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The Erdős–Turán property for a class of bases

by

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1. Introduction. Given a set $A \subset \mathbb{N}$ let $r_A(n)$ denote the number of ordered pairs $(a, a') \in A \times A$ such that $a + a' = n$. Then A is an *asymptotic additive basis of order* 2 (a basis for short in what follows) if there is $n_0 = n_0(A)$ such that $r_A(n) \geq 1$ for each positive integer $n \geq n_0$. It was conjectured by Erdős and Turán in 1941 [5] that if A is a basis, then

$$\limsup_{n\to\infty} r_A(n) = \infty.$$

An excellent account of this and related problems is given by Sárközy and Sós [11] (see also [3]).

Not much is known about this famous conjecture. It can be easily shown that if a set of positive integers A satisfies $\limsup_{n\to\infty} |A(n)|n^{-1/2} = \infty$, where $A(n) = A \cap [1, n]$, then $r_A(n)$ cannot be bounded. Indeed, if $r_A(n) \leq g$ for all $n \in \mathbb{N}$ and $A_n(x) = \sum_{a\in A(n)} x^a$, then $A_n^2(x) = \sum_{i\geq 0} r_{A(n)}(i)x^i$ and

$$|A(n)|^2 = A_n^2(1) = \sum_{i\geq 1} r_{A(n)}(i) \leq 2ng.$$

On the other hand, the counting function of a basis must satisfy $|A(n)| = \Omega(n^{1/2})$, since $|A(n)|^2 > |A(n) + A(n)| \geq n - n_0(A)$ for some $n_0(A)$. More explicit quantitative expressions of both facts can be found for instance in [7]. Therefore, the Erdős–Turán conjecture is open for bases whose counting function satisfies $0 < \limsup |A(n)|/n^{1/2} < \infty$, called *thin* bases. The first examples of thin bases were given by Cassels, Stöhr and Raikov (see e.g. [6]). More recent constructions were provided by Hofmeister [7]. In all these examples the constructed bases either contain arbitrarily large arithmetic progressions or contain elements of the form $\sum_{i\in E} x_i d^i$, $d \geq 2$, for arbitrarily large subsets $E \subset \mathbb{N}$ and all possible choices of $x_i \in \{0, r\}$ for some $r \neq 0$. In both cases we trivially see that $r_A(n)$ is not bounded.

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Erdős [2] showed that the function $r_A(n)$ can grow slowly. More precisely, he showed that there exist bases A for which

$$c_1 \log n \leq r_A(n) \leq c_2 \log n, \qquad n \geq n_0(A)$$

for some constants $c_1, c_2 > 0$. This result may explain the difficulties involved in the Erdős–Turán conjecture.

In this note we prove the validity of the Erdős–Turán conjecture for a class of "bounded" bases. The *binary support* of a positive integer n is the subset $S(n) \subset \mathbb{N} \cup \{0\}$ of its binary expansion

$$n = \sum_{i \in S(n)} 2^i.$$

We say that $A \subset \mathbb{N}$ is *bounded* if there is a function f such that for each $n \in A + A$ there is a pair $x, y \in A$ with

$$n = x + y, \qquad |S(x) \cup S(y)| \leq f(|S(n)|).$$

For instance, if the binary expansion of each element in A has no two consecutive 1's, then A is a bounded set with $f(n) = n$.

We prove the following result.

Theorem 1. *Let A be a bounded asymptotic basis of order* 2. *Then*

$$\limsup_{n \to \infty} r_A(n) = \infty.$$

In fact we prove a more general form of Theorem 1. Let us fix a positive integer $h \geq 2$ and let $r_A^{(h)}(n)$ denote the number of solutions of $n = a_1 + \cdots + a_h$ with $a_1, \ldots, a_h \in A$. We say that A has the *h-Erdős–Turán property* (h-ET for short) if $r_A^{(h)}(n)$ is unbounded. The set A is an *asymptotic additive basis of order h* if every sufficiently large integer can be expressed as the sum of h elements in $A \cup \{0\}$ and h is the minimum positive integer with this property. We shall omit the reference to h in the above definitions when $h = 2$. We prove the next result, for which the precise (and more technical) notion of (d, h)-bounded basis is explained in Section 2.

Theorem 2. *Let A be an asymptotic additive basis for $\mathbb{N}$ of order $h \geq 2$ and $d \geq 2$. If A is (d, h)-bounded then it has the h-Erdős–Turán property.*

This generalizes Theorem 1, since we shall see later that every $(2, 2)$-basis is a bounded basis.

