On Waring’s problem with polynomial summands II: Addendum

by

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Let $f_k(x)$ be an integral-valued polynomial of degree $k$ represented by

$$f_k(x) = a_k F_k(x) + \ldots + a_1 F_1(x),$$

where $F_i(x) = x(x-1)\ldots(x-i+1)/i!$ (1 $\leq i \leq k$), and $a_1, \ldots, a_k$ are integers satisfying $(a_1, \ldots, a_k) = 1$ and $a_k > 0$. Let $G(f_k)$ be the least $s$ such that the equation

$$f_k(x_1) + \ldots + f_k(x_s) = n$$

is soluble for all sufficiently large integers $n$. In [1] we proved, among other things, that

$$G(f_k) \leq 2^k - \frac{1}{2}(1 - (-1)^k)$$

for $k \geq 6$, and, when $k$ is odd, equality holds if and only if $f_k(x)$ satisfies

$$2 \nmid f_k(1) \quad \text{and} \quad f_k(x) \equiv (-1)^{k-1} f_k(1) H_k(x) \pmod{2^k}$$

for any $x$, where $H_k(x) = \sum_{i=1}^{k} (-1)^{k-i} 2^{i-1} F_i(x)$.

When $k$ is even, however, it would be somewhat difficult to classify those polynomials $f_k(x)$ for which equality holds in (2). From Theorem 1 of [1] we see that the point of this problem is to determine $G(f_k)$ when $f_k(x)$ satisfies (3). In this case we let

$$E_k(x) = 2^{-k} f_k(2x) \quad \text{and} \quad O_k(x) = 2^{-k} (f_k(2x+1) - f_k(1)).$$

Then both $E_k(x)$ and $O_k(x)$ are integral-valued polynomials, and at least one of $E_k(x)$ and $O_k(x)$ is not constant modulo 2 (cf. [1, Section 3]).

We will prove the following result.

**Theorem.** Suppose that $k \geq 6$ is even and that $f_k(x)$ satisfies (3). If one of $E_k(x)$ and $O_k(x)$ is constant modulo 2, then $G(f_k) = 2^k$; otherwise $G(f_k) = 2^k - 1$.  

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Combining this with the second assertion of Theorem 1 of [1], we have

**Corollary.** Suppose \( k \geq 6 \) is even. Then equality holds in (2) if and only if \( f_k(x) \) satisfies (3) and one of \( E_k(x) \) and \( O_k(x) \) is constant modulo 2.

For the proof of the Theorem we begin with some preliminaries. From [1, (1.7) and Section 2] it is easy to see that we need only consider the solutions of (1) in 2-adic integers. Thus by (3) we may assume that \( a_1 = f_k(1) = -1 \), and so

\[
a_i = (-1)^i 2^{i-1} + 2^k u_i, \quad u_i \text{ integers } (2 \leq i \leq k).
\]

Write

\[
E_k(x) = \sum_{i=1}^{k} d_i F_i(x), \quad O_k(x) = \sum_{i=1}^{k} d'_i F_i(x).
\]

Then by (4)–(6) and [1, (3.14) and (3.15)] we have

\[
d_i = 2^{-k} \sum_{l=i}^{\min(2i,k)} (-1)^j 2^{2i-1} \binom{i}{l-i} + \sum_{l=i}^{\min(2i,k)} u_l 2^{2i-l} \binom{i}{l-i}
= A_i + B_i,
\]
say; and

\[
d'_i = d_i + 2^{-k} \sum_{l=i}^{\min(2i,k-1)} (-1)^j 2^{2i} \binom{i}{l-i} + \sum_{l=i}^{\min(2i,k-1)} u_{l+1} 2^{2i-l} \binom{i}{l-i}
= d_i + A'_i + B'_i,
\]
say. From (7) and (8) we have

\[
d_{k/2} \equiv u_k \pmod{2} \quad \text{and} \quad d'_{k/2} \equiv u_k + 1 \pmod{2}.
\]

Thus at least one of \( E_k(x) \) and \( O_k(x) \) is not constant modulo 2.

If \( E_k(x) \) is constant modulo 2, then by (4) \( f_k(x) \) takes only three different values, 0, -1, and \(-1 + 2^k\), mod 2\(^k+1\). Thus the congruence

\[
\sum_{i=1}^{2^k-1} f_k(x_i) \equiv n \pmod{2^{k+1}}
\]

is unsolvable for \( n \equiv 2^k \pmod{2^{k+1}} \). Hence \( G(f_k) \geq 2^k \), and so equality holds in view of (2). Similarly, if \( O_k(x) \) is constant modulo 2, then the congruence (10) is unsolvable for \( n \equiv 1 \pmod{2^{k+1}} \). Thus \( G(f_k) = 2^k \) also in this case. This proves the first statement of the Theorem.
We now suppose that neither $E_k(x)$ nor $O_k(x)$ is constant modulo 2 and adopt the notation of [1, Section 2]. By Theorem 1 of [1] we see that to prove the second assertion of the Theorem it suffices to prove $G(f_k) \leq 2^k - 1$. We shall do this by showing that (cf. [1, (1.7) and Section 2])

$$M_{2^k-1}(f_k, 2^l, n) \geq 2^{(2^k-2)(l-2k)}$$

for all $n$ and $l \geq 2k$.

