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Generalized multiple Dirichlet series and generalized multiple polylogarithms

by

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1. Introduction and statement of results. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

Let $s = \sigma + i\tau$ be a complex variable, and

$$\psi(s) = \sum_{n=0}^{\infty} \frac{a(n)}{(\beta + nw)^s} \tag{1.1}$$

be a function with complex coefficients $a(n)$, where $\beta, w \in \mathbb{R}$ with $0 < \beta \le w$. We assume the following:

Assumption I. *There exists a $q > 0$ such that $\psi(s)$ is absolutely convergent for $\sigma > q$.*

Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$ and let $u \in \mathbb{R}$ with $1 \le u \le 1 + \delta$. We let

$$\psi(s; u) = \sum_{n=0}^{\infty} \frac{a(n)u^{-n}}{(\beta + nw)^s}. \tag{1.2}$$

By Assumption I, we can check that if $1 < u \le 1 + \delta$ then the right-hand side of (1.2) is absolutely convergent for any $s \in \mathbb{C}$, so $\psi(s; u)$ is holomorphic for all $s \in \mathbb{C}$. Corresponding to $\psi(s; u)$, let

$$G_1(t; \psi; u) = \sum_{n=0}^{\infty} a(n)u^{-n}e^{(\beta+nw)t}, \tag{1.3}$$

where t is a complex variable. By Assumption I, the series (1.3) is convergent when $\Re t < 0$. We further assume the following:

Assumption II. *$\psi(s)$ can be continued analytically to the whole complex plane $\mathbb{C}$, and is holomorphic for all $s \in \mathbb{C}$. In any fixed strip $\sigma_1 \le \sigma \le \sigma_2$,*

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$\psi(s;u)$ is uniformly convergent to $\psi(s)$ as $u \to 1+0$. Furthermore there exists a $\theta_0 = \theta_0(\sigma_1,\sigma_2) \in \mathbb{R}$ with $0 \le \theta_0 < \pi/2$ such that $\psi(s;u) = O(e^{\theta_0|\tau|})$ as $|\tau| \to \infty$.

Assumption III. *There exists a $\varrho = \varrho(\psi) > 0$ such that $G_1(t;\psi;u)$ can be continued holomorphically to*

$$\mathcal{D}(\varrho) = \{t \in \mathbb{C} \mid |t| < \varrho\} \tag{1.4}$$

for any $u \in [1, 1+\delta]$.

We will give typical examples which satisfy Assumptions I–III in Section 2 (see Example 2.2).

In the present paper, we consider generalized multiple Dirichlet series defined as follows. Let $(\alpha_0,\alpha_1,\dots,\alpha_r) \in \mathbb{R}^{r+1}$ and $(w_1,\dots,w_r) \in \mathbb{R}^r$ be such that $\alpha_0 = 0$ and $0 < \alpha_k - \alpha_{k-1} \le w_k$ $(1 \le k \le r)$. Let $\mathcal{P}_r = \{\psi_1,\dots,\psi_r\}$, where

$$\psi_k(s) = \sum_{n=0}^{\infty} \frac{a_k(n)}{(\alpha_k - \alpha_{k-1} + nw_k)^s}. \tag{1.5}$$

We assume that $\psi_k(s)$ and the associated series $\psi_k(s;u)$, $G_1(t;\psi_k;u)$ (defined similarly to (1.2) and (1.3)) satisfy Assumptions I–III $(1 \le k \le r)$. By Assumptions I and III, there exist $\{q_k = q(\psi_k)\ (>0) \mid 1 \le k \le r\}$ and $\{\varrho_k = \varrho(\psi_k)\ (>0) \mid 1 \le k \le r\}$. We let

$$\eta_r = \min_{1 \le k \le r} \{\varrho_k / 2^{r-1}\}. \tag{1.6}$$

We define the generalized multiple Dirichlet series associated with $\mathcal{P}_r$ by

$$\varPsi_r(s_1,\dots,s_r;u) = \sum_{n_1,\dots,n_r=0}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r) u^{-\sum_{\nu=1}^r n_\nu}}{\prod_{j=1}^r (\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu)^{s_j}} \tag{1.7}$$

for $s_1,\dots,s_r \in \mathbb{C}$ and $u \in [1,1+\delta]$. The special case $u=1$ and $a_j(n) = 1$ $(1 \le j \le r)$ has been studied by the first author in [12, 13]; it can be regarded as a generalization of both the Euler–Zagier multiple zeta function and the Barnes multiple zeta function. On the other hand, the special case $u = 1$, $\alpha_j = j$ and $w_j = 1$ $(1 \le j \le r)$ has also been studied before: see Arakawa–Kaneko [2] when a_js are periodic functions on $\mathbb{Z}$, and Matsumoto–Tanigawa [14] for more general a_js.

First we prove the following result by using the method introduced by Matsumoto–Tanigawa [14] (see also [11–13]). Indeed, this can be regarded as a generalization of Theorem 2 in [14].

Theorem 1.1. *For $s_1,\dots,s_r \in \mathbb{C}$ and $u \in [1,1+\delta]$, $\varPsi_r(s_1,\dots,s_r;u)$ is absolutely convergent for $s_j = \sigma_j + i\tau_j \in \mathbb{C}$ $(1 \le j \le r)$ with each $\sigma_j > q_j$. Furthermore $\varPsi_r(s_1,\dots,s_r;u)$ can be continued analytically to the* *whole complex space* $\mathbb{C}^r$ *and is holomorphic on* $\mathbb{C}^r$, *and satisfies*

$$\lim_{u\to 1+0} \Psi_r(s_1,\ldots,s_r;u) = \Psi_r(s_1,\ldots,s_r;1) \tag{1.8}$$

for any $(s_1,\ldots,s_r)\in\mathbb{C}^r$.

REMARK 1.2. The meromorphic continuation of $\Psi_r(s_1,\ldots,s_r;u)$ can be proved even if $\psi_k(s)$ has poles. When $u>1$, the multiple series (1.7) is absolutely convergent, hence holomorphic, for any $(s_1,\ldots,s_r)\in\mathbb{C}^r$. When $u=1$, if we assume that $\psi_k(s)$ has a pole of order at most one at $s=q_k$ and is holomorphic elsewhere (and satisfies $\psi_k(s)=O(e^{\theta_0|\tau|})$) for $1\le k\le r$, then we can show the following result, which generalizes Theorem 1 in [14]:

The function $\Psi_r(s_1,\ldots,s_r;1)$ *can be continued meromorphically to the whole space* $\mathbb{C}^r$, *and its possible singularities are located only on the subsets of* $\mathbb{C}^r$ *defined by one of the following equations*:

$$s_j+\cdots+s_r = q_j+\delta_{j+1}q_{j+1}+\cdots+\delta_r q_r - n$$
$$(1\le j\le r,\ \delta_k = 0 \text{ or } 1\ (2\le k\le r),\ n\in\mathbb{N}_0).$$

Moreover, (i) *if* $j=r\ge 2$ *and* $q_r\in\mathbb{N}$, *then* $n\le q_r-1$, (ii) *if* $2\le j\le r-1$, $q_j\in\mathbb{N}$ *and* $\delta_{j+1}=\cdots=\delta_r=1$, *then* $n\le q_r-1$, (iii) *if* $j=r=1$ *or if* $j=1$ *and* $\delta_2=\cdots=\delta_r=1$, *then* $n=0$.

The proof uses the method of proof of Theorem 1 in [14].