We tested bounded bases in the context of other bases-related results, both additive and multiplicative. For example we have the following:

The Erdős–Newman problem [4] asks for the existence of a set A with $\limsup_{n \to \infty} r_A(n) = k$ such that, for every finite partition $A = A_1 \cup \cdots \cup A_r$, we have $\limsup r_{A_i}(n) = k$ for some i. The problem has a positive answer as proved by Nešetřil and Rödl [10]. A similar question can be asked when

$\limsup r_A(n) = \infty$. The method we use in the proof of Theorem 2 gives the following result.

THEOREM 3. *Let $A = A_1 \cup \cdots \cup A_r$ be a partition of a (d,h)-bounded additive basis of order $h \geq 2$ into a finite number of parts. Then one of the parts has the h-ET property.*

Erdős also proposed the following strengthening of the Erdős–Turán conjecture: if A is a set of positive integers satisfying $|A(n)| = \Omega(n^{1/2})$ then A has the ET property, whether it is a basis or not. If that is true then Theorem 3 follows for a basis of order 2.

The method we use to prove Theorems 2 and 3 is inspired by the proof that Nešetřil and Rödl [10] gave for the multiplicative analog of the Erdős–Turán conjecture. This was first proved by Erdős [1] in 1964:

THEOREM 4. *Suppose that every positive integer can be written as a product of two elements in a set $A \subset \mathbb{N}$. Then, for every $m \in \mathbb{N}$, there exists a positive integer n which can be written as a product of two elements in A in at least m different ways.*

It is worth noting that the proof of Theorem 4 given in [10] can be extended to multiplicative bases of $\mathbb{Z}$. However the situation is strikingly different for additive bases of $\mathbb{Z}$ or, more generally, in any linearly ordered Abelian group. Nathanson [9] proves that for an arbitrary function $r : \mathbb{Z} \cup \{\infty\} \to \mathbb{N}_0$ with $r^{-1}(0)$ finite there is an additive basis A of order 2 such that $r_A(x) = r(x)$ for all but a finite number of $x \in \mathbb{Z}$. As shown by Nathanson, a greedy algorithm provides a basis for $\mathbb{Z}$ with $r_A(n) = 1$ for all $n \in \mathbb{Z}$. A similar result holds when restricted to bounded bases.

THEOREM 5. *There is a bounded basis of $\mathbb{Z}$ satisfying $r_A(n) = 1$ for each $n \in \mathbb{Z}$.*

Thus while the definition of bounded basis and the proofs of Theorems 2 and 3 were inspired by the multiplicative versions of the Erdős–Turán conjecture which is valid both in $\mathbb{N}$ and $\mathbb{Z}$, this notion is sensitive enough to separate $\mathbb{N}$ and $\mathbb{Z}$ in the additive case.

In the same vein, Theorem 3 is no longer true in $\mathbb{Z}$. We can construct an additive basis A for $\mathbb{Z}$ with a prescribed function $r_A(n)$, even with the ET property, but which can be partitioned into two B_2^1 sequences.

THEOREM 6. *Let $r : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be a given function. There is a set A of integers such that $r_A(n) = r(n)$ for each integer n and a partition $A = A_1 \cup A_2$ of A such that $r_{A_i}(n) = 1$ for each n, $i = 1, 2$.*

2. Proofs of Theorems 2 and 3. Let $d \geq 2$ be an integer. We consider the d-adic expansion of each positive integer $n = \sum_{i \geq 0} n_i d^i$, $0 \leq n_i < d$.

The *d-support* of n is the subset $S_d(n) \subset \mathbb{N}_0$ such that

$$n = \sum_{i \in S_d(n)} n_i d^i, \qquad 0 < n_i < d.$$

Let $0 \in A$ be an additive basis of order $h \geq 2$, that is, every sufficiently large integer can be expressed as a sum of h elements from A. We say that A is a *(d, h)-bounded set on* $X \subset \mathbb{N}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in X$ there is an h-tuple $\tau(n) = (a_1, \ldots, a_h) \in A^h$ satisfying

$$|S_d(a_1) \cup \cdots \cup S_d(a_h)| \leq f(|S_d(n)|). \tag{1}$$

We call a set X of positive integers (N, d)-*good* if there is an infinite set $Y \subset \mathbb{N}_0$ all of whose N-subsets are supports of some element in X, that is, for every N-subset $K \subset Y$ there is $n \in X$ such that $S_d(n) = K$. The set X is *d-good* if it is (N, d)-good for infinitely many values of N. A basis A is (d, h)-*bounded* if it is (d, h)-bounded on some d-good set X. In particular, a 2-bounded basis A is $(2, 2)$-bounded on $X = [n_0(A), \infty)$. This explains the notions involved in Theorem 2 and it also shows that Theorem 2 implies Theorem 1.

