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# Integers free of prime divisors from an interval, I

by

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**1. Introduction.** Let $\Gamma(x,y,z)$ be the number of positive integers not exceeding $x$ which are free of prime divisors from the interval $(z,y]$. Our interest in this function is two-fold. On the one hand it constitutes a generalization of the functions

$$\Psi(x,z)=\sum_{\substack{n\le x\\ P^+(n)\le z}} 1 \quad \text{and} \quad \Phi(x,y)=\sum_{\substack{n\le x\\ P^-(n)>y}} 1,$$

where $P^+(n)$ and $P^-(n)$ denote the largest and smallest prime divisor of $n$, respectively. These functions play an important role in various applications of number theory, such as an improvement of the Selberg sieve by Jurkat and Richert [7], a study of the distribution of $k$th power residues of primes by Davenport and Erdős [5] and a general convolution method by Daboussi [4]. The behaviour of $\Psi$ and $\Phi$ has been studied intensively by several authors (see, for example, [1, 2]). For an overview on results on $\Psi$ and $\Phi$ see Tenenbaum [14].

On the other hand, the behaviour of $\Gamma(x,y,z)$ is worth studying in its own right. The quantity $\Gamma(x,y,z)$ arises in applications such as the construction of large prime gaps (see Rankin [8, 9] or Schönhage [13]).

Throughout we will use the notation

$$u=\frac{\log x}{\log y} \quad \text{and} \quad v=\frac{\log x}{\log z}.$$

*Dickman's function* $\varrho$ is defined to be the unique continuous solution to the difference-differential equation

$$v\varrho'(v)+\varrho(v-1)=0 \quad (v>1),$$

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together with the initial condition

$$\varrho(v) = 1 \qquad (0 \le v \le 1).$$

Let $\varrho(v) = 0$ for $v < 0$ and extend $\varrho'(v)$ to $v = 1$ by right continuity.

For $u > 1$, the *Buchstab function* $\omega(u)$ is defined as the unique continuous solution to the difference-differential equation

$$(u\omega(u))' = \omega(u-1) \qquad (u > 2)$$

with initial condition

$$u\omega(u) = 1 \qquad (1 \le u \le 2).$$

Let $\omega(u) = 0$ for $u < 1$. Define $\omega$ at 1 and $\omega'$ at 1 and 2 by right continuity.

The functions $\varrho(v)$ and $\omega(u)$ arise in well known estimates for the functions $\Psi(x,z)$ and $\Phi(x,y)$. We will make use of the following two results. The first estimate is an easy consequence of a well known estimate of de Bruijn [2] (for a slightly weaker estimate see Theorem III.5.6 in Tenenbaum [14]). The function $\Phi$ was first studied by Buchstab [3] and later by de Bruijn [1]. For the estimate on $\Phi$ below see also Theorem III.6.3 in Tenenbaum [14].

THEOREM 1.1. *Uniformly for $x \ge z \ge 2$, we have*

$$\Psi(x,z) = x\varrho(v) + O\left(\frac{x \log z}{\log^2 x}\right).$$

THEOREM 1.2. *Uniformly for $x \ge y \ge 2$, we have*

$$\Phi(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

Theorems 1.1 and 1.2 allow us to derive an asymptotic formula for $\Gamma(x,y,z)$. This formula involves a function $\eta(u,v)$, defined for $0 < u \le v$ by

$$\eta(u,v) := \varrho(v) + \int_0^u \varrho(tv/u)\omega(u-t)\,dt \qquad (0 < u \le v). \tag{1}$$

For $0 < u \le 1$, the integral vanishes and we have $\eta(u,v) = \varrho(v)$. We extend the definition of $\eta(u,v)$ to all real values of $u$ and $v$ by setting $\eta(0,v) := \varrho(v)$, $\eta(u,v) := 0$ for $u < 0$ or $v < 0$ and finally $\eta(u,v) = 1$ for $0 \le v < u$. Note that $\eta(u,v)$ is continuous for $0 < u < v$ since $\varrho(v)$ is continuous for $v > 0$ and the integrand is uniformly bounded. The function $\eta(u,v)$ is also continuous at $0 < u = v$ due to the well known convolution identity (see [14], p. 420)

$$1 = \varrho(u) + \int_0^u \varrho(t)\omega(u-t)\,dt.$$

Also, $\eta(u,v) > 0$ for $0 \le u \le v$.

A function closely related to $\Gamma(x,y,z)$ is the function $\theta(x,y,z)$, which denotes the number of positive integers not exceeding $x$ all of whose prime divisors are in the interval $(z, y]$. The latter function has been studied by Friedlander [6] and, more recently, by Saias [10–12]. Like $\Gamma(x, y, z)$, $\theta(x, y, z)$ is also a generalization of the functions $\Psi$ and $\Phi$.