We further consider generalized multiple polylogarithms related to (1.5). Let $\mathbf{d}_r=(d_1,\ldots,d_r)\in\mathbb{C}^r$ with $\Re d_j>q_j$ for each j. With the above notation, and for $u\in[1,1+\delta]$, let

$$\begin{aligned} &F_r(t_1,\ldots,t_r;\mathbf{d}_r;\mathcal{P}_r;u) \\ &\quad= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{l=1}^r n_l}\prod_{j=1}^r e^{(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)t_j}}{\prod_{j=1}^r(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)^{d_j}}. \end{aligned} \tag{1.9}$$

This multiple series is convergent when $\Re t_j\le 0$ $(1\le j\le r)$. If we formally let $\psi_k(s)=\zeta(s)$, the Riemann zeta function, and $d_k\in\mathbb{N}$ $(1\le k\le r)$ in (1.9), then $F_r(\log x_1,\ldots,\log x_r;\mathbf{d}_r;\mathcal{P}_r;1)$ is the multiple polylogarithm defined by Goncharov [6] (see also [4]). However, $\zeta(s)$ does not satisfy Assumption II, so we will not consider the Goncharov multiple polylogarithms in this paper. Instead, we prove the following result.

THEOREM 1.3. *For* $\mathbf{d}_r\in\mathbb{C}^r$ *with each* $\Re d_j>q_j$ $(1\le j\le r)$ *and* $u\in[1,1+\delta]$, $F_r(t_1,\ldots,t_r;\mathbf{d}_r;\mathcal{P}_r;u)$ *is holomorphic for all* $(t_1,\ldots,t_r)\in\mathcal{D}(\eta_r)^r$, *and satisfies*, *for* $(t_1,\ldots,t_r)\in\mathcal{D}(\eta_r)^r$,

$$F_r(t_1,\ldots,t_r;\mathbf{d}_r;\mathcal{P}_r;u) = \sum_{N_1,\ldots,N_r=0}^{\infty} \Psi_r(d_1-N_1,\ldots,d_r-N_r;u)\,\frac{t_1^{N_1}\cdots t_r^{N_r}}{N_1!\cdots N_r!}. \tag{1.10}$$

Furthermore, *for any* $\xi\in\mathbb{R}$ *with* $0<\xi<\eta_r$, (1.10) *is uniformly convergent with respect to* $(t_1,\ldots,t_r,u)\in\overline{\mathcal{D}}(\xi)^r\times[1,1+\delta]$, *where* $\overline{\mathcal{D}}(\xi)=\{t\in\mathbb{C}\mid |t|\le\xi\}$.

The special case $\psi_j(s)=\sum_{n\ge1}(-1)^n n^{-s}$ $(1\le j\le r)$, $\mathbf{d}_r\in\mathbb{N}^r$ and $t_1=\cdots=t_{r-1}=0$ has been studied by the second author. Indeed, $F_r(0,\ldots,0,t;\mathbf{d}_r;\mathcal{P}_r;u)$ played an important role in giving some evaluation formulas for Euler–Zagier sums (see [15]). In order to prove Theorem 1.3 and Proposition 2.1 (see below), we make use of the technique introduced in [15].

As applications, using Theorem 1.3, we prove certain estimates for $\Psi_r(d_1-N_1,\ldots,d_r-N_r;1)$ (see Proposition 5.1 and Example 5.2). We further give certain multiple analogues of both Berndt's and Katsurada's formulas for Dirichlet L-functions proved in [3, 9] (see Example 5.3).

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2. Generalized polylogarithms. First we consider the case of $r=1$. Let $\psi(s)$ be as defined in (1.1) and $F_1(t;d;\psi;u)$ as defined in (1.9). With the notation of Section 1, we can prove the following.

PROPOSITION 2.1. *For* $d\in\mathbb{C}$ *with* $\Re d>q$ *and* $u\in[1,1+\delta]$, $F_1(t;d;\psi;u)$ *is holomorphic for all* $t\in\mathcal{D}(\varrho)$, *and satisfies*, *for* $t\in\mathcal{D}(\varrho)$,

$$F_1(t;d;\psi;u)=\sum_{N=0}^{\infty}\psi(d-N;u)\,\frac{t^N}{N!}. \tag{2.1}$$

Furthermore, *for any* $\xi\in\mathbb{R}$ *with* $0<\xi<\varrho$, (2.1) *is uniformly convergent with respect to* $(t,u)\in\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$.

Proof. By Assumption III, we can let

$$G_1(t;\psi;u)=\sum_{n=0}^{\infty}\mathfrak{B}_n(\psi;u)\,\frac{t^n}{n!} \tag{2.2}$$

for $|t|<\varrho$. We use the method of contour integrals (see, for example, [16, proof of Theorem 4.2]). We consider the path Υ which consists of the positive real axis $[\varepsilon,\infty]$ (top side), a circle C_ε around 0 of radius ε, and the positive real axis $[\varepsilon,\infty]$ (bottom side), where $0<\varepsilon<\varrho$. Note that we interpret t^s to mean $\exp(s\log t)$, where the imaginary part of $\log t$ varies from 0 (on the top side of the real axis) to 2π (on the bottom side). Let

$$\begin{aligned}(2.3)\qquad H_1(s;\psi;u) &= \int_{\Upsilon} G_1(-t;\psi;u)t^{s-1}\,dt \\ &= (e^{2\pi is}-1)\int_{\varepsilon}^{\infty} G_1(-t;\psi;u)t^{s-1}\,dt + \int_{C_\varepsilon} G_1(-t;\psi;u)t^{s-1}\,dt,\end{aligned}$$

which, in view of (1.3), is holomorphic for all $s \in \mathbb{C}$ if $0 < \varepsilon < \varrho$. Putting $s = -n$ for $n \in \mathbb{N}_0$ and $\varepsilon = \xi$ with $0 < \xi < \varrho$ in (2.3) and using (2.2), we have

$$H_1(-n;\psi;u) = \int_{C_\xi} G_1(-t;\psi;u)t^{-n-1}\,dt = \frac{(2\pi i)\mathfrak{B}_n(\psi;u)(-1)^n}{n!}.$$

From Assumption III, $G_1(t;\psi;u)$ is continuous for all $(t,u)\in\mathcal{D}(\varrho)\times[1,1+\delta]$. Hence the value $\mathcal{M}_\xi=\max\{|G_1(-t;\psi;u)| \mid (t,u)\in\{t\in\mathbb{C} \mid |t| = \xi\}\times[1,1+\delta]\}$ exists. By the above equation, we have

$$(2.4)\qquad \frac{|\mathfrak{B}_n(\psi;u)|}{n!} \le \frac{1}{2\pi}\int_{C_\xi} |G_1(-t;\psi;u)|\,|t|^{-n-1}\,|dt| \le \frac{\mathcal{M}_\xi}{\xi^n}$$

for any $n \in \mathbb{N}_0$ and $u \in [1, 1+\delta]$, where ξ is an arbitrary real number with $0 < \xi < \varrho$.