Proof of Theorem 2. Let $0 \in A$ be an additive basis of order $h \geq 2$ which is (d, h)-bounded on a good set X. Given N for which X is (N, d)-good, let $Y' \subset \mathbb{N}_0$ be an infinite set each of whose N-subsets is the d-support of some element in X. By the definition, there is a function f such that for each N-subset $K \subset Y'$ there is $n \in X$ and an h-tuple $\tau(n) = (a_1, \ldots, a_h)$ satisfying inequality (1).

Set $M = \lceil \log_d h \rceil$ and consider an infinite subset $Y = \{y_1 < y_2 < \cdots\} \subset Y'$ such that

$$|y_{r+1} - y_r| > \max\{f(N), M\}, \qquad r \geq 1.$$

We color the N-subsets of Y as follows. For each N-subset $K = \{k_1 < \cdots < k_N\} \subset Y$, choose $n = n(K)$ with $S_d(n) = K$ and let $\tau(n) = (a_1, \ldots, a_h)$. Note that

$$\max S_d(n) \leq \max\{S_d(a_1), \ldots, S_d(a_h)\} + M.$$

Since $k_{r+1} - k_r > M$, we have

$$(S_d(a_1) \cup \cdots \cup S_d(a_h)) \cap [k_r - M, k_r] \neq \emptyset, \qquad r = 1, \ldots, N.$$

We may therefore assume that $S_d(a_1)$ intersects $s \geq N/h$ of the intervals $[k_r - M, k_r]$. On the other hand, since $k_{r+1} - k_r > f(N)$ we have $S_d(a_1) \subset \bigcup_{r=1}^{N} [k_r - M, k_r]$. Let $a_1 = \sum_{i \geq 0} a_{1i} d^i$ be the d-adic expansion of a_1. Then we define the coloring $\mathbf{c}$ on the N-subsets of Y as follows:

$$c_r(K) = (a_{1i}, i \in [k_r - M, k_r]), \qquad \mathbf{c}(K) = (c_1(K), \ldots, c_N(K)). \tag{2}$$

By the definition of $\mathbf{c}$, if $c_r(K) = (x_{r0}, \dots, x_{rM})$, $r = 1, \dots, N$, then

$$x(K, \mathbf{c}) = \sum_{i=k_1-M}^{k_1} x_{1i} d^{k_1-M+i} + \dots + \sum_{i=k_N-M}^{k_N} x_{Ni} d^{k_N-M+i} = a_1 \in A. \tag{3}$$

The coloring $\mathbf{c}$ uses less than Nd^{M+1} colors. By Ramsey's theorem, there is an infinite subset $Y_0 \subset Y$ all of whose N-subsets receive the same color $\beta = (\beta_1, \dots, \beta_N)$. Let $J \subset \{1, \dots, N\}$ be the set of subscripts for which $\beta_i \neq (0, \dots, 0)$. By the choice of a_1 in the definition of $\mathbf{c}$ we have $|J| \geq s \geq N/h$. Choose a subset $S = \{z_1 < \dots < z_{2s}\} \subset Y_0$ of cardinality $2s$ such that between any two consecutive elements of S there are N elements of Y_0. Set $V_j = \{z_{2j-1}, z_{2j}\}$, $j = 1, \dots, s$.

Let $U' \subset S$ be a subset of cardinality s obtained by picking out one element in each V_j. Since between any two elements in S there are N elements of Y_0, U' can be completed to a set $U = \{u_1 < \dots < u_N\}$ such that $u_i \in U'$ whenever $i \in J$ and $u_i \in Y_0 \setminus S$ otherwise. By the choice of Y_0, the set U has color β. Therefore, the element $x(U, \mathbf{c})$ as defined in (3) belongs to the basis A. By the same token, the complement $W' = S \setminus U'$ of U' in S can be completed to an N-subset $W = \{w_1 < \dots < w_N\}$ of Y_0 by adding $N - s$ elements in $Y_0 \setminus S$ which fill positions with value $(0, \dots, 0)$ in β. Therefore, the element $x(W, \mathbf{c})$ belongs to the basis A as well.