The function

$$\sigma(u,v) := \frac{u}{v}\,\varrho'(u) + \int_0^\infty \varrho\left(u - \frac{u}{v}t\right) d\omega(t) \qquad (v \ge u > 0, u \ne 1) \tag{2}$$

arises in the study of $\theta(x, y, z)$. For results on $\sigma(u, v)$ see Friedlander [6] and Saias [10].

In this paper we establish an asymptotic estimate for $\Gamma(x, y, z)$ by expressing this function in terms of the functions $\Psi$ and $\Phi$ and estimating the latter functions by means of Theorems 1.1 and 1.2. In a sequel to this paper (see [15]), we will use complex integration together with the saddle-point method to sharpen many of the results obtained here.

**2. Some observations.** Let $P = \prod_{z<p\le y} p$. The Möbius inversion formula allows us to write the characteristic function of the set of integers $n$ which are free of prime divisors from the interval $(z, y]$ as

$$\sum_{\substack{d\mid P\\ d\mid n}} \mu(d).$$

Summing over $n \le x$, we obtain

$$\begin{aligned} \Gamma(x,y,z) &= \sum_{d\mid P} \mu(d)\left[\frac{x}{d}\right] = \sum_{d\mid P} \mu(d)\frac{x}{d} - \sum_{d\mid P} \mu(d)\left\{\frac{x}{d}\right\} \\ &= x \prod_{z<p\le y}\left(1 - \frac{1}{p}\right) - \sum_{d\mid P} \mu(d)\left\{\frac{x}{d}\right\}. \end{aligned} \tag{3}$$

Let us first consider the trivial estimate

$$\left|\sum_{d\mid P} \mu(d)\left\{\frac{x}{d}\right\}\right| \le \sum_{d\mid P} 1 \le 2^{y-z}.$$

If $y - z \le \log x$ we have $2^{y-z} \le x^{\log 2} < x^{3/4}$. The main term in (3) is $\gg x/\log x$, by Mertens' formula. This gives

$$\Gamma(x,y,z) \sim x \prod_{z<p\le y}\left(1 - \frac{1}{p}\right) \tag{4}$$

for $y - z \le \log x$.

With $u = \log x/\log y$, the fundamental lemma of the combinatorial sieve (see [14], Theorem I.4.3) implies that, uniformly for $x \ge y \ge z \ge 3/2$,

$$(5) \qquad \Gamma(x,y,z) = x \prod_{z<p\le y} \left(1-\frac{1}{p}\right)\{1+O(u^{-u/2})\} + O(\theta(x,y,z)).$$

Since $\theta(x,y,z) \le \Psi(x,y) \ll xe^{-u/2}$ (see Theorem III.5.1 of [14]), it follows from (5) that (4) holds in the domain $1 \le z \le y \le \exp\{c\log x/\log_2 x\}$, for some suitable constant $c$.

It is easy to see, however, that (4) does not hold uniformly for $1 \le z \le y \le \sqrt{x}$. To this end choose $z = 1$ and $y = \sqrt{x}$. Then the prime number theorem gives

$$\Gamma(x,y,z) \sim \frac{x}{\log x},$$

but Mertens' formula shows that

$$x \prod_{1<p\le\sqrt{x}} \left(1-\frac{1}{p}\right) \sim \frac{2xe^{-\gamma}}{\log x},$$

where $\gamma$ denotes Euler's constant.

**3. An asymptotic formula for $\Gamma(x,y,z)$.** In this section we will derive the following result for $\Gamma(x,y,z)$.

THEOREM 3.1. *Uniformly for $x \ge y \ge z \ge 3/2$, we have*

$$\Gamma(x,y,z) = x\eta(u,v) + O\left(\frac{x}{\log y}\right).$$

Theorem 3.1 and the lower bound on $\eta(u,v)$ in Lemma 3.4 (see below) will allow us to deduce the following asymptotic result.

THEOREM 3.2. *Uniformly for $y \ge z \ge 3/2$, we have*

$$\Gamma(x,y,z) = \begin{cases} x\eta(u,v)\left\{1+O\left(\dfrac{1}{\log z}\right)\right\} & \textit{for } x \ge yz, \\[2ex] x\eta(u,v)\left\{1+O\left(\dfrac{1}{\log(x/y)+\varrho(v)\log x}\right)\right\} & \textit{for } y \le x < yz. \end{cases}$$

The proof of Theorem 3.1 is based on the following identity for $\Gamma(x,y,z)$, which allows us to derive the result from the known estimates for $\Psi(x,z)$ and $\Phi(x,y)$ given in Theorems 1.1 and 1.2.