On the other hand, let $s \in \mathbb{C}$ with $\Re s > \max(1,q)$. Then the second term on the right-hand side of (2.3) tends to 0 as $\varepsilon \to 0$. Hence

$$\begin{aligned}(2.5)\qquad H_1(s;\psi;u) &= (e^{2\pi is}-1)\int_0^{\infty} G_1(-t;\psi;u)t^{s-1}\,dt \\ &= (e^{2\pi is}-1)\sum_{n=0}^{\infty} a(n)u^{-n}\int_0^{\infty} t^{s-1}e^{-(\beta+nw)t}\,dt \\ &= (e^{2\pi is}-1)\Gamma(s)\psi(s;u),\end{aligned}$$

where the interchange of summation and integration is valid because $\Re s > q$. Hence

$$(2.6)\qquad \psi(s;u) = \frac{1}{(e^{2\pi is}-1)\Gamma(s)}\,H_1(s;\psi;u) = \frac{\Gamma(1-s)}{2\pi i\,e^{\pi is}}\,H_1(s;\psi;u),$$

because

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s} = \frac{2\pi i}{e^{\pi is}-e^{-\pi is}}.$$

The relation (2.6) is valid for all $s \in \mathbb{C}$ by analytic continuation.

Next, for $d \in \mathbb{C}$ with $\Re d > q$ and $N \in \mathbb{N}_0$, we put $s = d - N$ in (2.3). Then

$$H_1(d-N;\psi;u) = (e^{2\pi i d} - 1)\int_{\varepsilon}^{\infty} G_1(-t;\psi;u)t^{d-N-1}\,dt + \int_{C_\varepsilon} G_1(-t;\psi;u)t^{d-N-1}\,dt =: I_1 + I_2. \tag{2.7}$$

Note that if $N \geq \Re d + 1$ then

$$\Big|\int_{\varepsilon}^{\infty} e^{-(\beta+nw)t}t^{d-N-1}\,dt\Big| \leq \frac{e^{-(\beta+nw)\varepsilon}\varepsilon^{\Re d-N-1}}{\beta+nw}. \tag{2.8}$$

Hence

$$|I_1| \leq \varepsilon^{\Re d-N-1}|e^{2\pi i d}-1|\sum_{n=0}^{\infty}\frac{|a(n)|e^{-(\beta+nw)\varepsilon}}{\beta+nw}.$$

On the other hand, by using the fact that

$$\int_{C_\varepsilon} t^p\,dt = \begin{cases} 2\pi i & (p=-1), \\ \varepsilon^{p+1}\,\dfrac{e^{2\pi i p}-1}{p+1} & (p \neq -1), \end{cases} \tag{2.9}$$

for $p \in \mathbb{C}$ and by (2.2), we have

$$I_2 = \begin{cases} (2\pi i)\mathfrak{B}_{N-d}(\psi;u)\,\dfrac{(-1)^{N-d}}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{d-N}(e^{2\pi i d}-1)\displaystyle\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(\psi;u)(-1)^n\varepsilon^n}{(n+d-N)n!} & (\text{otherwise}). \end{cases} \tag{2.10}$$

Note that the above infinite series is convergent because of the assumption $\varepsilon < \varrho$ and (2.4). Hence

(2.11)
$$|I_2| \leq \begin{cases} 2\pi\,\dfrac{|\mathfrak{B}_{N-d}(\psi;u)|}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{\Re d-N}|e^{2\pi i d}-1|\left|\displaystyle\sum_{n=0}^{\infty}\mathfrak{B}_n(\psi;u)\,\frac{(-1)^n\varepsilon^n}{(n+d-N)n!}\right| & (\text{otherwise}). \end{cases}$$

From (2.4) with $\xi = \varepsilon$, the first case of (2.11) yields

$$|I_2| \leq 2\pi\mathcal{M}_\varepsilon\varepsilon^{d-N}.$$

In the second case of (2.11), we let $\gamma_d = \min\{|d-m| \mid m \in \mathbb{Z}\}$. Using (2.4) with ξ such that $0 < \varepsilon < \xi < \varrho$, we see that the second case of (2.11) yields

$$|I_2| \leq \varepsilon^{\Re d-N}|e^{2\pi i d}-1|\,\frac{\mathcal{M}_\xi}{\gamma_d(1-\varepsilon/\xi)}.$$

Hence it follows from (2.6)–(2.11) that there exists a constant $M > 0$ which depends on ε, d and ψ but is independent of N and u such that

$$\left|\frac{\psi(d-N;u)}{\Gamma(1+N-d)}\right| = \frac{1}{2\pi|e^{\pi id}|}\,|H_1(d-N;\psi;u)| \le M\varepsilon^{-N} \tag{2.12}$$

for $N \in \mathbb{N}_0$ with $N \ge \Re d + 1$. Note that we can take ε arbitrary such that $0 < \varepsilon < \varrho$. Since $|s| \le |\Re s| + |\Im s|$ for $s \in \mathbb{C}$, we have

$$\begin{aligned}|\Gamma(1+N-d)| &= |(N-d)(N-d-1)\cdots([\Re d]+1-d)\Gamma([\Re d]+1-d)|\\ &\le (N-[\Re d]+[|\Im d|]+1)!\,|\Gamma([\Re d]+1-d)|\end{aligned}$$

for $N \in \mathbb{N}_0$ with $N \ge \Re d + 1$. Hence

$$\begin{aligned}&\frac{|\psi(d-N;u)|}{N!}\\ &\quad\le \frac{(N-[\Re d]+[|\Im d|]+1)!\,|\Gamma([\Re d]+1-d)|}{N!}\left|\frac{\psi(d-N;u)}{\Gamma(1+N-d)}\right|\\ &\quad\le \frac{(N-[\Re d]+[|\Im d|]+1)!\,|\Gamma([\Re d]+1-d)|}{N!}\,M\varepsilon^{-N}.\end{aligned} \tag{2.13}$$

Suppose $u \in (1, 1+\delta]$ and $t = i\theta$ with $\theta \in (-\varrho, \varrho) \subset \mathbb{R}$. Then there exists an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \varrho$ and $|\theta| < \varepsilon$. From the definition (1.9) we have

$$\begin{aligned}F_1(i\theta;d;\psi;u) &= \sum_{n=0}^{\infty}\frac{a(n)u^{-n}}{(\beta+nw)^d}\sum_{N=0}^{\infty}\frac{(\beta+nw)^N(i\theta)^N}{N!}\\ &= \sum_{N=0}^{\infty}\psi(d-N;u)\,\frac{(i\theta)^N}{N!}.\end{aligned} \tag{2.14}$$

From (2.13) we can see that each side of (2.14) is uniformly convergent with respect to $u \in [1, 1+\delta]$ because $|\theta| < \varepsilon$. Hence we can let $u \to 1$ on each side of (2.14), so (2.14) holds for $u = 1$ when $\theta \in (-\varrho, \varrho)$. We can define

$$F_1(t;d;\psi;u) = \sum_{N=0}^{\infty}\psi(d-N;u)\,\frac{t^N}{N!}$$

for any $u \in [1, 1+\delta]$ and $t \in \mathbb{C}$ with $|t| < \varrho$. From (2.13), this is uniformly convergent with respect to $(t, u) \in \overline{\mathfrak{D}}(\xi) \times [1, 1+\delta]$ when $0 < \xi < \varrho$. Thus we have the assertion. ∎