Note that, by the choice of Y, the supports of $x(U, \mathbf{c})$ and $x(W, \mathbf{c})$ are disjoint. Moreover, for any other choice of an s-subset L of S obtained by picking one element in each V_j, we have

$$x(U) + x(S \setminus U) = x(L) + x(S \setminus L).$$

Therefore, we get $2^{s-1} \geq 2^{N/h-1}$ different expressions of $m = x(U) + x(S \setminus U)$ as sums of two elements in the basis. Since there are infinitely many choices for N, the basis A has the ET property. ∎

Proof of Theorem 3. The proof is similar to the above proof of Theorem 2. Given n and $\tau(n) = (a_1, \dots, a_h)$ consider the following coloring of $S(n)$:

$$\mathbf{c}'(S(n)) = (\mathbf{c}(S(n)); \alpha(n)),$$

where $\mathbf{c}(S(n))$ is the coloring given in (2) and $\alpha(n) = i$ if $a_1 \in A_i$. By the same argument as in the proof of Theorem 2, for each $N > 0$ there is an infinite set $Y_0 \subset \mathbb{N}$ such that all its N-subsets have the same color $\beta' = (\beta_1, \dots, \beta_N; \alpha)$, where $\alpha = \alpha(N)$ identifies the set of the partition which the corresponding a_1 belongs to. Hence, there is a positive integer $m \in \mathbb{N}$ which admits at least $2^{N/h-1}$ different expressions as a sum of two elements in $A_{\alpha(N)}$. Since A is partitioned into a finite number of parts, there are infinitely many values of N with the same value $\alpha(N)$. ∎

Let us now consider additive bases in $\mathbb{Z}$. For $n \in \mathbb{Z}$ let $S(n)$ denote the 2-adic support of n in the sense that $|n| = \sum_{i\in S(n)} 2^i$. The notion of 2-bounded sets is extended to subsets of $\mathbb{Z}$ in the obvious way.

3. Additive bases for $\mathbb{Z}$. The notion of (d,h)-bounded basis extends naturally to bases in $\mathbb{Z}$. For simplicity we only consider $(2,2)$-bounded bases. In order to preserve the unicity of binary expansions, we define the binary support of a negative integer as $S(x) = -S(|x|)$, that is, $x = -\sum_{i\in S(x)} 2^{-i}$. A set B of integers is *bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in B+B$ there is a pair $x, y \in B$ satisfying

$$n = x + y, \qquad |S(x) \cup S(y)| \le f(|S(n)|).$$

We can now proceed to prove Theorem 5.

Proof of Theorem 5. We construct an increasing family $\{A_k : k \ge 1\}$ of sets of integers with the following properties:

(i) $r_{A_k}(n) \le 1$ for all $n \in \mathbb{Z}$.
(ii) $A_k + A_k \supseteq (-k/2, k/2)$ for all $k \ge 1$.
(iii) For each pair $x, y \in A_k$ we have $|S(x) \cup S(y)| \le 4|S(x+y)|$.

From (i) and (ii) it follows that $A = \bigcup_{k\ge 1} A_k$ is a unique representation basis of $\mathbb{Z}$ and, by (iii), it is 2-bounded.

For each integer $n \in \mathbb{Z}$ define $\sigma(n)$ by

$$S(\sigma(n)) = \{i \in S(n) : i-1 \notin S(n)\}.$$

Then $|S(n+\sigma(n))| = |S(\sigma(n))| \le |S(n)|$ and $S(n+\sigma(n))$ has no two consecutive integers.

Let $A_1 = \{0\}$, $A_2 = \{-10, 0, 9\}$ and suppose we have constructed A_k satisfying (i)–(iii) above and such that the support of each element in A_k has no two consecutive integers.

Let n_k be an element not in $A_k + A_k$ with smallest absolute value. Suppose that $n_k > 0$, the other case being similar. Choose a subset U_k of positive integers with cardinality $|U_k| = |S(n_k)|$ such that $\min U_k > 1 + \max_{x\in A_k+A_k} \max\{i \in S(|x|)\}$ and U_k has no two consecutive integers. Define

$$a_k = n_k + \sigma(n_k) + \sum_{i\in U_k} 2^i, \qquad b_k = n_k - a_k, \qquad A_{k+1} = A_k \cup \{a_k, b_k\}.$$