LEMMA 3.3. *For $x \ge y \ge z \ge 1$ we have*

$$\Gamma(x,y,z) = \Psi(x,z) - \Psi(x/y,z) + \sum_{\substack{n\le x/y \\ P^+(n)\le z}} \Phi(x/n,y).$$

*Proof.* If the integer $m \ge 1$ is counted in $\Gamma(x,y,z)$, we can write $m = nd$ with $P^+(n) \le z$ and $P^-(d) > y$. For each $n \le x/y$ there are $\Phi(x/n,y)$

possible choices for $d$. If $x/y < n \le x$ then $d = 1$ is the only choice and the contribution from these $n$ is $\Psi(x,z) - \Psi(x/y,z)$.

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* The result being trivial for bounded $y$, we assume that $y > y_0$ for a sufficiently large constant $y_0$. If $x \le yz$, it follows from Lemma 3.3 and Theorems 1.1 and 1.2 that

$$\begin{aligned}\Gamma(x,y,z) &= \Psi(x,z) - \Psi(x/y,z) + \sum_{n\le x/y} \Phi(x/n,y) \\ &= x\varrho(v) + O\left(\frac{x}{\log y}\right) \\ &\quad + \sum_{n\le x/y}\left\{\frac{\frac{x}{n}\omega\left(\frac{\log(x/n)}{\log y}\right) - y}{\log y} + O\left(\frac{x/n}{\log^2 y}\right)\right\}.\end{aligned} \tag{6}$$

Hence

$$\Gamma(x,y,z) = x\varrho(v) + \sum_{n\le x/y}\frac{x\omega\left(\frac{\log(x/n)}{\log y}\right)}{n\log y} + O\left(\frac{x}{\log y}\right), \tag{7}$$

for $x \le yz$. Now $\omega(\log(x/t)/\log y)/t$ is a monotonic function in $t$, since

$$\left(\frac{\omega\left(\frac{\log(x/t)}{\log y}\right)}{t}\right)' = \frac{1}{t^2}\left(-\omega'\left(\frac{\log(x/t)}{\log y}\right)(\log y)^{-1} - \omega\left(\frac{\log(x/t)}{\log y}\right)\right) < 0$$

for $y > y_0$. Thus, we can write

$$\sum_{n\le x/y}\frac{x\omega\left(\frac{\log(x/n)}{\log y}\right)}{n\log y} = \int_1^{x/y}\frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\,dt + O\left(\frac{x}{\log y}\right). \tag{8}$$

For $0 \le v \le 1$ we have $\varrho(v) = 1$. Thus (7) and (8) imply, for $x \le yz$,

$$\begin{aligned}\Gamma(x,y,z) &= x\varrho(v) + \int_1^{x/y}\frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\varrho\left(\frac{\log t}{\log z}\right)dt + O\left(\frac{x}{\log y}\right) \\ &= x\varrho(v) + x\int_0^{u-1}\omega(u-t)\,\varrho(tv/u)\,dt + O\left(\frac{x}{\log y}\right) \\ &= x\eta(u,v) + O\left(\frac{x}{\log y}\right).\end{aligned}$$

In the remainder of the proof we establish the theorem for $x > yz$. From Lemma 3.3 and Theorem 1.1 we have

$$\Gamma(x,y,z) = x\varrho(v) + \sum_{\substack{n\le x/y \\ P^+(n)\le z}}\Phi(x/n,y) + O\left(\frac{x}{\log y}\right). \tag{9}$$

Theorem 1.2 implies

$$(10)\quad \sum_{\substack{n\le x/y\\ P^+(n)\le z}} \Phi(x/n,y) = \sum_{\substack{n\le x/y\\ P^+(n)\le z}} \left\{ \frac{\frac{x}{n}\omega\left(\frac{\log(x/n)}{\log y}\right)-y}{\log y} + O\left(\frac{x/n}{\log^2 y}\right)\right\}$$

$$= \int_1^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, d\Psi(t,z) + O\left(\frac{x}{\log y}\right).$$

By Theorem 1.1, we have

$$\int_1^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, d\Psi(t,z)$$

$$= \int_1^{z} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, d[t] + \int_z^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, d\left\{t\varrho\left(\frac{\log t}{\log z}\right)+E(t)\right\}$$

$$= \int_1^{z} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\varrho\left(\frac{\log t}{\log z}\right) d(t-\{t\}) + \int_z^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\varrho\left(\frac{\log t}{\log z}\right) dt$$

$$+ \int_z^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\cdot \frac{\varrho'\left(\frac{\log t}{\log z}\right)}{\log z}\, dt + \int_z^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, dE(t),$$

where $E(t) = O((t\log z)/\log^2 t)$.