Example 2.2. Let $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ be such that $\sum_{a=1}^{m} f(a) = 0$. It can be regarded as a periodic function defined on $\mathbb{Z}$. For example, any non-trivial primitive Dirichlet character and any non-trivial additive character defined mod m satisfy this condition. We define

$$L(s;f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s} \tag{2.15}$$

and

$$G_1(t;L;u)=\sum_{n=1}^{\infty} f(n)u^{-n}e^{nt}=\sum_{a=1}^{m}\frac{f(a)u^{-a}e^{at}}{1-u^{-m}e^{mt}}$$

for $u\in[1,1+\delta]$. Then $L(s;f)$ and $G_1(t;L;u)$ satisfy Assumptions I–III. Note that $\varrho=2\pi/m$ and $q=1$ in this case. For $d\in\mathbb{C}$ with $\Re d>1$, let

$$F_1(t;d;L)=\sum_{n=1}^{\infty}\frac{f(n)e^{nt}}{n^d}.$$

It follows from Proposition 2.1 that $F_1(t;d;L)$ is holomorphic on $\mathcal{D}(2\pi/m)$ and satisfies

$$F_1(t;d;L)=\sum_{N=0}^{\infty} L(d-N;f)\,\frac{t^N}{N!}. \tag{2.16}$$

In particular, when f is a primitive Dirichlet character χ of conductor m, we know that $L(-2j-1,\chi)=0$ if $\chi(-1)=-1$ and $L(-2j,\chi)=0$ if $\chi(-1)=1$ for $j\in\mathbb{N}_0$ (see, for example, [16, Chap. 4]). Hence, applying (2.16) with $d=2k$ and $d=2k+1$ for $k\in\mathbb{N}$ and using $\cos x=(e^{ix}+e^{-ix})/2$, we obtain

$$\sum_{n=1}^{\infty}\frac{\chi(n)\cos(n\theta)}{n^{2k}}=\sum_{j=0}^{k-1}L(2k-2j,\chi)\,\frac{(i\theta)^{2j}}{(2j)!}\qquad(\chi(-1)=1),$$

$$\sum_{n=1}^{\infty}\frac{\chi(n)\cos(n\theta)}{n^{2k+1}}=\sum_{j=0}^{k}L(2k+1-2j,\chi)\,\frac{(i\theta)^{2j}}{(2j)!}\qquad(\chi(-1)=-1)$$

for $\theta\in(-2\pi/m,2\pi/m)$. These are typical examples of Berndt's result (see [3, Theorem 4.2]; see also [5, (1.2.12)]). Similarly, it follows from (2.16) that

$$\sum_{n=1}^{\infty}\frac{\chi(n)\cos(n\theta)}{n^{2k+1}}=\sum_{j=0}^{\infty}L(2k+1-2j,\chi)\,\frac{(i\theta)^{2j}}{(2j)!}\qquad(\chi(-1)=1),$$

$$\sum_{n=1}^{\infty}\frac{\chi(n)\cos(n\theta)}{n^{2k}}=\sum_{j=0}^{\infty}L(2k-2j,\chi)\,\frac{(i\theta)^{2j}}{(2j)!}\qquad(\chi(-1)=-1)$$

for $k\in\mathbb{N}$ and $\theta\in(-2\pi/m,2\pi/m)$. Using the functional equations for $L(s,\chi)$, we can confirm that these equations coincide with Katsurada's formulas for $L(s,\chi)$ (see [9, Theorem 3]).

3. Proof of Theorem 1.1. Using the method introduced in [14, Section 2] (see also [11–13]), we give the proof of Theorem 1.1 by induction on r. The case of $r=1$ can be directly obtained from Assumptions I and II.

Hence we assume that Theorem 1.1 holds for $r-1$, and aim to prove the case of r (≥ 2).

As in Section 1, let

$$\Psi_r(s_1,\ldots,s_r;u)=\Psi_r(s_1,\ldots,s_r;\psi_1,\ldots,\psi_r;u)$$

be the function defined by (1.7). Since each $\psi_k(s)$ defined by (1.5) converges absolutely for $\Re s>q_k$ $(1\leq k\leq r)$, we can easily check that $\Psi_r(s_1,\ldots,s_r;u)$ converges absolutely if $\sigma_k=\Re s_k>q_k$ $(1\leq k\leq r)$.

First we assume each $\sigma_k>q_k$ $(1\leq k\leq r)$. Recall the Mellin–Barnes formula

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z\,dz, \tag{3.1}$$

where $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, $-\Re s<c<0$, and the path of integration is the vertical line $\Re z=c$. By the above assumption, we may assume $-\sigma_r<c<-q_r$. Put $s=s_r$ and

$$\lambda=\frac{\alpha_r-\alpha_{r-1}+n_rw_r}{\alpha_{r-1}+n_1w_1+\cdots+n_{r-1}w_{r-1}}$$

in (3.1). Then multiply both sides by

$$\frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{\nu=1}^r n_\nu}}{\prod_{j=1}^{r-2}(\alpha_j+\sum_{\nu=1}^j n_\nu w_\nu)^{s_j}(\alpha_{r-1}+\sum_{\nu=1}^{r-1} n_\nu w_\nu)^{s_{r-1}+s_r}}$$

and sum up with respect to $n_1,\ldots,n_r$ to obtain

$$\begin{aligned}\Psi_r(s_1,\ldots,s_r;u)=\frac{1}{2\pi i}\int_{(c)}&\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\\&\times\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)\psi_r(-z;u)\,dz.\end{aligned} \tag{3.2}$$

Let $M\in\mathbb{N}$ and $\varepsilon\in\mathbb{R}$ be a small positive number. We shall shift the path to $\Re z=M-\varepsilon$. We see that

$$\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)=O(1)$$

in the region $c\leq\Re z\leq M-\varepsilon$ because $\sigma_k>q_k$ $(1\leq k\leq r-2)$, $-\sigma_r<c$ and

$$\sigma_{r-1}+\sigma_r+\Re z\geq\sigma_{r-1}+\sigma_r+c>\sigma_{r-1}.$$

From the well-known Stirling formula for $\Gamma(s)$, we have

$$|\Gamma(s)|=e^{-\pi|\tau|/2}(|\tau|+1)^{\sigma-1/2}\left(1+O\left(\frac{1}{|\tau|+1}\right)\right) \tag{3.3}$$

as $|\tau|\to\infty$, where $s=\sigma+i\tau$. Hence, by Assumption II, the integrand on the right-hand side of (3.2) tends to zero as $|\Im z| \to \infty$, so this shift is possible. By the inductive assumption, Ψ_{r-1} is holomorphic on $\mathbb{C}^{r-1}$ and ψ_r holomorphic on $\mathbb{C}$. Therefore we only have to count the residues of the poles of $\Gamma(-z)$ at $z = 0, 1, \ldots, M-1$. Since the residue of the pole of $\Gamma(s_r+z)\Gamma(-z)/\Gamma(s_r)$ at $z = k$ equals $-\binom{-s_r}{k}$, we obtain

$$(3.4) \quad \Psi_r(s_1, \ldots, s_r; u) = \sum_{k=0}^{M-1} \binom{-s_r}{k} \Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k; u)\psi_r(-k; u) + \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \times \Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z; u)\psi_r(-z; u)\, dz =: S_1 + S_2.$$

Now S_1 is holomorphic on the whole $\mathbb{C}^r$ by the inductive assumption. On the other hand, $\Gamma(s_r + z)$ has no pole on the path $(M - \varepsilon)$, when $\Re(-s_r) = -\sigma_r < M - \varepsilon$, so that $\sigma_r > -M + \varepsilon$. Using (3.3) and Assumption II, we see that S_2 is absolutely convergent, so it is holomorphic in the region

$$\{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \sigma_1 > q_1, \ldots, \sigma_{r-1} > q_{r-1},\, \sigma_r > -M + \varepsilon\},$$

where M is arbitrary.

Next we fix $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, and consider the continuation with respect to s_k for $1 \le k \le r-1$. Since Ψ_{r-1} is holomorphic on $\mathbb{C}^{r-1}$, the integrand in S_2 is holomorphic for all $(s_1, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}$. So, if we prove that S_2 converges absolutely for any $(s_1, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, then $\Psi_r(s_1, \ldots, s_r; u)$ is holomorphic on the whole $\mathbb{C}^r$ because M is arbitrary. In order to prove this result, we need the following lemma.