For any $n$ let $r_n$ be the integer satisfying $n \equiv -r_n \pmod{2^k}$ and $0 \leq r_n < 2^k$. When $1 \leq r_n \leq 2^k - 2$, we have proved in [1, Section 3] that $I^*(f_k, 2^l, n) \leq r_n + 1 \leq 2^k - 1$, thus (11) holds in these cases (cf. [1, proof of Theorem 3(i)]). To deal with the remaining cases the crucial step is to establish the following result.

**Lemma.** If $k \geq 6$ is even, then (3) does not hold with $f_k(x)$ replaced by $E_k(x)$ or $O_k(x)$.

By the Lemma, we may apply [1, Theorem 3(ii)] to $E_k(x)$. Since $2^k - 1 > 2^k - 1 + 4(k - 1)$ for $k \geq 6$, we thus have $N_{2^k-1}(E_k, 2^\gamma(E_k), m) \geq 1$ for any $m$, which implies that (11) holds in the case of $r_n = 0$ (cf. [1, Section 2]). Similarly, [1, Theorem 3(ii)] applied to $O_k(x)$ shows that (11) also holds when $r_n = 2^k - 1$. The proof of the Theorem is now complete.

It remains to verify the Lemma. We distinguish two cases.

(i) $k$ is not a power of 2. Write $k = 2^\beta v$ with $\beta \geq 1$, $2 \not| v$ and $v \geq 3$. Let $r = k/2 + h$ and $h = 2^{\beta-1}$.

We first prove the assertion for $E_k(x)$. Clearly, we may assume that $d_1 = E_k(1)$ is odd, otherwise the result is trivial. From (7) we have

$$A_r = -2^{2h-1} \sum_{l=k+1}^{2r} (-1)^l \binom{r}{l-r} = -2^{2h-1} \sum_{l=0}^{2h-1} (-1)^l \binom{r}{l}.$$  

By Lucas’ test we see that

$$\binom{r}{l} \equiv (2^{\beta-1}(v+1)} \pmod{2}$$

for $1 \leq l \leq 2h - 1 (= 2^{\beta} - 1)$. It follows from (12) that $A_r \equiv 2^{2h-1} \pmod{2^h}$. Also, $2^{2h} \mid B_r$ by (7). Hence $2^{2h-1} \mid d_r$. From this and from $2 \mid d_1$ and $2^{2h} \mid 2^{r-1}$ we have $d_r \equiv (1)^{r-1} 2^{r-1} d_1 \pmod{2^k}$, which implies that the assertion holds for $E_k(x)$.

Moreover, by (8) it is easily seen that $2^{2h} \mid (A'_r, B'_r)$. Thus $2^{2h-1} \mid d'_r$, and the assertion for $O_k(x)$ also holds as above.

(ii) $k$ is a power of 2. Write $k = 2^\beta$ with $\beta \geq 3$. Let $r = k/2 + h$ and $h = 2^{\beta-2}$.
By (7) and Vandermonde’s identity, we have
\[
A_r = -2^{2h-1} \sum_{l=0}^{2h-1} (-1)^l \binom{r}{l} = -2^{2h-1} \sum_{l=0}^{2h-1} (-1)^l \sum_{j=0}^{l} \binom{k/2}{j} \binom{h}{l-j} \\
= -2^{2h-1} \sum_{l=1}^{2h-1} (-1)^l \sum_{j=1}^{l} \binom{k/2}{j} \binom{h}{l-j}.
\]

It is easily verified that \( \binom{k/2}{j} = \binom{2^{\beta-1}}{j} \equiv 0 \pmod{4} \) for \( 2 \nmid j \). Thus we have
\[
(13) \quad A_r \equiv -2^{2h-1} \binom{k/2}{h} \sum_{l=h}^{2h-1} (-1)^l \binom{h}{l-h} = 2^{2h-1} \binom{2^{\beta-1}}{2^{\beta-2}} \\
= 2^h \binom{2^{\beta-1}-1}{2^{\beta-2}-1} \equiv 2^h \pmod{2^{2h+1}}.
\]

Also, by (7) we get
\[
(14) \quad B_r \equiv 2^{2h} u_k \binom{r}{k-r} = 2^{2h} u_k \binom{3 \cdot 2^{\beta-2}}{2^{\beta-2}} \equiv 2^{2h} u_k \pmod{2^{2h+1}}.
\]

Furthermore, by (8) and (13) we have
\[
(15) \quad A'_r = -2A_r + 2^h \binom{r}{k-r} \equiv 2^h \pmod{2^{2h+1}}
\]
and
\[
(16) \quad B'_r \equiv 0 \pmod{2^{2h+1}}.
\]

The Lemma can now be proved easily. If \( 2 \nmid u_k \), then the result for \( E_k(x) \) is trivial by (9). If \( 2 \mid u_k \), then by (7), (13) and (14), we have \( 2^h \parallel d_r \). Thus, in view of \( 2^{2h+1} \mid 2^{r-1} \), the assertion for \( E_k(x) \) holds as in case (i).

When \( 2 \mid u_k \) the assertion for \( O_k(x) \) is trivial (again by (9)). When \( 2 \nmid u_k \), by (7), (8) and (13) to (16), we have \( 2^h \parallel d'_r \). Thus the result for \( O_k(x) \) also holds. This completes the proof of the Lemma.

References


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