Use integration by parts on the first and last integral, noting that $\omega$ and $\omega'$ are uniformly bounded, and observe that

$$\int_1^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\varrho\left(\frac{\log t}{\log z}\right) dt = x\int_0^{u-1} \omega(u-t)\varrho(tv/u)\, dt = x(\eta(u,v)-\varrho(v)).$$

We obtain

$$\int_1^{x/y} \frac{x\omega\left(\frac{\log(x/t)}{\log y}\right)}{t\log y}\, d\Psi(t,z)$$

$$= x\eta(u,v) - x\varrho(v) + O\left(\frac{x}{\log y}\right) + O\left(\frac{x}{\log y}\int_z^{x/y} \frac{\varrho'\left(\frac{\log t}{\log z}\right)}{t\log z}\, dt\right)$$

$$+ O\left(\frac{x}{\log y}\int_z^{x/y} \frac{t\log z}{\log^2 t}\, d\left\{\frac{\omega\left(\frac{\log(x/t)}{\log y}\right)}{t}\right\}\right)$$

$$= x\eta(u,v) - x\varrho(v) + O\bigg(\frac{x}{\log y}\bigg) + O\bigg(\frac{x}{\log y}\varrho\bigg(\frac{\log t}{\log z}\bigg)\bigg|_z^{x/y}\bigg)$$

$$+ O\bigg(\frac{x}{\log y}\int_z^{x/y}\frac{t\log z}{\log^2 t}\cdot\frac{1}{t^2}\,dt\bigg).$$

Thus,

$$\int_1^{x/y}\frac{x\omega\big(\frac{\log(x/t)}{\log y}\big)}{t\log y}\,d\Psi(t,z) = x\eta(u,v) - x\varrho(v) + O\bigg(\frac{x}{\log y}\bigg). \tag{11}$$

So (10) and (11) show that for $x > yz$ we have

$$\sum_{\substack{n\le x/y\\ P^+(n)\le z}}\Phi(x/n,y) = x\eta(u,v) - x\varrho(v) + O\bigg(\frac{x}{\log y}\bigg).$$

Together with (9) this gives

$$\Gamma(x,y,z) = x\eta(u,v) + O\bigg(\frac{x}{\log y}\bigg),$$

for $x > yz$, and hence for all $x \ge y \ge z \ge 3/2$. This completes the proof of Theorem 3.1.

LEMMA 3.4. *If $u \le v$, then*

(i) $$\eta(u,v) \le e^\gamma\frac{u}{v} \qquad \textit{for } u \ge 1,$$

(ii) $$\eta(u,v) \ge \frac{u}{2v} \qquad \textit{for } u \ge \frac{v}{v-1}.$$

*Proof.* (i) Since $\omega(u) \le 1$ we have, for $u \ge 1$,

$$\eta(u,v) = \varrho(v) + \int_0^{u-1}\varrho(tv/u)\omega(u-t)\,dt \le \varrho(v) + \int_0^{u-1}\varrho(tv/u)\,dt$$

$$= \varrho(v) + \frac{u}{v}\int_0^{v(1-1/u)}\varrho(t)\,dt \le \varrho(v) + \frac{u}{v}e^\gamma - \frac{u}{v}\int_{v(1-1/u)}^{\infty}\varrho(t)\,dt$$

$$\le \varrho(v) + \frac{u}{v}e^\gamma - u\frac{1}{v}\int_{v-1}^{v}\varrho(t)\,dt = \frac{u}{v}\,e^\gamma + \varrho(v) - u\varrho(v) \le \frac{u}{v}e^\gamma.$$

(ii) Since $\omega(u) \ge 1/2$ for $u \ge 1$, we have, for $u \ge v/(v-1)$,

$$\eta(u,v) \ge \int_0^{u-1}\varrho(tv/u)\omega(u-t)\,dt \ge \frac{u}{2v}\int_0^{v-v/u}\varrho(t)\,dt \ge \frac{u}{2v}\int_0^1\varrho(t)\,dt = \frac{u}{2v}.$$

*Proof of Theorem 3.2.* If $y \le x < yz$, then $1 \le u < v/(v-1)$ and

$$\eta(u,v) = \varrho(v) + \int_0^u \varrho(tv/u)\omega(u-t)\,dt = \varrho(v) + \int_0^{u-1} \frac{1}{u-t}\,dt$$
$$= \varrho(v) + \log u > \varrho(v) + (u-1)/2,$$

which gives

$$\eta(u,v)\log y > (\log y)\varrho(v) + \frac{\log(x/y)}{2} > \frac{1}{2}((\log x)\varrho(v) + \log(x/y)).$$

If $x \ge yz$ then $u \ge v/(v-1)$ and Lemma 3.4 shows that

$$\eta(u,v)\log y \ge (\log y)u/(2v) = (\log z)/2.$$

Theorem 3.1 yields

$$\Gamma(x,y,z) = x\eta(u,v)\left\{1 + O\left(\frac{1}{\eta(u,v)\log y}\right)\right\},$$

which concludes the proof of Theorem 3.2.