LEMMA 3.1. *For $r \in \mathbb{N}$ with $r \ge 2$, there exists a polynomial $P_r(X) \in \mathbb{R}[X]$ such that*

$$(3.5) \qquad \Psi_r(s_1, \ldots, s_r; u) = O(P_r(|\tau_r|)e^{\theta_0|\tau_r|}) \qquad (|\tau_r| \to \infty)$$

for any $(s_1, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $u \in [1, 1+d]$, where the constant implied by the O-symbol depends on $\tau_1, \ldots, \tau_{r-1}$.

Proof. We denote (3.5) by

$$\Psi_r(s_1, \ldots, s_r; u) \ll P_r(|\tau_r|)e^{\theta_0|\tau_r|}.$$

We prove this lemma by induction on r (≥ 2). First we consider the case of

$r=2$. It follows from Assumption II and (3.4) that

$$
\begin{aligned}
(3.6)\qquad & |\Psi_2(s_1,s_2;u)| \\
&\le \sum_{k=0}^{M-1}\left|\binom{-s_2}{k}\right| |\Psi_1(s_1+s_2+k;u)\psi_2(-k;u)| \\
&\quad + \frac{1}{2\pi}\left|\int_{(M-\varepsilon)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\Psi_1(s_1+s_2+z;u)\psi_2(-z;u)\,dz\right| \\
&\ll \sum_{k=0}^{M-1}\left|\binom{-s_2}{k}\right| e^{\theta_0|\tau_2|} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\right| e^{\theta_0|\tau_2+y|}e^{\theta_0|y|}\,dy,
\end{aligned}
$$

where $z=x+iy$. For simplicity, we denote the last term on the right-hand side of (3.6) by I. Using (3.3), we have

$$
\begin{aligned}
(3.7)\qquad & I \ll e^{\pi|\tau_2|/2}(|\tau_2|+1)^{-\sigma_2+1/2} \\
&\times \int_{-\infty}^{\infty} e^{(\theta_0-\pi/2)|\tau_2+y|}e^{(\theta_0-\pi/2)|y|}(|\tau_2+y|+1)^{\sigma_2+x-1/2}(|y|+1)^{-x-1/2}\,dy.
\end{aligned}
$$

Now Lemma 4 in [12] applied with $A=B=\theta_0-\pi/2$, $p=\sigma_2+x-1/2$ and $q=-x-1/2$ yields

$$
\begin{aligned}
(3.8)\qquad & I \ll e^{\pi|\tau_2|/2}(|\tau_2|+1)^{-\sigma_2+1/2} \\
&\quad\times [\{1+(|\tau_2|+1)^{\sigma_2+x-1/2}\}(|\tau_2|+1)^{-x+1/2}e^{(\theta_0-\pi/2)|\tau_2|} \\
&\quad + \{1+(|\tau_2|+1)^{\sigma_2+x-1/2}\}e^{(\theta_0-\pi/2)|\tau_2|}].
\end{aligned}
$$

Combining (3.6) and (3.8), we see that there exists $P_2(X)\in\mathbb{R}[X]$ such that

$$
\Psi_2(s_1,s_2;u) \ll P_2(|\tau_2|)e^{\theta_0|\tau_2|} \qquad (|\tau_2|\to\infty).
$$

Thus we have the assertion for $r=2$.

Assume that the assertion holds for $r-1$. Substituting the assumed bounds into (3.4) and using Assumption II, we have

$$
\begin{aligned}
\Psi_r(s_1,\dots,s_r;u) &\ll \sum_{k=0}^{M-1}\left|\binom{-\sigma_r+i\tau_r}{k}\right| P_{r-1}(|\tau_{r-1}+\tau_r|)e^{\theta_0|\tau_{r-1}+\tau_r|} \\
&\quad + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left|\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\right| \\
&\quad \times P_{r-1}(|\tau_{r-1}+\tau_r+y|)e^{\theta_0|\tau_{r-1}+\tau_r+y|}e^{\theta_0|y|}\,dy.
\end{aligned}
$$

By the same method as above, we can see that there exists $P_r(X)\in\mathbb{R}[X]$ such that

$$
\Psi_r(s_1,\dots,s_r;u) \ll P_r(|\tau_r|)e^{\theta_0|\tau_r|}.
$$

This finishes the proof of Lemma 3.1. ∎

Now we can complete the proof of Theorem 1.1 as follows. If we fix any $(s_1,\ldots,s_r)\in\mathbb{C}^r$, then it follows from Lemma 3.1 that

$$\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)\ll P_{r-1}(|\tau_{r-1}+\tau_r+y|)e^{\theta_0|\tau_{r-1}+\tau_r+y|}$$

as $|y|\to\infty$, where $z=x+iy$. Since s_{r-1} is fixed, this can be written as

$$\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)\ll\widetilde{P}_{r-1}(|\tau_r+y|)e^{\theta_0|\tau_r+y|}\quad(|y|\to\infty), \tag{3.9}$$

where $\widetilde{P}_{r-1}(X)\in\mathbb{R}[X]$. Recall that S_2 is the second term on the right-hand side of (3.4). Then, by using (3.3), and by (3.8) and Assumption II, we have

$$\begin{aligned} S_2 &\ll \int_{-\infty}^{\infty}\widetilde{\widetilde{P}}_{r-1}(y)e^{-\pi|y|/2-\pi|\tau_r+y|/2}e^{\theta_0|\tau_r+y|}e^{\theta_0|y|}\,dy \\ &= \int_{-\infty}^{\infty}\widetilde{\widetilde{P}}_{r-1}(y)e^{(\theta_0-\pi/2)(|\tau_r+y|+|y|)}\,dy \end{aligned}$$

for some $\widetilde{\widetilde{P}}_{r-1}(X)\in\mathbb{R}[X]$. Since $0\le\theta_0<\pi/2$, S_2 converges absolutely for any $(s_1,\ldots,s_r)\in\mathbb{C}^r$. By (3.4), $\Psi_r(s_1,\ldots,s_r;u)$ is holomorphic on $\mathbb{C}^r$.

Lastly, we prove (1.8). More precisely, we prove that (1.8) holds uniformly with respect to s_j $(1\le j\le r)$ in any fixed strip $\sigma_{1j}\le\Re s_j\le\sigma_{2j}$ as $u\to 1{+}0$. The case of $r=1$ follows from Assumption II. Hence we assume that the case of $r-1$ holds and prove the case of r (≥ 2). Let $u\to 1+0$ in (3.4). From the inductive assumption, the integrand in S_2 is uniformly convergent with respect to z in any fixed strip $\sigma_1\le\Re z\ (=M-\varepsilon)\ \le\sigma_2$ as $u\to 1+0$. Exchanging $\lim_{u\to 1+0}$ and the integral, and using the inductive assumption, we see that the right-hand side of (3.4) tends to

$$\begin{aligned} &\sum_{k=0}^{M-1}\binom{-s_r}{k}\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+k;1)\psi_r(-k;1) \\ &\qquad+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \\ &\qquad\times\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;1)\psi_r(-z;1)\,dz \end{aligned} \tag{3.10}$$

as $u\to 1+0$. It is clear that this convergence is uniform with respect to s_j in any fixed strip $\sigma_{1j}\le\Re s_j\le\sigma_{2j}$ $(1\le j\le r)$. From (3.4), we see that (3.10) coincides with $\Psi_r(s_1,\ldots,s_r;1)$. Hence the assertion in the case of r holds. This completes the proof of Theorem 1.1.

