The exponential sum over squarefree integers

by

JAN-CHRISTOPH SCHLAGE-PUCHTA (Freiburg)

Denote by \( r_\nu(N) \) the number of representations of \( N \) as the sum of \( \nu \) squarefree numbers. In a series of papers Evelyn and Linfoot \([3]-[8]\) proved that

\[
r_\nu(N) = S_\nu(N) N^{\nu-1} + O(N^{\nu-1-\theta(\nu)+\varepsilon}),
\]

where

\[
S_\nu(N) = \frac{1}{(\nu-1)!} \left( \frac{6}{\pi^2} \right)^\nu \prod_{p^2 \mid N} \left( 1 - \frac{1}{(1-p^2)^\nu} \right) \prod_{p^2 \mid N} \left( 1 - \frac{1}{(1-p^2)^{\nu-1}} \right),
\]

and

\[
\theta(2) = \theta(3) = \frac{1}{3}, \quad \theta(\nu) = \frac{1}{2} - \frac{1}{2\nu} \quad (\nu \geq 4).
\]

Mirsky \([9]\) improved the error term for \( \nu \geq 3 \) to

\[
\theta(\nu) = \frac{1}{2} - \frac{1}{4\nu - 2}.
\]

Using a new approach to bound the minor arc integral developed by Brüdern, Granville, Perelli, Vaughan and Wooley \([1]\), Brüdern and Perelli \([2]\) showed that \( \theta = 1/2 \) for all \( \nu \geq 3 \), and that any further improvement would imply a quasiriemannian hypothesis. Moreover, assuming the generalized riemannian hypothesis, they proved that \( \theta(3) = 3/4 + 1/14 \) and \( \theta(\nu) = 3/4 \) for all \( \nu \geq 4 \). These results are optimal apart from the summand 1/14; in a personal communication Brüdern conjectured that \( \theta(3) = 3/4 \) should hold true. It is the aim of this note to prove this conjecture.

Define \( S(\alpha) = \sum_{n \leq N} \mu^2(n)e(an) \). For integers \( N \) and \( Q \) satisfying \( 1 \leq Q < N^{1/2}/2 \), let \( \mathcal{M}(Q) \) be the union of all intervals \( \{ \alpha : |\alpha q - a| \leq QN^{-1} \} \), where \( q \leq Q \), and \( (a,q) = 1 \). Set \( m(Q) = [QN^{-1}, 1 - QN^{-1}] \setminus \mathcal{M}(Q) \). With this notation we will prove the following.

**Theorem 1.** We have \( S(\alpha) \ll N^{1+\varepsilon}Q^{-1} \) for all \( \alpha \in m(Q) \), provided that \( Q \leq N^{1/2} \).

---

2000 Mathematics Subject Classification: 11L07, 11N25, 11P55.

Key words and phrases: squarefree integers, exponential sum, circle method.
Under the restriction $Q \leq N^{3/7}$, this was proven in [2, Theorem 4]. As already remarked in [2, Sec. 5], the weakening of the assumption on $Q$ implies the following.

**Theorem 2.** Assume the generalized riemannian hypothesis. Then
\[ r_3(N) = \mathcal{G}(N)N^2 + O(N^{5/4+\varepsilon}). \]

By Dirichlet’s theorem on diophantine approximation, for every $\alpha \in \mathcal{m}(Q)$ there exist coprime integers $a, q$ with $q \leq NQ^{-1}$ such that $|q\alpha - a| \leq N^{-1}Q$. By the definition of $\mathcal{m}(Q)$, we necessarily have $q > Q$. Hence, Theorem 1 is essentially equivalent to the following.

**Theorem 3.** Define $S(\alpha)$ as above, and let $q$ be an integer satisfying $|q\alpha - a| \leq q^{-1}$. Then
\[ |S(\alpha)| \ll N^{1+\varepsilon}q^{-1} + N^{\varepsilon}q. \]

We approach Theorem 3 by the following lemma, which replaces Lemma 1 in [2].