We have

$$A_{k+1} + A_{k+1} = \{n_k\} \cup (A_k + A_k) \cup (A_k + a_k) \cup (A_k + b_k),$$

where, by the choice of a_k and b_k, the four sets on the right hand side are pairwise disjoint. Hence A_{k+1} satisfies (i). Since $n_k \in A_{k+1} + A_{k+1}$, (ii) is also satisfied. Let us check that (iii) also holds. Since $|S(a_k)| \le 2|S(n_k)|$ and

$|S(b_k)| \le 2|S(n_k)|$, it follows that

$$|S(a_k) \cup S(b_k)| \le 4|S(n_k)| = 4|S(a_k + b_k)|.$$

Let $x \in A_k$. Since $S(x)$ has no two consecutive integers, we have

$$|S(x) \cup S(a_k)| \le |S(x + a_k)| \quad \text{if } x > 0,$$

and it can be easily checked that

$$|S(x) \cup S(a_k)| \le 4|S(x + a_k)| \quad \text{if } x < 0.$$

Indeed, let $V = S(x)$ and $V' = S(|x|) \setminus S(a_k)$. Then

$$\begin{aligned} |S(x) \cup S(a_k)| &\le |V| + 2|S(n_k)| \le |V'| + 3|S(n_k)| \le 4(|V'| + |U_k| - 1) \\ &\le 4|S(x + a_k)|, \end{aligned}$$

where in the last inequality we used the facts that V' has no two consecutive integers, that $\max S(|x|) < \min U_k$, and that for any positive integers r and s, $2^{r+s} - 2^r = 2^{r+s-1} + \dots + 2^r$. Similar arguments show that

$$|S(x) \cup S(b_k)| \le 4|S(x + b_k)|. \ \blacksquare$$

Proof of Theorem 6. Set $Z = r^{-1}(\infty)$ and $Z' = \mathbb{Z} \setminus Z$. Consider the set $F \subset \mathbb{Z} \times \mathbb{N}$ of ordered pairs

$$F = \{(i,j) : i \in Z,\ j = 1, 2, \ldots\} \cup \{(i,j) : i \in Z',\ j = 1, \ldots, r(i)\},$$

with the ordering defined as

$$(i,j) \preceq (i',j') \Leftrightarrow \begin{cases} |i| + j < |i'| + j', \text{ or} \\ |i| + j = |i'| + j' \text{ and } j < j', \text{ or} \\ |i| + j = |i'| + j' \text{ and } j = j' \text{ and } |i| < |i'|, \text{ or} \\ |i| + j = |i'| + j' \text{ and } j = j' \text{ and } |i| = |i'| \text{ and } i < i', \end{cases}$$

so that $F = \{f_1 = (0,1), f_2 = (-1,1), f_3 = (1,1), f_4, \ldots\}$ in this ordering.

We construct a non-decreasing family $\{A_k : k \ge 1\}$ of sets of integers satisfying

(a) $r_{A_k}(n) = \max\{j : (n,j) \preceq f_k\}$ if $(n,1) \preceq f_k$ and $r_{A_k}(n) \le 1$ otherwise,

(b) $r_{A_k^+}(n) \le 1$ and $r_{A_k^-}(n) \le 1$ for all $n \in \mathbb{Z}$, where $A_k^+ = A_k \cap [0,\infty)$ and $A_k^- = A_k \setminus A_k^+$.

Once such a family of sets has been constructed, the set $A = \bigcup_{k \ge 1} A_k$ satisfies

$$r_A(n) = \lim_{k \to \infty} r_{A_k}(n) = r(n) \quad \text{for all } n \in \mathbb{Z}$$

while both $A^+ = A \cap [0,\infty)$ and $A^- = A \setminus A^+$ are B_2^1 sequences.

Let $A_1 = \{0\}$ and suppose that A_{k-1} satisfies (a) and (b) above. Let $f_k = (i,j)$. Then $r_{A_{k-1}}(i) = j - 1$ if $j > 1$ and $r_{A_{k-1}}(i) \le 1$ if $j = 1$.

If $r_{A_{k-1}}(i) = j = 1$ then we define $A_k = A_{k-1}$.