**4. The difference-differential equations for $\eta(u,v)$.** Like $\Psi(x,z)$ and $\Phi(x,y)$, $\Gamma(x,y,z)$ also satisfies functional equations.

PROPOSITION 4.1. (i) *Let $z \le y_1 \le y_2 \le x$. Then*

$$\Gamma(x,y_1,z) - \Gamma(x,y_2,z) = \sum_{y_1 < p \le y_2} \Gamma(x/p, p-1, z).$$

(ii) *Let $z_1 \le z_2 < y < x$. Then*

$$\Gamma(x,y,z_2) - \Gamma(x,y,z_1) = \sum_{z_1 < p \le z_2} \Gamma(x/p, y, p).$$

*Proof.* Part (i) follows from grouping the integers contributing to the left hand side according to their smallest prime divisor in $[y_1, x]$. Part (ii) follows from grouping the integers contributing to the left hand side according to their largest prime divisor in $[2, z_2]$.

The smooth versions of the functional equations in Proposition 4.1 are the following difference-differential equations.

PROPOSITION 4.2. *For $1 < u < v$ the partial derivatives $\partial\eta/\partial u$ and $\partial\eta/\partial v$ are continuous and satisfy*

$$u\frac{\partial}{\partial u}\eta(u,v) = \eta(u-1, v(1-u^{-1})) \tag{12}$$

*and*

$$v\frac{\partial}{\partial v}\eta(u,v) = -\eta(u(1-v^{-1}), v-1). \tag{13}$$

*Proof.* From the definition of $\eta(u,v)$ in (1) we have

$$\eta(u,v)=\varrho(v)+\int_0^1\varrho(v(1-t))u\omega(ut)\,dt.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial u}\eta(u,v)&=\frac{1}{u}\int_0^1\varrho(v(1-t))\,d(ut\omega(ut))\\&=\frac{1}{u}\varrho(v(1-u^{-1}))+\int_{1/u}^1\varrho(v(1-t))\omega(ut-1)\,dt\\&=\frac{1}{u}\varrho(v(1-u^{-1}))+\frac{1}{u}\int_0^{u-1}\varrho(tv/u)\omega(u-1-t)\,dt\\&=\frac{1}{u}\eta(u-1,v(1-u^{-1})),\end{aligned}$$

which proves (12). To show (13) we write

$$\begin{aligned}v\frac{\partial}{\partial v}\eta(u,v)&=v\varrho'(v)+v\int_0^u\varrho'(tv/u)tu^{-1}\omega(u-t)\,dt\\&=-\varrho(v-1)-\int_0^u\varrho(tv/u-1)\omega(u-t)\,dt\\&=-\varrho(v-1)-\int_0^{u(1-v^{-1})}\varrho(tv/u)\omega(u(1-v^{-1})-t)\,dt\\&=-\eta(u(1-v^{-1}),v-1).\end{aligned}$$

Remark 4.3. Note that we could have defined $\eta(u,v)$ for $0<u<v$ as the unique continuous solution of the difference-differential equation (12) for $u>1$ together with the initial condition $\eta(u,v)=\varrho(v)$ for $0<u\le 1$. The natural way to obtain an asymptotic formula for $\Gamma(x,y,z)$ would then be to employ the functional equation in Proposition 4.1 and then induct on the variable $u$. However, the explicit definition in (1) together with Lemma 3.3 allows us to make direct use of well known results on $\Psi(x,z)$ and $\Phi(x,y)$.

Remark 4.4. With the definition of $\eta(u,v)$ extended to the whole $(u,v)$-plane, one easily verifies that the difference-differential equations (12) and (13) hold for all $u,v\in\mathbb{R}$ except for $u\in\{0,1\}$, $v\in\{0,1\}$ or $u=v$.

In the following we will use the notation $\eta_r(u)=\eta(u,u/r)$ for $0<r\le 1$. For $r=0$ we define $\eta_0(u)=0$.

Lemma 4.5. *Let $\eta_r'(u) = \frac{\partial}{\partial u}\eta_r(u)$. For $0 \le r \le 1$ and $u > 1$ we have*

$$u\eta_r'(u) = \eta_r(u-1) - \eta_r(u-r). \tag{14}$$

*Proof.* The result is trivial when $r = 0, 1$ since both sides vanish. By the previous results $\partial\eta/\partial u$ and $\partial\eta/\partial v$ exist and are continuous for $v > u > 1$. The result follows from the chain rule.

