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On the two-dimensional theta functions of the Borweins

by

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1. Introduction. Let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of nonnegative integers, positive integers, integers, real numbers and complex numbers respectively. Jonathan and Peter Borwein [4], in their work on a cubic counterpart of Jacobi's theta function identity and a cubic analogue of the arithmetic-geometric mean iteration of Gauss and Legendre, introduced the following three 2-dimensional theta functions:

$$a(q) := \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2}, \quad q \in \mathbb{C},\ |q| < 1, \tag{1.1}$$

$$b(q) := \sum_{(m,n)\in\mathbb{Z}^2} \omega^{m-n} q^{m^2+mn+n^2}, \quad q \in \mathbb{C},\ |q| < 1, \tag{1.2}$$

$$c(q) := \sum_{(m,n)\in\mathbb{Z}^2} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}, \quad q \in \mathbb{C},\ |q| < 1, \tag{1.3}$$

where $\omega = e^{2\pi i/3}$. Note that in (1.3) principal values of the cube roots are taken. We observe that

$$a(0) = 1, \quad b(0) = 1, \quad c(0) = 0. \tag{1.4}$$

The Borweins [4] (and together with F. G. Garvan [5]) proved the beautiful cubic identity

$$a(q)^3 = b(q)^3 + c(q)^3. \tag{1.5}$$

The Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C},\ |q| < 1. \tag{1.6}$$

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We have

$$\varphi(0)=1. \tag{1.7}$$

Set

$$p(q):=\frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}, \tag{1.8}$$

$$k(q):=\frac{\varphi^3(q^3)}{\varphi(q)}. \tag{1.9}$$

Clearly

$$p(0)=0, \quad k(0)=1. \tag{1.10}$$

When there is no risk of confusion we write p for $p(q)$ and k for $k(q)$. Making use of identities proved in [1], [2] and [5], we prove the following parametric representations of $a(q^m)$, $b(q^m)$, $c(q^m)$ ($m\in\{1,2,3,4,6,8,12\}$), as well as of $a(-q)$, $b(-q)$, $c(-q)$, in terms of p and k. Since $b(q^m)$ can be determined from $a(q^m)$ and $c(q^m)$ by means of (1.5) (with q replaced by q^m), we only give the values of $b(q^m)$ when they can be expressed in terms of p and k in a fairly simple manner.

THEOREM 1.

$$\begin{aligned}
a(q)&=(1+4p+p^2)k,\\
b(q)&=2^{-1/3}((1-p)^4(1+2p)(2+p))^{1/3}k,\\
c(q)&=2^{-1/3}3(p(1+p)^4)^{1/3}k.
\end{aligned}$$

THEOREM 2.

$$\begin{aligned}
a(q^2)&=(1+p+p^2)k,\\
b(q^2)&=2^{-2/3}((1-p)(1+2p)(2+p))^{2/3}k,\\
c(q^2)&=2^{-2/3}3(p(1+p))^{2/3}k.
\end{aligned}$$

THEOREM 3.

$$\begin{aligned}
a(q^3)&=3^{-1}(1+4p+p^2+2^{2/3}((1-p)^4(1+2p)(2+p))^{1/3})k,\\
c(q^3)&=3^{-1}(1+4p+p^2-2^{-1/3}((1-p)^4(1+2p)(2+p))^{1/3})k.
\end{aligned}$$

THEOREM 4.

$$\begin{aligned}
a(q^4)&=\left(1+p-\tfrac{1}{2}p^2\right)k,\\
b(q^4)&=2^{-4/3}((1-p)(1+2p)(2+p)^4)^{1/3}k,\\
c(q^4)&=2^{-4/3}3(p^4(1+p))^{1/3}k.
\end{aligned}$$

THEOREM 5.