If $r_{A_{k-1}}(i) = j - 1$, let $x_k = \max\{|x| : x \in A_{k-1} + A_{k-1}\} + |A_{k-1}| + |i|$ and define

$$a_k = i + x_k, \quad b_k = -x_k, \quad A_k = A_{k-1} \cup \{a_k, b_k\}.$$

Then

$$A_k + A_k = ((A_{k-1} + A_{k-1}) \cup \{i\}) \cup (A_{k-1} + a_k) \cup (A_{k-1} + b_k),$$

where, by the choice of x_k, the three sets on the right hand side are pairwise disjoint. Therefore $r_{A_k}(n) \le r_{A_{k-1}}(n) + 1$ for all $n \in \mathbb{Z}$, and equality holds if either $n = i$, so that $r_{A_k}(i) = j$, or $n \in (A_{k-1} + a_k) \cup (A_{k-1} + b_k)$, in which case $f_k \prec (n, 1)$ and $r_{A_k}(n) = 1$. Thus A_k satisfies (a). Moreover, since x_k is large enough, both $A_k^+ = A_{k-1}^+ \cup \{a_k\}$ and $A_k^- = A_{k-1}^- \cup \{b_k\}$ are still B_2^1 sets. This completes the proof. ∎

4. Remarks and open problems. It is pleasing to note that the multiplicative version of Erdős–Turán conjecture is valid both in $\mathbb{N}$ and in $\mathbb{Z}$. However the additive version for 2-bounded bases is true in $\mathbb{N}$ while it fails to be true in $\mathbb{Z}$ even in the stronger form of Theorem 6 (we note that the basis in this theorem can be required to be bounded as well). The proof of Theorem 5, though, involves constructing a very sparse basis for the set of all integers. Thicker bases for the integers without the ET property have been constructed by Nathanson [9] and Łuczak and Schoen [8] with counting function of order $n^{1/3}$. It might be true that unboundedness of the representation function appears when the lower density of the set has order $n^{1/2}$. In that respect the following may be an easier problem.

Problem 1. *Let A be an additive basis of $\mathbb{N}$ and assume that it has the ET property. Is it true that in any finite partition of A one of the parts still has the ET property?*

Perhaps the following perspective may contribute to a better understanding of the Erdős–Turán conjecture. Given a function $f : \mathbb{N} \to \mathbb{N}$, we say that a basis $A \subset \mathbb{Z}$ is *f-restricted* if for any $n \in \mathbb{Z}$ there exist $x, y \in X$ such that

$$n = x + y, \quad \max\{|x|, |y|\} \le f(n).$$

A basis for the set of positive integers is f-restricted with $f(n) = n$, so that this case for bases of $\mathbb{Z}$ seems to be closely related to the Erdős–Turán conjecture. One possible formulation of the link between the two problems is the following:

Problem 2. *Is it true that for every thin additive basis A of $\mathbb{N}$ there exists a basis B of $\mathbb{N}$ and a constant c which bounds the number of solutions $n = a - b$, $a \in A$, $b \in B$, for every $n \in \mathbb{N}$?*

A positive answer implies that proving the ET property for f-restricted bases in $\mathbb{Z}$ with $f(n) = n$ gives a proof of the Erdős–Turán conjecture.

On the one side of the spectrum the function $f(n) = n/2 + c$ yields a positive solution of the Erdős–Turán conjecture for f-restricted bases.

PROPOSITION 1. *For any positive integer c an f-restrictive basis in $\mathbb{Z}$ with $f(n) = n/2 + c$ has the Erdős–Turán property.*

Proof. Let A be an f-restricted basis of $\mathbb{Z}$. Define sets A_i, $i = 0, 1, \ldots, 2c$, by $n \in A_i$ iff there exists $x \in A$ such that $n/2 - x = i/2$. By the van der Waerden Theorem, one of the sets A_i contains an arithmetic progression of any length. Since an arithmetic progression of length k yields $k/2$ pairs of elements with the same sum, the basis A has the Erdős–Turán property. ∎

Of course the same proof gives the Erdős–Turán property for f-restricted bases for functions $f(n) = n/2 + \varepsilon(n)$, where $\varepsilon(n)$ is a very slowly growing function (essentially the inverse function to the van der Waerden's function $W(k, k)$; by Shelah and Gowers this is not as astronomically slowly growing as it seems at first glance). These remarks also apply to f-restricted bases in $\mathbb{N}$.

By using a greedy algorithm as in the above proof of Theorem 5 we easily get examples of f-restricted bases for $\mathbb{Z}$ without the ET property for any function $f(n) > 2^n$. It would be interesting to find examples for more slowly growing functions as suggested by the following problem.

PROBLEM 3. *Fix a positive $c \in \mathbb{R}$. Let A be a basis for $\mathbb{Z}$ such that for each sufficiently large $n \in A + A$ there are $x, y \in A$ with $n = x + y$ and $\max\{|x|, |y|\} \leq cn$. Is it true that A has the ET property?*

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