By differentiating (14) $n-1$ times we obtain

Corollary 4.6. *Let $0 \le r \le 1$ and let*

$$D_{n,r} = \{i + jr : i, j \ge 0,\ i + j \le n\}.$$

*Then $\eta_r(u)$ is $n$ times continuously differentiable on $\mathbb{R} \setminus D_{n,r}$. Furthermore*

$$u\eta_r^{(n)}(u) = \eta_r^{(n-1)}(u-1) - \eta_r^{(n-1)}(u-r) - (n-1)\eta_r^{(n-1)}(u) \quad \text{for } n \ge 1,\ 0 \le r \le 1,\ u \in \mathbb{R} \setminus D_{n,r}. \tag{15}$$

We define $\eta_r^{(n)}(u)$ on $D_{n,r}$ by right continuity. Then (15) is valid for all $u \in \mathbb{R}$.

## 5. Results on $\eta(u, v)$ derived from the difference-differential equations

Lemma 5.1. *We have, for $0 \le r \le 1$,*

(i)
$$\lim_{u\to\infty} \eta_r(u) = r,$$

(ii)
$$\lim_{u\to\infty} \eta_r^{(n)}(u) = 0 \quad \text{for } n \ge 1.$$

*Proof.* If $r = 0$, then (i) is trivial. Let $0 < r \le 1$. Now $\omega(u) \to e^{-\gamma}$ and $\varrho(u) \to 0$ as $u \to \infty$. Thus

$$\eta_r(u) = \varrho(u/r) + \int_0^u \varrho(t/r)\omega(u-t)\,dt$$
$$= e^{-\gamma}\int_0^u \varrho(t/r)\,dt + o(1) = re^{-\gamma}\int_0^{u/r} \varrho(t)\,dt + o(1)$$

as $u$ tends to infinity. Part (i) follows, since $\int_0^\infty \varrho(t)\,dt = e^\gamma$ by Theorem III.5.7 in [14]. Part (ii) follows from (i) and Corollary 4.6 by induction on $n$.

With $\sigma_r(u) := \sigma(u, u/r)$, $0 < r \le 1$ it follows from the definition (2) of $\sigma(u, v)$ that, for $0 \le r \le 1$, $u > 1$, $u \ne 2r, 1 + r, 2$,

$$(u\sigma_r(u))' = \sigma_r(u-r) - \sigma_r(u-1).$$

This allows us to bound $\eta_r'(u)$ in terms of $\sigma_r(u)$. The behaviour of $\sigma_r(u)$ is well understood due to the work of Friedlander [6] and Saias [10].

LEMMA 5.2. *For $1 \le u \le 3$ and $r \le 1/3$ we have $\sigma_r(u) \gg 1$.*

*Proof.* This follows directly from Theorem 2 and Theorem 5 in [6].

PROPOSITION 5.3. *Let $\eta_r' = \partial\eta_r/\partial u$. Then, for $0 \le r \le 1$,*

(i) $$\eta_r'(u) \ll r\sigma_r(u) \quad \textit{for } u \ge 2,$$

(ii) $$\eta_r'(u) \ll \sigma_r(u) \quad \textit{for } u > 1.$$

*Proof.* (i) We first consider the case $r \le 1/3$. For $2 \le u \le 3$ and $r \le 1/3$ we have $\sigma_r(u) \gg 1$, by Lemma 5.2. Lemmas 4.5 and 3.4 imply $\eta_r'(u) = (\eta_r(u-1) - \eta_r(u-r))/u \ll r$ for $2 \le u \le 3$. Thus we have $\eta_r'(u) \ll r\sigma_r(u)$ for $2 \le u \le 3$ and $r \le 1/3$. Now assume that $\eta_r'(u) \le cr\sigma_r(u)$ for some positive constant $c$ and $u \le N$, where $N \ge 3$. For $N < u \le N + r$ we have

$$|\eta_r'(u)| = \left|\frac{1}{u}\int_{u-1}^{u-r} \eta_r'(t)\,dt\right| \le \frac{cr}{u}\left|\int_{u-1}^{u-r} \sigma_r(t)\,dt\right| = cr\sigma_r(u),$$

which proves (i) in the case $r \le 1/3$.

