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On Waring's problem with polynomial summands II: Addendum

by

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Let $f_k(x)$ be an integral-valued polynomial of degree k represented by

$$f_k(x) = a_k F_k(x) + \ldots + a_1 F_1(x),$$

where $F_i(x) = x(x-1)\ldots(x-i+1)/i!$ $(1 \le i \le k)$, and $a_1, \ldots, a_k$ are integers satisfying $(a_1, \ldots, a_k) = 1$ and $a_k > 0$. Let $G(f_k)$ be the least s such that the equation

$$(1) \qquad f_k(x_1) + \ldots + f_k(x_s) = n \quad \text{with } x_i \ge 0$$

is soluble for all sufficiently large integers n. In [1] we proved, among other things, that

$$(2) \qquad G(f_k) \le 2^k - \tfrac{1}{2}(1 - (-1)^k) \quad \text{for } k \ge 6,$$

and, when k is odd, equality holds if and only if $f_k(x)$ satisfies

$$(3) \quad 2 \nmid f_k(1) \quad \text{and} \quad f_k(x) \equiv (-1)^{k-1} f_k(1) H_k(x) \pmod{2^k} \quad \text{for any } x,$$

where $H_k(x) = \sum_{i=1}^{k} (-1)^{k-i} 2^{i-1} F_i(x)$.

When k is even, however, it would be somewhat difficult to classify those polynomials $f_k(x)$ for which equality holds in (2). From Theorem 1 of [1] we see that the point of this problem is to determine $G(f_k)$ when $f_k(x)$ satisfies (3). In this case we let

$$(4) \qquad E_k(x) = 2^{-k} f_k(2x) \quad \text{and} \quad O_k(x) = 2^{-k}(f_k(2x+1) - f_k(1)).$$

Then both $E_k(x)$ and $O_k(x)$ are integral-valued polynomials, and at least one of $E_k(x)$ and $O_k(x)$ is not constant modulo 2 (cf. [1, Section 3]).

We will prove the following result.

THEOREM. *Suppose that $k \ge 6$ is even and that $f_k(x)$ satisfies (3). If one of $E_k(x)$ and $O_k(x)$ is constant modulo 2, then $G(f_k) = 2^k$; otherwise $G(f_k) = 2^k - 1$.*

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Combining this with the second assertion of Theorem 1 of [1], we have

COROLLARY. *Suppose $k \geq 6$ is even. Then equality holds in* (2) *if and only if $f_k(x)$ satisfies* (3) *and one of $E_k(x)$ and $O_k(x)$ is constant modulo* 2.

For the proof of the Theorem we begin with some preliminaries. From [1, (1.7) and Section 2] it is easy to see that we need only consider the solutions of (1) in 2-adic integers. Thus by (3) we may assume that $a_1 = f_k(1) = -1$, and so

$$a_i = (-1)^i 2^{i-1} + 2^k u_i, \qquad u_i \text{ integers } (2 \leq i \leq k). \tag{5}$$

Write

$$E_k(x) = \sum_{i=1}^{k} d_i F_i(x), \qquad O_k(x) = \sum_{i=1}^{k} d_i' F_i(x). \tag{6}$$

Then by (4)–(6) and [1, (3.14) and (3.15)] we have

$$\begin{aligned} d_i &= 2^{-k} \sum_{l=i}^{\min(2i,k)} (-1)^l 2^{2i-1} \binom{i}{l-i} + \sum_{l=i}^{\min(2i,k)} u_l 2^{2i-l} \binom{i}{l-i} \\ &= A_i + B_i, \end{aligned} \tag{7}$$

say; and

$$\begin{aligned} d_i' = d_i &+ 2^{-k} \sum_{l=i}^{\min(2i,k-1)} (-1)^{l+1} 2^{2i} \binom{i}{l-i} \\ &+ \sum_{l=i}^{\min(2i,k-1)} u_{l+1} 2^{2i-l} \binom{i}{l-i} \\ = d_i &+ A_i' + B_i', \end{aligned} \tag{8}$$

say. From (7) and (8) we have

$$d_{k/2} \equiv u_k \pmod 2 \quad \text{and} \quad d_{k/2}' \equiv u_k + 1 \pmod 2. \tag{9}$$

Thus at least one of $E_k(x)$ and $O_k(x)$ is not constant modulo 2.

If $E_k(x)$ is constant modulo 2, then by (4) $f_k(x)$ takes only three different values, 0, -1, and $-1 + 2^k$, mod 2^{k+1}. Thus the congruence

$$\sum_{i=1}^{2^k - 1} f_k(x_i) \equiv n \pmod{2^{k+1}} \tag{10}$$

is unsolvable for $n \equiv 2^k \pmod{2^{k+1}}$. Hence $G(f_k) \geq 2^k$, and so equality holds in view of (2). Similarly, if $O_k(x)$ is constant modulo 2, then the congruence (10) is unsolvable for $n \equiv 1 \pmod{2^{k+1}}$. Thus $G(f_k) = 2^k$ also in this case. This proves the first statement of the Theorem.

We now suppose that neither $E_k(x)$ nor $O_k(x)$ is constant modulo 2 and adopt the notation of [1, Section 2]. By Theorem 1 of [1] we see that to prove the second assertion of the Theorem it suffices to prove $G(f_k) \le 2^k - 1$. We shall do this by showing that (cf. [1, (1.7) and Section 2])

$$M_{2^k-1}(f_k, 2^l, n) \ge 2^{(2^k-2)(l-2k)} \quad \text{for all } n \text{ and } l \ge 2k. \tag{11}$$

For any n let r_n be the integer satisfying $n \equiv -r_n \pmod{2^k}$ and $0 \le r_n < 2^k$. When $1 \le r_n \le 2^k - 2$, we have proved in [1, Section 3] that $\Gamma^*(f_k, 2^\gamma, n) \le r_n + 1 \le 2^k - 1$, thus (11) holds in these cases (cf. [1, proof of Theorem 3(i)]). To deal with the remaining cases the crucial step is to establish the following result.