$$\begin{aligned}
a(q^6)&=3^{-1}(1+p+p^2+2^{1/3}((1-p)(1+2p)(2+p))^{2/3})k,\\
c(q^6)&=3^{-1}(1+p+p^2-2^{-2/3}((1-p)(1+2p)(2+p))^{2/3})k.
\end{aligned}$$

THEOREM 6.

$$a(q^8) = 2^{-2}(1 + p + p^2 + 3((1 - p)(1 + p)(1 + 2p))^{1/2})k.$$

Theorems 1–5 are proved in Sections 2–6 respectively. Theorem 6 is proved in Section 12 by applying the "duplication principle" (Theorem 9) to Theorem 4. We omit the complicated expressions for $b(q^8)$ and $c(q^8)$. We could also determine $a(q^9)$ and $c(q^9)$ by applying the "triplication principle" (Theorem 10) to Theorem 3. However the resulting expressions for $a(q^9)$ and $c(q^9)$ are complicated so we do not give them here.

THEOREM 7.

$$a(q^{12}) = 3^{-1}\left(1 + p - \tfrac{1}{2}p^2 + 2^{-1/3}((1 - p)(1 + 2p)(2 + p)^4)^{1/3}\right)k,$$
$$c(q^{12}) = 3^{-1}\left(1 + p - \tfrac{1}{2}p^2 - 2^{-4/3}((1 - p)(1 + 2p)(2 + p)^4)^{1/3}\right)k.$$

Theorem 7 is proved in Section 7. Alternative proofs can be given by applying the duplication principle to Theorem 5 and by applying the triplication principle to Theorem 4.

THEOREM 8.

$$a(-q) = (1 - 2p - 2p^2)k,$$
$$b(-q) = 2^{-1/3}((1 - p)(1 + 2p)^4(2 + p))^{1/3}k,$$
$$c(-q) = -2^{1/3}3(p(1 + p))^{1/3}k.$$

Theorem 8 is proved in Section 8.

From Theorems 1, 2 and 4, we obtain the duplication principle for p and k, namely

THEOREM 9 (Duplication principle).

$$p(q^2) = \frac{1 + p - p^2 - ((1 - p)(1 + p)(1 + 2p))^{1/2}}{p^2},$$
$$k(q^2) = \frac{(1 + p - p^2 + ((1 - p)(1 + p)(1 + 2p))^{1/2})k}{2}.$$

Theorem 9 is proved in Section 9.

From Theorems 1, 2, 3 and 5, we obtain the triplication principle for p and k, namely

THEOREM 10 (Triplication principle).

$$\begin{aligned} p(q^3) = 3^{-1}((-4 - 3p + 6p^2 + 4p^3) &+ 2^{2/3}(1 - 2p - 2p^2)((1 - p)(1 + 2p)(2 + p))^{1/3} \\ &+ 2^{1/3}(1 + 2p)((1 - p)(1 + 2p)(2 + p))^{2/3}), \\ k(q^3) = 3^{-2}(3 + 2^{2/3}(1 + 2p)((1 - p)(1 + 2p)(2 + p))^{1/3} &+ 2^{4/3}((1 - p)(1 + 2p)(2 + p))^{2/3})k. \end{aligned}$$

Theorem 10 is proved in Section 11.

From Theorems 1 and 9, we obtain the "change of sign principle" for p and k, namely

THEOREM 11 (Change of sign principle).

$$p(-q) = \frac{-p}{1+p}, \qquad k(-q) = (1+p)^2 k.$$

Theorem 11 is proved in Section 10.

