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The resolution of the diophantine equation $x(x+d)\dots(x+(k-1)d) = by^2$ for fixed d

by

P. Filakovszky (Nové Zámky) and L. Hajdu (Debrecen)

1. Introduction. A classical problem of number theory is to determine those finite arithmetical progressions for which the product of terms yields a perfect power, or an "almost" perfect one. Erdős and Selfridge proved in 1975 (cf. [2]) that the product of two or more consecutive positive integers is never a perfect power, i.e. the equation

$$x(x+1)\dots(x+k-1) = y^l$$

has no solutions with $k, l \geq 2$ and $x \geq 1$. There are many results in the literature concerning various generalizations of the above equation (see e.g. the extensive survey papers [8–11], or the recent papers [1], [4–7], and the references given there).

Let $P(b)$ denote the greatest prime factor of a positive integer $b > 1$, and put $P(1) = 1$. In this paper we investigate the following equation:

$$x(x+d)\dots(x+(k-1)d) = by^2 \quad \text{with } d > 1,\ k \geq 3, \quad (x,d) = 1,\ P(b) \leq k, \tag{1}$$

in positive integers x, d, k, b, y. In [7] Saradha proved that equation (1) has only the solutions

$$(x,d,k,b,y) = (2,7,3,2,12), (18,7,3,1,120), (64,17,3,2,504),$$

provided that $d \leq 22$. In fact she gave an algorithm for the resolution of (1) for fixed values of d, and used her method to compute all solutions with $1 < d < 23$. The main steps of her method are the following. Put $C = (k-1)^2 d^2/4$, and suppose first that for a solution (x,d,k,b,y) of (1) we have $x \geq C$. For such a solution Saradha derived an upper bound $k_0(d)$

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for k, which varies between 18 and 314 as d ranges through the interval $[7, 22]$. It is not guaranteed that her method provides an upper bound $k_0(d)$ for an arbitrary value of d. Subsequently she proved that $4 \le k \le 6$ if $d = 7$, $4 \le k \le 8$ if $d \in \{11, 13, 17, 19\}$, and that (1) has no solutions for other values of d with $1 < d < 23$. The remaining cases were verified by numerical calculations.

In [1] Brindza, Hajdu and Ruzsa proved the following result.

THEOREM A. *If* (x, d, k, b, y) *is a solution to* (1) *with* $k \ge 8$, *then* $x < D$, *where* $D = 4d^4(\log d)^4$.

This implies that we can take $k_0(d) = 8$ if $x \ge D$. This uniform bound makes it possible, at least in principle, to resolve equation (1) for any fixed d. This paper provides an algorithm to do that. We shall illustrate the algorithm by determining all solutions of (1) with $23 \le d \le 30$.

2. Result and description of the algorithm. The main steps of our method for the resolution of (1) with fixed d are the following. First we provide a simple search algorithm to find the solutions with small x. According to Theorem A we have $k \le 7$ for large solutions. We show that each such solution corresponds to a point on one among 16 elliptic curves. The elliptic equations can be resolved by a mathematical software package.

THEOREM. *Suppose that* $23 \le d \le 30$. *The only solutions to equation* (1) *are the following ones*:

- $d = 23$, $k = 3$: $(x, b, y) = (2, 6, 20)$, $(4, 6, 30)$, $(75, 6, 385)$, $(98, 2, 924)$, $(338, 3, 3952)$, $(3675, 6, 91805)$,
- $d = 23$, $k = 4$: $(x, b, y) = (75, 6, 4620)$,
- $d = 24$, $k = 3$: $(x, b, y) = (1, 1, 35)$.

REMARK. The above theorem provides a solution to (1) with $k > 3$, namely $(x, d, k, b, y) = (75, 23, 4, 6, 4620)$. This is not surprising, as it was pointed out by F. Beukers that equation (1) has infinitely many solutions with $k = 4$.

Proof of the Theorem. Suppose first that (x, d, k, b, y) is a solution to (1) with $23 \le d \le 30$ and $x < D$, where D is defined in Theorem A. By the estimate $k < 4d(\log d)^2$ due to Saradha [7], the left hand side of equation (1) is bounded by a constant depending only on d. Hence after fixing d, all solutions to (1) can be found by a simple search. However, as a huge amount of computation is needed, it is worth to be more economical.

Let d be fixed. A positive integer a is called a *bad number* if some prime p with $p \ge 4d(\log d)^2$ occurs in the prime factorization of a with an odd exponent. Suppose that $x+id$ is a bad number for some i with $0 \leq i \leq k-1$, and choose a prime p with the above properties for $a = x + id$. Then by Saradha's result we have $p > k$. By the condition $(x, d) = 1$, there is no other factor $x+jd$ which is divisible by p. Hence p divides the left-hand side with an odd exponent, which contradicts $P(b) \leq k$. This argument shows that no factor $x + id$ is bad.

We work with the residue classes (mod d) separately. Let m be a positive integer with $(m, d) = 1$, $m < d$. We make a list L_3 consisting of all those positive integers $x' < D$ with $x' \equiv m \pmod{d}$ for which none of the numbers x', $x' + d$, $x' + 2d$ is bad. Then we make a list L_4 of all $x' \in L_3$ with $x' + d \in L_3$. Subsequently we make a list L_5 of all the numbers $x' \in L_4$ with $x' + d \in L_4$ and so on. For $23 \leq d \leq 30$ the process stops around L_{15}. Observe that $x' \in L_i$ if and only if none of $x', x' + d, \ldots, x' + (i - 1)d$ is bad. Hence every solution (x, d, k, b, y) of (1) with $x < D$ satisfies $x \in L_k$. Finally, for each number $x' \in L_k$ we check if $x'(x' + d) \ldots (x' + (k - 1)d)$ has a square-free part which has a greatest prime factor $\leq k$, for all lists L_k. The numbers which pass this last test provide all the solutions with $x \equiv m \pmod{d}$. Finally we take the union over all m to collect all solutions of (1) with $x < D$.

Now suppose that (x, d, k, b, y) is a solution to (1) with $x \geq D$. Then, by Theorem A, $k \leq 7$. Write now $x + id = a_i x_i^2$ $(i = 0, \ldots, k - 1)$ with square-free a_i's and suppose that $P(a_i) > k$ for some i. By the assumption $(x, d) = 1$ this implies $P(b) > k$, which is a contradiction. This shows that $P(a_i) \leq k$. Hence we get

$$x(x + d)(x + 2d) = cz^2, \tag{2}$$

where c and z are positive integers with $P(c) \leq k$, c square-free. Moreover, by the assumption $(x, d) = 1$ we find that $(c, d) = 1$ in (2). Hence $c \in \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$. Thus for each d we have to resolve 16 elliptic equations of the form

$$u^3 - c^2 d^2 u = v^2 \quad \text{in } u, v \in \mathbb{Z},$$

where u and v are given by $u = c(x + d)$ and $v = c^2 z$, respectively. These elliptic equations can be resolved easily with the use of the program package SIMATH (cf. [12]). For a detailed description of the algorithm implemented in SIMATH, see e.g. [3].

The simple search method already yielded all the solutions mentioned in the theorem. As in these solutions $k \leq 7$, all of them, but no more, were also provided by the resolution of the elliptic equations. ∎

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Šoltésovej 21
94059 Nové Zámky, Slovakia

Institute of Mathematics and Informatics
University of Debrecen
P.O. Box 12
H-4010 Debrecen, Hungary
E-mail: hajdul@math.klte.hu

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