Remark 3.2. For any $N\in\mathbb{N}_0$, let $M=N+1$ and $s_r\to -N$ in (3.4). Then S_2 tends to 0 because $\Gamma(s_r)$ has a pole at $s_r=-N$. Hence we obtain

$$(3.11)\quad \Psi_r(s_1,\ldots,s_{r-1},-N;u) = \sum_{\nu=0}^{N}\binom{N}{\nu}\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+\nu-N;u)\psi_r(-\nu;u)$$

for $u\in[1,1+\delta]$ and $(s_1,\ldots,s_{r-1})\in\mathbb{C}^{r-1}$. In particular, let $\psi_j(s)=L(s;f_j)$ $(1\le j\le r)$ and $u=1$, where each f_j is defined mod m_j and satisfies a certain condition (see Example 2.2). Then we can check that Assumptions I–III hold. In this case, $\Psi_r(s_1,\ldots,s_r;1)$ coincides with the multiple L-function

$$L_r(s_1,\ldots,s_r;f_1,\ldots,f_r)=\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}(n_1+n_2)^{s_2}\cdots(n_1+\cdots+n_r)^{s_r}},$$

which has been studied in [2]. Hence (3.11) gives

$$(3.12)\quad L_r(s_1,\ldots,s_{r-1},-N;f_1,\ldots,f_r) = \sum_{\nu=0}^{N}\binom{N}{\nu}L_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+\nu-N;f_1,\ldots,f_{r-1})L_1(-\nu;f_r)$$

for $(s_1,\ldots,s_{r-1})\in\mathbb{C}^{r-1}$. This result was proved by Kamano (see [8]) by using the method introduced in [1]. This case can also be derived directly from the relation (2.3) in [14].

4. Proof of Theorem 1.3. In this section, we prove Theorem 1.3 by induction on r.

The case of $r=1$ is just what we proved in Proposition 2.1. Hence we assume that the assertion holds for $r-1$ and prove the case of r (≥ 2).

Let $\mathcal{P}_r=\{\psi_1,\ldots,\psi_r\}$ satisfy Assumptions I–III. Then we can take $\{q_k\}_{1\le k\le r}$ and $\{\varrho_k\}_{1\le k\le r}$, and define η_{r-1} and η_r by (1.6). Let

$$(4.1)\quad G_r(t_1,\ldots,t_r;\mathbf{d}_{r-1};\mathcal{P}_r;u) = F_{r-1}(t_1,\ldots,t_{r-2},t_{r-1}+t_r;\mathbf{d}_{r-1};\mathcal{P}_{r-1};u)G_1(t_r;\psi_r;u) = \sum_{n_1,\ldots,n_r=0}^{\infty}\frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{l=1}^r n_l}\prod_{j=1}^r e^{(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)t_j}}{\prod_{j=1}^{r-1}(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)^{d_j}},$$

which is convergent when $\Re t_j<0$ $(1\le j\le r)$. By the inductive assumption, $F_{r-1}(t_1,\ldots,t_{r-1}+t_r;\mathbf{d}_{r-1};\mathcal{P}_{r-1};u)$ is holomorphic for $(t_1,\ldots,t_r)\in\mathcal{D}(\eta_{r-1})^{r-2}\times\mathcal{D}(\eta_{r-1}/2)^2$, and $G_1(t_r;\psi_r;u)$ is holomorphic for $t_r\in\mathcal{D}(\varrho_r)$. Since $\eta_r\le\min(\eta_{r-1}/2,\varrho_r)$, we see that $G_r(t_1,\ldots,t_r;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is holomorphic for $(t_1,\ldots,t_r)\in\mathcal{D}(\eta_r)^r$. Therefore, if we fix $t_r\in\mathcal{D}(\eta_r)$ then the function of $r-1$ real variables $G_r(i\theta_1,\ldots,i\theta_{r-1},t_r;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is real-analytic for $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}\subset\mathbb{R}^{r-1}$ (see, for example, [10, Corollary 2.3.7]). Similarly, if we fix $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$, then $G_r(\{i\theta_k\},t_r;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is holomorphic for $t_r\in\mathcal{D}(\eta_r)$. Hence we define $\{\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\}_{n\geq 0}$ by

$$G_r(i\theta_1,\dots,i\theta_{r-1},t_r;\mathbf{d}_{r-1};\mathcal{P}_r;u)=\sum_{n=0}^{\infty}\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\,\frac{t_r^n}{n!}. \tag{4.2}$$

As in the proof of Proposition 2.1, we let

$$\begin{aligned}H_r(s;&i\theta_1,\dots,i\theta_{r-1};\mathbf{d}_{r-1};\mathcal{P}_r;u)\\&=\int_{\varUpsilon}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{s-1}\,dt\\&=(e^{2\pi is}-1)\int_{\varepsilon}^{\infty}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{s-1}\,dt\\&\qquad+\int_{C_\varepsilon}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{s-1}\,dt,\end{aligned}\tag{4.3}$$

which is holomorphic for all $s\in\mathbb{C}$ if we fix $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$ and $0<\varepsilon<\eta_r$.

Putting $s=-n$ for $n\in\mathbb{N}_0$ and $\varepsilon=\xi$ with $0<\xi<\eta_r$ in (4.3), and using (4.2), we have

$$\begin{aligned}H_r(-n;i\theta_1,\dots,i\theta_{r-1};\mathbf{d}_{r-1};\mathcal{P}_r;u)&=\int_{C_\xi}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{-n-1}\,dt\\&=\frac{(2\pi i)\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)(-1)^n}{n!}.\end{aligned}$$

By the inductive assumption and (4.1), we see that the Taylor expansion of $G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ around $t=0$ is uniformly convergent with respect to $(\theta_1,\dots,\theta_{r-1},t,u)\in[-\xi,\xi]^{r-1}\times\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$ when $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. In particular, $G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is continuous for $(\theta_1,\dots,\theta_{r-1},t,u)\in[-\xi,\xi]^{r-1}\times\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$. Hence the value

$$\begin{aligned}\widetilde{\mathcal{M}}_\xi=\max\{&|G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)|\ |\\&(t,u)\in[-\xi,\xi]^{r-1}\times\{|t|=\xi\}\times[1,1+\delta]\}\end{aligned}$$

exists when $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. By the above equation, we have

$$\frac{|\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)|}{n!}\leq\frac{\widetilde{\mathcal{M}}_\xi}{\xi^n}\tag{4.4}$$

for any $n\in\mathbb{N}_0$, $(\theta_1,\dots,\theta_{r-1})\in[-\xi,\xi]^{r-1}$ and $u\in[1,1+\delta]$.