For $n \in \mathbb{N}_0$ and $l, m \in \mathbb{N}$ we set

$$N(l,m;n) := \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid l(x^2+xy+y^2) + m(z^2+zt+t^2) = n\}. \tag{1.11}$$

Clearly $N(l,m;0) = 1$ and

$$\sum_{n=0}^{\infty} N(l,m;n) q^n = a(q^l) a(q^m). \tag{1.12}$$

As an application of Theorems 1–7, we determine $N(l,m;n)$ for certain small values of l and m. In preparation for doing this we prove in Sections 13–15 some results concerning the Eisenstein series

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

where

$$\sigma(m) = \begin{cases} \sum_{d \mid m} d & \text{if } m \in \mathbb{N}, \\ 0 & \text{if } m \notin \mathbb{N}. \end{cases}$$

THEOREM 12. *For* $n \in \mathbb{N}$,

$$N(1,1;n) = 12\sigma(n) - 36\sigma(n/3).$$

The proof of Theorem 12 is given in Section 16. A result equivalent to Theorem 12 was stated but not proved by Liouville [7]. An elementary proof was given by Huard, Ou, Spearman and Williams [6].

THEOREM 13. *For* $n \in \mathbb{N}$,

$$N(1,2;n) = 6\sigma(n) - 12\sigma(n/2) + 18\sigma(n/3) - 36\sigma(n/6).$$

The proof of Theorem 13 is given in Section 17. Liouville [8] gave a result equivalent to Theorem 13 but did not prove it.

THEOREM 14. *For* $n \in \mathbb{N}$,

$$N(1,3;n) = \begin{cases} 12\sigma(n) - 36\sigma(n/3) & \text{if } n \equiv 0 \pmod 3, \\ 6\sigma(n) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

The proof of Theorem 14 is given in Section 18. Liouville [11] stated a result equivalent to Theorem 14 but did not prove it.

THEOREM 15. *For* $n \in \mathbb{N}$,

$$N(1,4;n) = \begin{cases} 12\sigma(n/4) - 36\sigma(n/12) & \text{if } n \equiv 0 \pmod 2, \\ 6\sigma(n) - 18\sigma(n/3) & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The proof of Theorem 15 is given in Section 19. Liouville did not consider the evaluation of $N(1,4;n)$ and the result appears to be new.

THEOREM 16. *For* $n \in \mathbb{N}$,

$$N(1,6;n) = \begin{cases} -6\sigma(n)+12\sigma(n/2)+30\sigma(n/3)-60\sigma(n/6) & \text{if } n \equiv 0 \pmod 3, \\ 6\sigma(n)-12\sigma(n/2) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem 16 is proved in Section 20. Liouville [9] stated but did not prove a result equivalent to Theorem 16.

THEOREM 17. *For* $n \in \mathbb{N}$,

$$N(2,3;n) = \begin{cases} -6\sigma(n)+12\sigma(n/2)+30\sigma(n/3)-60\sigma(n/6) & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \equiv 1 \pmod 3, \\ 6\sigma(n)-12\sigma(n/2) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem 17 is proved in Section 21. Liouville [10] stated but did not prove a result equivalent to Theorem 17.

We close our introduction by noting that for small $|q|$ we have

$$a(q) = 1 + 6q + O(q^2), \tag{1.13}$$

$$b(q) = 1 - 3q + O(q^2), \tag{1.14}$$

$$c(q) = 3q^{1/3}(1 + q + O(q^2)), \tag{1.15}$$

$$\varphi(q) = 1 + 2q + O(q^4), \tag{1.16}$$

$$p(q) = 2q + 2q^2 + O(q^3), \tag{1.17}$$

$$k(q) = 1 - 2q + O(q^3). \tag{1.18}$$

Hence

$$\frac{c(q^2)}{c^2(q)} = \frac{3q^{2/3}(1+O(q^2))}{9q^{2/3}(1+O(q))} = \frac{1}{3}\,(1+O(q)),$$

so

$$\lim_{q\to 0^+} \frac{c(q^2)}{c^2(q)} = \frac{1}{3}. \tag{1.19}$$

2. Proof of Theorem 1. For $n \in \mathbb{N}$ set

$$z_n = z_n(q) = \varphi^2(q^n). \tag{2.1}$$

Also set

$$m = m(q) = \frac{z_1}{z_3} = \frac{\varphi^2(q)}{\varphi^2(q^3)}. \tag{2.2}$$

From (1.8) and (2.2) we deduce

$$m = 2p + 1. \tag{2.3}$$

From (1.9), (2.1) and (2.2) we obtain

$$\sqrt{z_1 z_3} = k(2p + 1). \tag{2.4}$$

Berndt, Bhargava and Garvan [3, Lemma 2.1, p. 4168] (see also Berndt [2, Lemma 2.1, p. 94]) have shown that

