}.$$

As in the proof of Corollary 3 presented in the last section, we can write down a similar one through Proposition 5, which is essentially the same as the original proof due to Winquist [3].

By means of the linear factorization

$$9(1+2i)^3(1+6j)-(1+2i)(1+6j)^3$$
$$= 4(1+2i)(1+6j)(1+3i-3j)(2+3i+3j)$$

we can reformulate the identity of Proposition 5 as follows:

$$4q^5(q;q)_\infty^{10} = \sum_{i,j}(-1)^{i+j}(1+3i-3j)(2+3i+3j) \times (1+2i)(1+6j)q^{3\binom{i+1}{2}+3\binom{j}{2}+2j+5}. \tag{9}$$

Let K be a multiple of 11. The exponent of q^K in this double series expansion can be expressed as

$$0 \equiv 2K = 10+3i+j+3i^2+3j^2 \equiv (3+6i)^2+(1+6j)^2 \pmod{11},$$

which can happen only when

$$1+6j \equiv 3+6i \equiv 0 \pmod{11}.$$

With the help of the linear combinations

$$2(1+3i-3j) = (3+6i)-(1+6j),$$
$$2(2+3i+3j) = (3+6i)+(1+6j),$$

we see that the coefficient of q^K in (9) is divisible by 11^4. This is the same as what we have obtained before.

It should be pointed out that the quintuple product identity makes the real difference between Theorems 1 and 4. In fact, the former expresses a product of ten infinite factorials as the difference between two products of ten infinite factorials. However, this cannot be done for the latter because there do not exist product expressions for the bilateral sums with respect to j appearing in Winquist's identity (8). From this point of view, our proof of Ramanujan's partition congruence modulo 11 may be more elementary than Winquist's.

References

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