Define

$$(4.5)\quad \mathcal{Z}_r(\mathbf{d}_{r-1}, s; i\theta_1, \ldots, i\theta_{r-1}; \mathcal{P}_r; u) = F_r(i\theta_1, \ldots, i\theta_{r-1}, 0; \mathbf{d}_{r-1}, s; \mathcal{P}_r; u)$$
$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{\nu=1}^r n_\nu} \prod_{j=1}^{r-1} e^{(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)i\theta_j}}{\prod_{j=1}^{r-1}(\alpha_j+\sum_{\nu=1}^j n_\nu w_\nu)^{d_j}(\alpha_r+\sum_{\nu=1}^r n_\nu w_\nu)^s}$$

for $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, $s \in \mathbb{C}$ with $\Re s > q_r$ and $u \in [1, 1+\delta]$. Assuming $\Re s > \max(1, q_r)$ and using the same method as in the proof of Proposition 2.1, we have

$$(4.6)\quad \mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u) = \frac{1}{(e^{2\pi i s}-1)\Gamma(s)} H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$$
$$= \frac{\Gamma(1-s)}{2\pi i\, e^{\pi i s}} H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u).$$

Note that $H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for all $s \in \mathbb{C}$ if we fix $\{\theta_k\} \in (-\eta_r, \eta_r)^{r-1}$ (as mentioned above), and the poles of $\Gamma(1-s)$ coincide with $\mathbb{N} = \{1, 2, \ldots\}$. Since $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ is absolutely convergent for $s \in \mathbb{C}$ with $\Re s > q_r$, it follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ is defined and holomorphic for all $s \in \mathbb{C} \setminus \{1, 2, \ldots, [q_r]\}$ if we fix $\{\theta_k\} \in (-\eta_r, \eta_r)^{r-1}$.

Furthermore, we can prove that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ has no pole as follows. Fix $s \in \mathbb{C}$. If $1 < u \le 1+\delta$ then from (1.7) and (4.5), and by substituting the Taylor expansion for each $\exp((\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu)i\theta_j)$ and changing the order of summations, we have

$$(4.7)\quad \mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$$
$$= \sum_{N_1,\ldots,N_{r-1}=0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_{r-1} - N_{r-1}, s; u) \frac{(i\theta_1)^{N_1}\cdots(i\theta_{r-1})^{N_{r-1}}}{N_1!\cdots N_{r-1}!}.$$

We see that (4.3) is uniformly convergent with respect to $(\theta_1, \ldots, \theta_{r-1}, u) \in [-\xi, \xi]^{r-1} \times [1, 1+\delta]$, for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Hence, for $u \in [1, 1+\delta]$, $H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$. Put $\theta_1 = \cdots = \theta_{r-1} = \theta$. Then for $u \in [1, 1+\delta]$, $H_r(s; \{i\theta\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $\theta \in (-\eta_r, \eta_r)$, and its Taylor expansion around $\theta = 0$ is uniformly convergent with respect to $(\theta, u) \in [-\xi, \xi] \times [1, 1+\delta]$. It follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta\}; \mathcal{P}_r; u)$ also has these properties. Hence, for any $u \in [1, 1+\delta]$, we define the one-variable complex function $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)$ which is holomorphic for $t \in \mathcal{D}(\eta_r)$ and its Taylor expansion around $t = 0$ is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1+\delta]$. In particular, $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)$ is continuous for $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1+\delta]$. Putting $\xi = \varepsilon$ with $0 < \varepsilon < \eta_r$ shows the existence of

$$\mathcal{M}'_\varepsilon = \max\{|\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)| \mid (t, u) \in \{t \in \mathbb{C} \mid |t| = \varepsilon\} \times [1, 1+\delta]\}.$$

Using the same method as in the proof of (4.4) and by (4.7) and the continuity of $\Psi_r(d_1-N_1,\dots,d_{r-1}-N_{r-1},s;u)$ in $u\in[1,1+\delta]$ (see Theorem 1.1), we see that

$$\left|\sum_{N_1+\cdots+N_{r-1}=n}\frac{\Psi_r(d_1-N_1,\dots,d_{r-1}-N_{r-1},s;u)}{N_1!\cdots N_{r-1}!}\right|\le\frac{\mathcal{M}'_\varepsilon}{\varepsilon^n}\tag{4.8}$$

for $u\in[1,1+\delta]$ and $n\in\mathbb{N}_0$, where ε is an arbitrary real number with $0<\varepsilon<\eta_r$. This means that the right-hand side of (4.7) is uniformly convergent with respect to $(\theta_1,\dots,\theta_{r-1},u)\in[-\xi,\xi]^{r-1}\times[1,1+\delta]$ for any $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. Hence we can let $u\to 1$ in (4.7), so (4.7) holds for $u\in[1,1+\delta]$. Since s is an arbitrary complex number, $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)$ has no pole, so it is holomorphic for all $s\in\mathbb{C}$ when $u\in[1,1+\delta]$, and real-analytic for $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$ when $s\in\mathbb{C}$ and $u\in[1,1+\delta]$.

For $d_r\in\mathbb{C}$ with $\Re d_r>q_r$ and $N\in\mathbb{N}_0$ with $N\ge\Re d_r+1$, we put $s=d_r-N$ in (4.3). Then we have

$$\begin{aligned}H_r(d_r-N;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)&=(e^{2\pi i d_r}-1)\int_\varepsilon^\infty G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{d_r-N-1}\,dt\\&\quad+\int_{C_\varepsilon}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{d_r-N-1}\,dt=:J_1+J_2.\end{aligned}\tag{4.9}$$

Since $N\ge\Re d_r+1$, we have

$$\left|\int_\varepsilon^\infty e^{-(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)t}t^{d_r-N-1}\,dt\right|\le\frac{e^{-(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)\varepsilon}|\varepsilon^{d_r-N-1}|}{\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu}.$$

Hence

$$\begin{aligned}|J_1|&\le\varepsilon^{\Re d_r-N-1}|e^{2\pi i d_r}-1|\\&\quad\times\sum_{n_1,\dots,n_r=0}^\infty\frac{|a_1(n_1)\cdots a_r(n_r)|e^{-(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)\varepsilon}}{\prod_{j=1}^{r-1}(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)^{\Re d_j}(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)}.\end{aligned}\tag{4.10}$$

On the other hand, by using (2.9), we have

$$J_2=\begin{cases}(2\pi i)\mathfrak{B}_{N-d_r}(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\dfrac{(-1)^{N-d_r}}{(N-d_r)!} & (N-d_r\in\mathbb{N}_0),\\[2ex] \varepsilon^{d_r-N}(e^{2\pi i d_r}-1)\displaystyle\sum_{n=0}^\infty\frac{\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)(-1)^n\varepsilon^n}{(n+d_r-N)n!} & (\text{otherwise}).\end{cases}$$

The last series is uniformly convergent with respect to $(\theta_1,\dots,\theta_{r-1},u)\in$

$[-\varepsilon,\varepsilon]^{r-1}\times[1,1+\delta]$ because of the assumption $\varepsilon<\eta_r$ and (4.4). Hence either

$$|J_2|\le 2\pi\,\frac{|\mathfrak{B}_{N-d_r}(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)|}{(N-d_r)!}\qquad (N-d_r\in\mathbb{N}_0) \tag{4.11}$$

or

$$|J_2|\le\varepsilon^{\Re d_r-N}|e^{2\pi i d_r}-1|\left|\sum_{n=0}^{\infty}\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\,\frac{(-1)^n\varepsilon^n}{(n+d_r-N)n!}\right| \tag{4.12}$$
$$\text{(otherwise).}$$

Just as (2.12), it follows from (4.4), (4.6), (4.8)–(4.12) that there exists a constant $M>0$ independent of N and $\{\theta_k\}$ such that

$$\begin{aligned}&\left|\frac{\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N;\{i\theta_k\};\mathcal{P}_r;u)}{\Gamma(1+N-d_r)}\right|\\&\qquad\le\frac{1}{2\pi|e^{\pi i d_r}|}\,|H_r(d_r-N;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)|\le M\varepsilon^{-N}\end{aligned} \tag{4.13}$$

for $N\in\mathbb{N}$ with $N\ge\Re d_r+1$. Note that we can take ε arbitrary such that $0<\varepsilon<\eta_r$. As in the case of (2.13), we have

$$\begin{aligned}&\frac{|\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N;\{i\theta_k\};\mathcal{P}_r;u)|}{N!}\\&\qquad\le\frac{(N-[\Re d_r]+[|\Im d_r|]+1)!\;|\Gamma([\Re d_r]+1-d_r)|}{N!}\,M\varepsilon^{-N}\end{aligned} \tag{4.14}$$

for $N\in\mathbb{N}$ with $N\ge\Re d_r+1$ and $u\in[1,1+\delta]$.