$$a(q) = \sqrt{z_1 z_3}\,\frac{(m^2 + 6m - 3)}{4m}, \tag{2.5}$$

$$b(q) = \sqrt{z_1 z_3}\,\frac{(3 - m)(9 - m^2)^{1/3}}{4m^{2/3}}, \tag{2.6}$$

$$c(q) = \sqrt{z_1 z_3}\,\frac{3(m + 1)(m^2 - 1)^{1/3}}{4m}. \tag{2.7}$$

Theorem 1 now follows on using (2.3) and (2.4) in (2.5)–(2.7).

3. Proof of Theorem 2. Berndt, Bhargava and Garvan [3, eqn. (5.16), p. 4184] (see also Berndt [2, eqn. (5.16), p. 112]) have shown (with q^2 replaced by q) that

$$a(q) = \frac{1}{3}\left(\frac{c^2(q)}{c(q^2)} + 4\,\frac{c^2(q^2)}{c(q)}\right). \tag{3.1}$$

Using the expressions for $a(q)$ and $c(q)$ given in Theorem 1 in (3.1), we obtain

$$(1 + 4p + p^2)k = \frac{2^{-2/3}3(p(1 + p)^4)^{2/3}k^2}{c(q^2)} + \frac{2^{7/3}3^{-2}c^2(q^2)}{(p(1 + p)^4)^{1/3}k}. \tag{3.2}$$

If we set

$$x = x(q) = \frac{c(q^2)}{2^{-2/3}3(p(1 + p))^{2/3}k}, \tag{3.3}$$

then equation (3.2) becomes after rearrangement

$$2px^3 - (1 + 4p + p^2)x + (1 + 2p + p^2) = 0. \tag{3.4}$$

Solving the cubic equation (3.4) for x, we find

$$x = 1,\ \frac{1}{2}\left(\sqrt{2p + 5 + \frac{2}{p}} - 1\right) \text{ or } -\frac{1}{2}\left(\sqrt{2p + 5 + \frac{2}{p}} + 1\right). \tag{3.5}$$

From (1.17) and (1.18) we see that as $q \to 0^+$ we have $p \to 0^+$ and $k \to 1^-$. Thus from (3.5) we deduce

$$\lim_{q\to 0^+} x = 1,\ +\infty \text{ or } -\infty \tag{3.6}$$

respectively. From Theorem 1 and (3.3) we obtain

$$x = 3(1+p)^2 k\,\frac{c(q^2)}{c^2(q)}. \tag{3.7}$$

Hence by (1.19) we have

$$\lim_{q\to 0^+} x = 1. \tag{3.8}$$

From (3.5), (3.6) and (3.8) we deduce that $x = 1$. Hence by (3.3) we have

$$c(q^2) = 2^{-2/3} 3(p(1+p))^{2/3} k \tag{3.9}$$

as asserted.

Now Borwein, Borwein and Garvan [5, eqn. (2.28), p. 44] have shown that

$$a(q) = \frac{1}{2}\frac{c^2(q)}{c(q^2)} - \frac{1}{2}\frac{b^2(q)}{b(q^2)}. \tag{3.10}$$