**Lemma 1.** Let $\alpha \in (0,1)$ be a real number, and assume that $|q\alpha - a| < 1/q$. Let $D$ be an integer, and denote by $W(D, z)$ the number of integers $d \leq D$ satisfying $\|d^2\alpha\| \leq z$. Then, for $D^2 > \frac{1}{4}q$, we have
\[ W(D, z) \ll D^2q^{-1} + D^{1+\varepsilon}z^{1/2}. \]

**Proof.** Decompose the interval $[1, D^2]$ into $K = [D^2q^{-1}] + 1$ intervals of length $q$, where the last interval may be shorter. For $k \leq K$, let $a_k$ be the number of integers $d$ such that $\|d^2\alpha\| \leq z$ and $kq \leq d^2 < (k+1)q$. Then $\sum_{k \leq K} a_k = W(D, z)$, and by the arithmetic-quadratic mean inequality, $\sum_{k \leq K} a_k^2 \geq W(D, z)^2K^{-1}$. Denote by $\mathcal{D}$ the set of all pairs $(d_1, d_2)$ with $\|d_1^2\alpha\| \leq z$ and $1 \leq |d_1^2 - d_2^2| \leq q$. Then either $W(D, z) \leq 2K$, which is sufficiently small, or we can bound $|\mathcal{D}|$ from below via
\[ |\mathcal{D}| \geq \sum_k \left( \frac{a_k}{2} \right) \gg \sum_k a_k^2 - \sum_k a_k \gg \sum_k a_k^2 \gg W(D, z)^2K^{-1}. \]

Denote by $\mathcal{N} \subseteq [1, q]$ the set of all values of $|d_1^2 - d_2^2|$, where $d_1, d_2$ range over all pairs in $\mathcal{D}$. Then every pair in $\mathcal{D}$ gives rise to an element of $\mathcal{N}$, and the number of different pairs $d_1, d_2$ having the same difference $d_1^2 - d_2^2 = n$ is bounded above by the number of divisors of $n$, and therefore $\ll q^{\varepsilon}$. Hence, we deduce
\[ W(D, z)^2 \ll |\mathcal{D}|K \ll |\mathcal{N}|Kq^{\varepsilon}. \]

On the other hand, for every $n \in \mathcal{N}$, we have $\|n\alpha\| \leq \|d_1^2\alpha\| + \|d_2^2\alpha\| \leq 2z$, hence
\[ W(D, z)^2 \ll D^2q^{\varepsilon-1}|\{n \leq q : \|\alpha n\| \leq 2z\}| \ll D^2q^{\varepsilon-1}(qz + 1). \]
From this, in the case $W(D, z) > 2K$ we obtain
\[ W(D, z) \ll D^{1+\varepsilon} z^{1/2} + D^{1+\varepsilon} q^{-1/2}, \]
which is again of the right size, since $D > \frac{1}{2}q^{1/2}$. ■

Proof of Theorem 3. Write
\[
S(\alpha) = \sum_{d \leq \sqrt{N}} \mu(d) \sum_{m \leq N d^{-2}} e(\alpha d^2 m)
\]
\[
\ll \log N \max_{1 \leq D \leq \sqrt{N}/2} \sum_{D \leq d < 2D} \min \left( \frac{N}{D^2}, \| \alpha d^2 \|^{-1} \right)
\]
\[
= \log N \max_{1 \leq D \leq \sqrt{N}/2} \Upsilon(\alpha, D),
\]
say. To prove Theorem 3, it suffices to show that $\Upsilon(\alpha, D) \ll N^{1+\varepsilon} q^{-1}$ for all $D \leq \sqrt{N}/2$. For $D > \frac{1}{2}q^{1/2}$, we have
\[
\Upsilon(\alpha, D) \ll \log N \max_{z > N/D^2} z^{-1} W(D, z)
\]
\[
\ll \log N \max_{z > N/D^2} (z^{-1} D^2 q^{-1} + D^{1+\varepsilon} z^{-1/2}) \ll N^{1+\varepsilon} q^{-1} + N^{1/2+\varepsilon}.
\]
For $D \leq \frac{1}{4}q^{1/2}$, we argue as in the proof of [2, Lemma 1]. We have
\[
\left| \alpha d^2 - \frac{ad^2}{q} \right| \leq 4D^2 \left| \alpha - \frac{a}{q} \right| \leq 4D^2 q^{-2} \leq \frac{1}{4q},
\]
and therefore
\[
|\Upsilon(\alpha, D)| \leq 2 \sum_{D \leq d < 2D} \left\| \frac{ad^2}{q} \right\| \ll q \log q \ll N^{\varepsilon} q.
\]
Taking these estimates together, we find that
\[
S(\alpha) \ll N^{1+\varepsilon} q^{-1} + N^{1/2+\varepsilon} + N^{\varepsilon} q,
\]
and the second term is always dominated by either the first or the last one, which implies our theorem. ■

References


Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg, Germany
E-mail: jcp@mathematik.uni-freiburg.de

Received on 10.12.2003