Assume $r > 1/3$. For $0 < u \le 1$ we have $\sigma_r(u) = \sigma(u,v) = \omega(v)$ from the definition of $\sigma(u,v)$ in (2). Also, $\varrho'(v) \ll \omega(v)$. Indeed, $\varrho'(v) = 0 = \omega(v)$ for $0 \le v < 1$, and $|\varrho'(v)| = \varrho(v-1)/v \le 1/v$ and $1/2 \le \omega(v)$ for $1 \le v$. Thus

$$\eta_r'(u) = \frac{1}{r}\varrho'\left(\frac{u}{r}\right) \le 3\varrho'(v) \ll \omega(v) = \sigma_r(u) \le r\sigma_r(u).$$

Now assume that $\eta_r'(u) \le cr\sigma_r(u)$ for some positive constant $c$ and $u \le N$, where $N \ge 1$. For $N < u \le N + r$ we have

$$|\eta_r'(u)| = \left|\frac{1}{u}\int_{u-1}^{u-r} \eta_r'(t)\,dt\right| \le \frac{cr}{u}\left|\int_{u-1}^{u-r} \sigma_r(t)\,dt\right| = cr\sigma_r(u).$$

This shows that, in the case $r > 1/3$, we have $\eta_r'(u) \ll r\sigma_r(u)$ for $u > 0$, which completes the proof of part (i).

(ii) In view of (i), it suffices to consider $1 < u < 2$ and $r \le 1/3$. We have $\sigma_r(u) \gg 1$, by Lemma 5.2. On the other hand we clearly have $\eta_r'(u) \ll 1$ for $u > 1$. This concludes the proof of Proposition 5.3.

COROLLARY 5.4. *For $u \ge 1$, $0 \le r \le 1$, we have*

(i) $$\eta_r(u) = r\Big(1 + O\Big(\int_u^{\infty} \sigma_r(t)\,dt\Big)\Big) \quad \textit{for } u \ge 2,$$

(ii) $$\eta_r(u) = r(1 + O(\varrho(u))) \quad \textit{for } u \ge 1.$$

*Proof.* (i) follows from Proposition 5.3 and Lemma 5.1.

(ii) For $u \geq 2$ we have

$$\int_u^\infty \sigma_r(t)\,dt \ll \int_u^\infty \varrho(t)\,dt \ll \varrho(u).$$

If $1 \leq u < 2$, (ii) follows from Lemma 3.4.

THEOREM 5.5. *We have, for* $0 < u \leq v$,

$$\eta(u,v) = e^\gamma \omega(u)\frac{u}{v} + O\biggl(\frac{\log v}{v^2}\max\biggl(1, \frac{1}{u-1}, u-1\biggr)\biggr).$$

*Proof.* For $0 < u \leq 1$ the left hand side is equal to $\varrho(v) < 1/\Gamma(v+1) = O(1/v^2)$. On the right side we have $\omega(u) = 0$ in this case. For $1 < u \leq 2$ we have

$$\begin{aligned}
\eta(u,v) = \varrho(v) + \int_0^{u-1} \varrho(tv/u)\,\frac{dt}{u-t} &= \varrho(v) + \int_0^{v(1-1/u)} \varrho(s)\,\frac{ds}{v-s} \\
&= \varrho(v) + \int_0^{\log v} \varrho(s)\,\frac{ds}{v-s} + \int_{\log v}^{v(1-1/u)} \varrho(s)\,\frac{ds}{v-s} \\
&\leq \varrho(v) + \frac{1}{v-\log v} e^\gamma + \frac{1}{v/2}\int_{\log v}^\infty \varrho(s)\,ds \\
&\leq \varrho(v) + \frac{e^\gamma}{v} + e^\gamma\,\frac{\log v}{v(v-\log v)} + O\biggl(\frac{1}{v^2}\biggr) \leq \frac{e^\gamma}{v} + O\biggl(\frac{\log v}{v^2}\biggr).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\eta(u,v) = \varrho(v) + \int_0^{v(1-1/u)} \varrho(s)\,\frac{ds}{v-s} &\geq \varrho(v) + \frac{1}{v}\int_0^{v(1-1/u)} \varrho(s)\,ds \\
&= \varrho(v) + \frac{e^\gamma}{v} - \frac{1}{v}\int_{v(1-1/u)}^\infty \varrho(s)\,ds \geq \varrho(v) + \frac{e^\gamma}{v} - \frac{1}{v^2(1-1/u)} \\
&\geq \varrho(v) + \frac{e^\gamma}{v} - \frac{1}{v^2(u-1)}.
\end{aligned}$$

This proves the result for $1 < u \leq 2$.