Appealing to Theorem 1 for the values of $a(q)$, $b(q)$ and $c(q)$, and (3.9) for the value of $c(q^2)$, from (3.10) we obtain

$$b(q^2) = 2^{-2/3}((1-p)(1+2p)(2+p))^{2/3} k. \tag{3.11}$$

Then, from the identity

$$a(q)a(q^2) = b(q)b(q^2) + c(q)c(q^2) \tag{3.12}$$

(see [5, Theorem 2.6, p. 40]), we obtain

$$a(q^2) = (1+p+p^2)k. \tag{3.13}$$

4. Proof of Theorem 3. The following two identities are proved in [5, Lemma 2.1(ii), p. 36 and eqn. (2.1), p. 37]:

$$a(q^3) = \tfrac{1}{3}a(q) + \tfrac{2}{3}b(q), \tag{4.1}$$

$$c(q^3) = \tfrac{1}{3}a(q) - \tfrac{1}{3}b(q). \tag{4.2}$$

The values of $a(q^3)$ and $c(q^3)$ then follow by Theorem 1.

5. Proof of Theorem 4. Berndt, Bhargava and Garvan [3, eqn. (5.16), p. 4184] (see also Berndt [2, eqn. (5.16), p. 112]) have shown that

$$a(q^2) = \frac{1}{3}\left(\frac{c^2(q^2)}{c(q^4)} + 4\,\frac{c^2(q^4)}{c(q^2)}\right). \tag{5.1}$$

Using the values of $a(q^2)$ and $c(q^2)$ from Theorem 2 in (5.1), we obtain

$$3(1+p+p^2)k = \frac{2^{-4/3}3^2(p(1+p))^{4/3}k^2}{c(q^4)} + 2^{8/3}3^{-1}(p(1+p))^{-2/3}k^{-1}c^2(q^4). \tag{5.2}$$

Set

$$y = y(q) = \frac{c(q^4)}{2^{-4/3}3(p^4(1+p))^{1/3}k}. \tag{5.3}$$

Then (5.2) becomes after some rearrangement

$$p^2y^3 - (1+p+p^2)y + (1+p) = 0. \tag{5.4}$$

Solving the cubic equation (5.4) for y, we find

$$y = 1,\ \frac{1}{p} \text{ or } \frac{-(p+1)}{p}. \tag{5.5}$$

From (5.3) and (5.5) we deduce

$$c(q^4) = 2^{-4/3}3(p^4(1+p))^{1/3}k,\ 2^{-4/3}3(p(1+p))^{1/3}k \text{ or } -2^{-4/3}3(p(1+p)^4)^{1/3}k. \tag{5.6}$$

Berndt, Bhargava and Garvan [3, eqn. (6.3), p. 4188] (see also Berndt [2, eqn. (6.3), p. 116]) have shown that

$$a(q) - a(q^2) = 2\,\frac{c^2(q^2)}{c(q)}. \tag{5.7}$$

Thus

$$a(q^4) = a(q^2) - 2\,\frac{c^2(q^4)}{c(q^2)}. \tag{5.8}$$

From Theorem 2, (5.6) and (5.8), we obtain

$$a(q^4) = \left(1+p-\tfrac{1}{2}p^2\right)k,\ \left(-\tfrac{1}{2}+p+p^2\right)k \text{ or } \left(-\tfrac{1}{2}-2p-\tfrac{1}{2}p^2\right)k.$$

As $a(0)=1$, $p(0)=0$ and $k(0)=1$, we deduce that

$$a(q^4) = \left(1+p-\tfrac{1}{2}p^2\right)k, \tag{5.9}$$

$$c(q^4) = 2^{-4/3}3(p^4(1+p))^{1/3}k. \tag{5.10}$$

Finally, by (1.5) (with q replaced by q^3), (5.9) and (5.10), we obtain

$$b(q^4)^3 = a(q^4)^3 - c(q^4)^3 = 2^{-4}(1-p)(1+2p)(2+p)^4k^3,$$

so that

$$b(q^4) = \varepsilon 2^{-4/3}((1-p)(1+2p)(2+p)^4)^{1/3}k$$

for some cube root of unity ε. As $b(0)=1$, $p(0)=0$, $k(0)=1$, we must have $\varepsilon = 1$. Thus,