Suppose $1<u\le 1+\delta$ and $\theta_r\in(-\eta_r,\eta_r)$. Then by (1.9), and using the Taylor expansion for $\exp((\alpha_r+\sum_{\nu=1}^{r}n_\nu w_\nu)i\theta_r)$, we have

$$\begin{aligned}&F_r(i\theta_1,\dots,i\theta_{r-1},i\theta_r;\mathbf{d}_r;\mathcal{P}_r;u)\\&\qquad=\sum_{N_r=0}^{\infty}\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N_r;\{i\theta_k\};\mathcal{P}_r;u)\,\frac{(i\theta_r)^{N_r}}{N_r!}.\end{aligned} \tag{4.15}$$

By (4.14), the right-hand side of (4.15) is uniformly convergent with respect to $(\theta_r,u)\in[-\xi,\xi]\times[1,1+\delta]$ when $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$ and $0<\xi<\eta_r$. Hence (4.15) holds for $u=1$. As mentioned above, (4.7) holds for any $s\in\mathbb{C}$, $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$, and $u\in[1,1+\delta]$. Consequently

$$\begin{aligned}&\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N_r;\{i\theta_k\};\mathcal{P}_r;u)\\&\qquad=\sum_{N_1,\dots,N_{r-1}=0}^{\infty}\Psi_r(d_1-N_1,\dots,d_r-N_r;u)\,\frac{(i\theta_1)^{N_1}\cdots(i\theta_{r-1})^{N_{r-1}}}{N_1!\cdots N_{r-1}!}\end{aligned} \tag{4.16}$$

for $u \in [1, 1+\delta]$. Hence (4.15) can also be written as

$$F_r(i\theta_1, \ldots, i\theta_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_1, \ldots, N_r = 0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{(i\theta_1)^{N_1} \cdots (i\theta_r)^{N_r}}{N_1! \cdots N_r!} \tag{4.17}$$

for $u \in [1, 1+\delta]$, and (4.17) is uniformly convergent with respect to $(\theta_1, \ldots, \theta_r, u) \in [-\xi, \xi]^r \times [1, 1+\delta]$ for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Therefore, for $u \in [1, 1+\delta]$, we can define

$$F_r(t_1, \ldots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_1, \ldots, N_r = 0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{t_1^{N_1} \cdots t_r^{N_r}}{N_1! \cdots N_r!}, \tag{4.18}$$

which is uniformly convergent with respect to $(t_1, \ldots, t_r, u) \in \overline{\mathcal{D}}(\xi)^r \times [1, 1+\delta]$ and holomorphic for $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_r)^r$ (see, for example, [7, Section 2.2]). Thus we obtain the case of r. This completes the proof of Theorem 1.3.

5. Some applications. First we prove the following estimates for $\Psi_r(d_1 - N_1, \ldots, d_r - N_r; u)$ by using the same method as in the proof of Proposition 2.3.10 in [10].

PROPOSITION 5.1. *With the same notation as in Theorem* 1.3,

$$\limsup_{N_1 + \cdots + N_r \to \infty} \left\{ \frac{|\Psi_r(d_1 - N_1, \ldots, d_r - N_r; u)|}{N_1! \cdots N_r!} \right\}^{1/(N_1 + \cdots + N_r)} \le \frac{1}{\eta_r}. \tag{5.1}$$

Proof. Assume otherwise. Then we take $\kappa \in \mathbb{R}$ with $\kappa > 1/\eta_r$ such that there exist infinitely many $(N_1, \ldots, N_r) \in \mathbb{N}_0^r$ such that

$$\frac{|\Psi_r(d_1 - N_1, \ldots, d_r - N_r; u)|}{N_1! \cdots N_r!} > \kappa^{N_1 + \cdots + N_r}.$$

This means that the right-hand side of (1.10) does not converge absolutely at $(1/\kappa, \ldots, 1/\kappa) \in \mathcal{D}(\eta_r)^r$, which is a contradiction. ∎

EXAMPLE 5.2. Let $\psi_j(s) = L(s; f_j)$ $(1 \le j \le r)$ as considered in Remark 3.2. Then (5.1) gives

$$\limsup_{N_1 + \cdots + N_r \to \infty} \left\{ \frac{|L_r(d_1 - N_1, \ldots, d_r - N_r; f_1, \ldots, f_r)|}{N_1! \cdots N_r!} \right\}^{1/(N_1 + \cdots + N_r)} \le \frac{1}{\eta_r}, \tag{5.2}$$

where each $\Re d_j > 1$ $(1 \le j \le r)$ and $\eta_r = \min_{1 \le k \le r} \{2\pi / 2^{r-1} m_k\}$.

Secondly we give certain multiple analogues of both Berndt's and Katsurada's formulas considered in Example 2.2.

EXAMPLE 5.3. As in the above example, let $\psi_j(s) = L(s; f_j)$ $(1 \le j \le r)$ and define a generalization of multiple polylogarithm by

$$\begin{aligned}(5.3)\quad &\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)\\ &= \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)\prod_{j=1}^{r} e^{(\sum_{\mu=1}^{j} n_\mu)t_j}}{n_1^{d_1}(n_1+n_2)^{d_2}\cdots(n_1+\cdots+n_r)^{d_r}}\end{aligned}$$

for $d_1,\ldots,d_r \in \mathbb{C}$ with $\Re d_j > 1$ $(1 \le j \le r)$. Theorem 1.3 with $\psi_j(s) = L(s; f_j)$ $(1 \le j \le r)$ and $u = 1$ shows that $\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)$ is defined and holomorphic for $(t_1,\ldots,t_r) \in \mathfrak{D}(\eta_r)^r$ such that

$$\begin{aligned}(5.4)\quad &\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)\\ &= \sum_{N_1,\ldots,N_r=0}^{\infty} L_r(d_1-N_1,\ldots,d_r-N_r;f_1,\ldots,f_r)\,\frac{t_1^{N_1}\cdots t_r^{N_r}}{N_1!\cdots N_r!},\end{aligned}$$

where $\eta_r = \min_{1\le k\le r}\{2\pi/2^{r-1}m_k\}$. Putting $t_1 = \cdots = t_{r-1} = 0$ and $t_r = \pm i\theta$ for $\theta \in (-\eta_r, \eta_r)$ in (5.4), we have

$$\begin{aligned}(5.5)\quad &\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)\cos((n_1+\cdots+n_r)\theta)}{n_1^{d_1}(n_1+n_2)^{d_2}\cdots(n_1+\cdots+n_r)^{d_r}}\\ &= \sum_{N=0}^{\infty} L_r(d_1,\ldots,d_{r-1},d_r-2N;f_1,\ldots,f_r)\,\frac{(i\theta)^{2N}}{(2N)!}.\end{aligned}$$

REMARK 5.4. In the case $f_j(n) = (-1)^n$ $(1 \le j \le r)$, the function $\mathfrak{F}_r(i\theta_1,\ldots,i\theta_r;f_1,\ldots,f_r)$ has recently been used to prove what is called the parity result for Euler–Zagier sums (see [15]).

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