For $2 < u \leq 3$ we use (12) to write

$$\eta(u,v) = \eta(2,v) + \int_2^u \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t}$$

$$= e^{\gamma}\omega(2)\frac{2}{v} + O\Big(\frac{\log v}{v^2}\Big) + \int_{2}^{2+2(\log v)/v} \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t}$$

$$+ \int_{2+2(\log v)/v}^{u} \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t}.$$

From Lemma 3.4 we know that $\eta(u,v) = O(u/v)$. Thus,

$$\int_{2}^{2+2(\log v)/v} \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t} = O\Big(\frac{\log v}{v^2}\Big).$$

Next, note that $t \geq 2 + 2(\log v)/v$ implies

$$v\Big(1-\frac{1}{t}\Big)\Big(1-\frac{1}{t-1}\Big) \geq \log\Big(v\Big(1-\frac{1}{t}\Big)\Big).$$

Thus, for $t \geq 2 + 2(\log v)/v$,

$$\begin{aligned}\eta(t-1, v(1-t^{-1})) &\geq \frac{e^{\gamma}}{v(1-t^{-1})} + \varrho(v(1-t^{-1})) \\ &\qquad - \frac{1}{v(1-t^{-1})}\int_{\log(v(1-t^{-1}))}^{\infty} \varrho(s)\,ds \\ &\geq \frac{e^{\gamma}}{v(1-t^{-1})} + O(1/v^2).\end{aligned}$$

Together with the upper bound for $\eta(u,v)$ in the region $1 < u \leq 2$ this implies, for $t \geq 2 + 2(\log v)/v$,

$$\eta(t-1, v(1-t^{-1})) = \frac{e^{\gamma}}{v(1-t^{-1})} + O\Big(\frac{\log(v(1-t^{-1}))}{v^2(1-t^{-1})^2}\Big).$$

Hence,

$$\begin{aligned}&\int_{2+2(\log v)/v}^{u} \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t} \\ &\qquad = \int_{2+2(\log v)/v}^{u} \Big(\frac{e^{\gamma}}{v(1-t^{-1})} + O\Big(\frac{\log v}{v^2}\Big)\Big)\frac{dt}{t} \\ &\qquad = \frac{e^{\gamma}}{v}\Big(\log(u-1) - \log\Big(1+\frac{2\log v}{v}\Big)\Big) + O\Big(\frac{\log v}{v^2}\Big) \\ &\qquad = \frac{e^{\gamma}}{v}\log(u-1) + O\Big(\frac{\log v}{v^2}\Big).\end{aligned}$$

Therefore,

$$\eta(u,v) = \frac{e^\gamma}{v}(1+\log(u-1)) + O\left(\frac{\log v}{v^2}\right) = e^\gamma \frac{u}{v}\omega(u) + O\left(\frac{\log v}{v^2}\right)$$

in the region $2 < u \le 3$.

For $u > 3$ we proceed by induction. Let $N \ge 3$ and assume that, for $N-1 < u \le N$, there exists a constant $c$ such that

$$\left|\eta(u,v) - e^\gamma \omega(u)\frac{u}{v}\right| \le c\frac{(N-2)\log v}{v^2}.$$

Then, for $N < u \le N+1$,

$$\begin{aligned}
&\left|\eta(u,v) - e^\gamma \omega(u)\frac{u}{v}\right| \\
&\qquad = \left|\eta(N,v) + \int_N^u \eta(t-1, v(1-t^{-1}))\,\frac{dt}{t} - e^\gamma\omega(u)\frac{u}{v}\right| \\
&\qquad = \Bigg|\eta(N,v) - e^\gamma\omega(N)\frac{N}{v} \\
&\qquad\quad + \int_N^u \left(\eta(t-1, v(1-t^{-1})) - e^\gamma\omega(t-1)\frac{t-1}{v(1-t^{-1})}\right)\frac{dt}{t} \\
&\qquad\quad + \int_N^u \frac{e^\gamma}{v}\omega(t-1)\,dt + e^\gamma\omega(N)\frac{N}{v} - e^\gamma\omega(u)\frac{u}{v}\Bigg|.
\end{aligned}$$

Since $\omega(u)$ satisfies $(u\omega(u))' = \omega(u-1)$, the last three terms in the previous expression combine to zero and we complete the proof of Theorem 5.5 as follows:

$$\begin{aligned}
\left|\eta(u,v) - e^\gamma\omega(u)\frac{u}{v}\right| &\le \left|\eta(N,v) - e^\gamma\omega(N)\frac{N}{v}\right| \\
&\quad + \int_N^u \left|\eta(t-1, v(1-t^{-1})) - e^\gamma\omega(t-1)\frac{t-1}{v(1-t^{-1})}\right|\frac{dt}{t} \\
&\le c\frac{(N-2)\log v}{v^2} + c\int_N^u \frac{(N-2)\log(v(1-t^{-1}))}{v^2(1-t^{-1})t}\,dt \\
&\le c\frac{(N-2)\log v}{v^2}\left(1 + \int_N^u \frac{dt}{t-2+t^{-1}}\right) \\
&\le c\frac{(N-2)\log v}{v^2}\left(1 + \frac{1}{N-2}\right) = c\frac{(N-1)\log v}{v^2}.
\end{aligned}$$

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