$$b(q^4) = 2^{-4/3}((1-p)(1+2p)(2+p)^4)^{1/3}k. \tag{5.11}$$

6. Proof of Theorem 5. Replacing q by q^2 in (4.1) and (4.2), we have

$$a(q^6) = \tfrac{1}{3}a(q^2) + \tfrac{2}{3}b(q^2), \tag{6.1}$$

$$c(q^6) = \tfrac{1}{3}a(q^2) - \tfrac{1}{3}b(q^2). \tag{6.2}$$

Appealing to Theorem 2 for the values of $a(q^2)$ and $b(q^2)$, we obtain the values of $a(q^6)$ and $c(q^6)$.

7. Proof of Theorem 7. Replacing q by q^2 in (6.1) and (6.2), we have

$$a(q^{12}) = \tfrac{1}{3}a(q^4) + \tfrac{2}{3}b(q^4), \tag{7.1}$$

$$c(q^{12}) = \tfrac{1}{3}a(q^4) - \tfrac{1}{3}b(q^4). \tag{7.2}$$

Appealing to Theorem 4 for the values of $a(q^4)$ and $b(q^4)$, we obtain the values of $a(q^{12})$ and $c(q^{12})$.

8. Proof of Theorem 8. This theorem follows from Theorems 1 and 4 and the relations

$$a(q) + a(-q) = 2a(q^4), \tag{8.1}$$

$$b(q) + b(-q) = 2b(q^4), \tag{8.2}$$

$$c(q) + c(-q) = 2c(q^4), \tag{8.3}$$

proved in [5, p. 40].

9. Proof of Theorem 9. For convenience we set

$$p_1 = p(q^2), \qquad k_1 = k(q^2). \tag{9.1}$$

By Theorems 1 and 2 we have

$$(1 + 4p_1 + p_1^2)k_1 = (1 + p + p^2)k \tag{9.2}$$

and by Theorems 2 and 4 we have

$$(1 + p_1 + p_1^2)k_1 = \left(1 + p - \tfrac{1}{2}p^2\right)k. \tag{9.3}$$

From (9.2) and (9.3) we deduce

$$\frac{1 + 4p_1 + p_1^2}{1 + p_1 + p_1^2} = \frac{1 + p + p^2}{1 + p - \frac{1}{2}p^2}. \tag{9.4}$$

Writing (9.4) as a quadratic equation in p_1, we obtain

$$p^2p_1^2 - 2(1 + p - p^2)p_1 + p^2 = 0. \tag{9.5}$$

Solving (9.5) for p_1 gives

$$p_1 = \frac{1 + p - p^2 + \lambda((1-p)(1+p)(1+2p))^{1/2}}{p^2}, \tag{9.6}$$

where $\lambda = \pm 1$. Taking $q = 0$ in

$$p^2p_1 = 1 + p - p^2 + \lambda((1-p)(1+p)(1+2p))^{1/2},$$

we obtain, as $p(0)=p_1(0)=0$, $0=1+\lambda$ so that $\lambda=-1$. Hence

$$p_1 = p(q^2) = \frac{1+p-p^2-((1-p)(1+p)(1+2p))^{1/2}}{p^2} \tag{9.7}$$

as claimed. Then, from (9.2) and (9.7), we obtain

$$k_1 = k(q^2) = \frac{(1+p-p^2+((1-p)(1+p)(1+2p))^{1/2})k}{2}. \tag{9.8}$$

10. Proof of Theorem 11. For convenience we set

$$p_2 = p(-q), \qquad k_2 = k(-q). \tag{10.1}$$

By Theorems 2 and 4 we have

$$(1+p_2+p_2^2)k_2 = (1+p+p^2)k, \tag{10.2}$$

$$\left(1+p_2-\tfrac{1}{2}p_2^2\right)k_2 = \left(1+p-\tfrac{1}{2}p^2\right)k. \tag{10.3}$$