Lemma. *If $k \ge 6$ is even, then* (3) *does not hold with $f_k(x)$ replaced by $E_k(x)$ or $O_k(x)$.*

By the Lemma, we may apply [1, Theorem 3(ii)] to $E_k(x)$. Since $2^k - 1 > 2^{k-1} + 4(k-1)$ for $k \ge 6$, we thus have $N_{2^k-1}(E_k, 2^{\gamma(E_k)}, m) \ge 1$ for any m, which implies that (11) holds in the case of $r_n = 0$ (cf. [1, Section 2]). Similarly, [1, Theorem 3(ii)] applied to $O_k(x)$ shows that (11) also holds when $r_n = 2^k - 1$. The proof of the Theorem is now complete.

It remains to verify the Lemma. We distinguish two cases.

(i) *k is not a power of* 2. Write $k = 2^\beta v$ with $\beta \ge 1$, $2 \nmid v$ and $v \ge 3$. Let

$$r = k/2 + h \quad \text{and} \quad h = 2^{\beta-1}.$$

We first prove the assertion for $E_k(x)$. Clearly, we may assume that $d_1 = E_k(1)$ is odd, otherwise the result is trivial. From (7) we have

$$A_r = -2^{2h-1} \sum_{l=k+1}^{2r} (-1)^l \binom{r}{l-r} = -2^{2h-1} \sum_{l=0}^{2h-1} (-1)^l \binom{r}{l}. \tag{12}$$

By Lucas' test we see that

$$\binom{r}{l} = \binom{2^{\beta-1}(v+1)}{l} \equiv 0 \pmod 2$$

for $1 \le l \le 2h - 1$ $(= 2^\beta - 1)$. It follows from (12) that $A_r \equiv 2^{2h-1} \pmod{2^{2h}}$. Also, $2^{2h} \mid B_r$ by (7). Hence $2^{2h-1} \parallel d_r$. From this and from $2 \nmid d_1$ and $2^{2h} \mid 2^{r-1}$ we have $d_r \not\equiv (-1)^{r-1} 2^{r-1} d_1 \pmod{2^k}$, which implies that the assertion holds for $E_k(x)$.

Moreover, by (8) it is easily seen that $2^{2h} \mid (A'_r, B'_r)$. Thus $2^{2h-1} \parallel d'_r$, and the assertion for $O_k(x)$ also holds as above.

(ii) *k is a power of* 2. Write $k = 2^\beta$ with $\beta \ge 3$. Let

$$r = k/2 + h \quad \text{and} \quad h = 2^{\beta-2}.$$

By (7) and Vandermonde's identity, we have

$$A_r = -2^{2h-1}\sum_{l=0}^{2h-1}(-1)^l\binom{r}{l} = -2^{2h-1}\sum_{l=0}^{2h-1}(-1)^l\sum_{j=0}^{l}\binom{k/2}{j}\binom{h}{l-j}$$
$$= -2^{2h-1}\sum_{l=1}^{2h-1}(-1)^l\sum_{j=1}^{l}\binom{k/2}{j}\binom{h}{l-j}.$$

It is easily verified that $\binom{k/2}{j} = \binom{2^{\beta-1}}{j} \equiv 0 \pmod 4$ for $2^{\beta-2}\nmid j$. Thus we have

$$A_r \equiv -2^{2h-1}\binom{k/2}{h}\sum_{l=h}^{2h-1}(-1)^l\binom{h}{l-h} = 2^{2h-1}\binom{2^{\beta-1}}{2^{\beta-2}} \tag{13}$$
$$= 2^{2h}\binom{2^{\beta-1}-1}{2^{\beta-2}-1} \equiv 2^{2h} \pmod{2^{2h+1}}.$$

Also, by (7) we get

$$B_r \equiv 2^{2h}u_k\binom{r}{k-r} = 2^{2h}u_k\binom{3\cdot 2^{\beta-2}}{2^{\beta-2}} \equiv 2^{2h}u_k \pmod{2^{2h+1}}. \tag{14}$$

Furthermore, by (8) and (13) we have

$$A'_r = -2A_r + 2^{2h}\binom{r}{k-r} \equiv 2^{2h} \pmod{2^{2h+1}} \tag{15}$$

and

$$B'_r \equiv 0 \pmod{2^{2h+1}}. \tag{16}$$

The Lemma can now be proved easily. If $2\nmid u_k$, then the result for $E_k(x)$ is trivial by (9). If $2\mid u_k$, then by (7), (13) and (14), we have $2^{2h}\parallel d_r$. Thus, in view of $2^{2h+1}\mid 2^{r-1}$, the assertion for $E_k(x)$ holds as in case (i).

When $2\mid u_k$ the assertion for $O_k(x)$ is trivial (again by (9)). When $2\nmid u_k$, by (7), (8) and (13) to (16), we have $2^{2h}\parallel d'_r$. Thus the result for $O_k(x)$ also holds. This completes the proof of the Lemma.

References

[1] H. B. Yu, *On Waring's problem with polynomial summands II*, Acta Arith. 86 (1998), 245–254.

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