From (10.2) and (10.3) we deduce

$$\frac{1+p_2+p_2^2}{1+p_2-\frac{1}{2}p_2^2} = \frac{1+p+p^2}{1+p-\frac{1}{2}p^2}. \tag{10.4}$$

Rewriting (10.4) as a quadratic equation in p_2, we obtain

$$(1+p)p_2^2 - p^2p_2 - p^2 = 0. \tag{10.5}$$

Hence $p_2 = p$ or $-p/(1+p)$. From (1.17) we have

$$p = p(q) = 2q + 2q^2 + O(q^3),$$
$$p_2 = p(-q) = -2q + 2q^2 + O(q^3),$$

for small $|q|$, so that $p_2 \neq p$. Hence

$$p_2 = p(-q) = \frac{-p}{1+p}. \tag{10.6}$$

Then, from (10.2) and (10.6), we have

$$k(-q) = k_2 = \frac{(1+p+p^2)k}{1-\dfrac{p}{1+p}+\dfrac{p^2}{(1+p)^2}} = (1+p)^2k.$$

11. Proof of Theorem 10. For convenience we set

$$p_3 = p(q^3), \qquad k_3 = k(q^3). \tag{11.1}$$

From Theorems 1–5 and 7 we deduce that

$$(11.2)\qquad (1+4p_3+p_3^2)k_3 = \tfrac{1}{3}(1+4p+p^2+2^{2/3}((1-p)^4(1+2p)(2+p))^{1/3})k,$$

$$(11.3)\qquad (1+p_3+p_3^2)k_3 = \tfrac{1}{3}(1+p+p^2+2^{1/3}((1-p)(1+2p)(2+p))^{2/3})k,$$

$$(11.4)\qquad \left(1+p_3-\tfrac{1}{2}p_3^2\right)k_3 = \tfrac{1}{3}\left(1+p-\tfrac{1}{2}p^2+2^{-1/3}((1-p)(1+2p)(2+p)^4)^{1/3}\right)k.$$

Set

$$X = ((1-p)(1+2p)(2+p))^{1/3}.$$

Then

$$\begin{aligned} k_3 &= -\tfrac{1}{3}(1+4p_3+p_3^2)k_3 + \tfrac{2}{3}(1+p_3+p_3^2)k_3 + \tfrac{2}{3}\left(1+p_3-\tfrac{1}{2}p_3^2\right)k_3 \\ &= -\tfrac{1}{9}(1+4p+p^2+2^{2/3}(1-p)X)k + \tfrac{2}{9}(1+p+p^2+2^{1/3}X^2)k \\ &\quad +\tfrac{2}{9}\left(1+p-\tfrac{1}{2}p^2+2^{-1/3}(2+p)X\right)k \end{aligned}$$

so that

$$(11.5)\qquad k(q^3) = k_3 = \tfrac{1}{9}(3+2^{2/3}(1+2p)X+2^{4/3}X^2)k$$

as asserted.

Next, from (11.2) and (11.3), we deduce

$$\begin{aligned} p_3k_3 &= \tfrac{1}{3}(1+4p_3+p_3^2)k_3 - \tfrac{1}{3}(1+p_3+p_3^2)k_3 \\ &= \tfrac{1}{9}(1+4p+p^2+2^{2/3}(1-p)X)k - \tfrac{1}{9}(1+p+p^2+2^{1/3}X^2)k \end{aligned}$$

so that

$$(11.6)\qquad p_3k_3 = \tfrac{1}{9}(3p+2^{2/3}(1-p)X-2^{1/3}X^2)k.$$

Hence, from (11.5) and (11.6), we obtain

$$\begin{aligned} p(q^3) = p_3 &= \frac{3p+2^{2/3}(1-p)X-2^{1/3}X^2}{3+2^{2/3}(1+2p)X+2^{4/3}X^2} \\ &= \tfrac{1}{3}(-4-3p+6p^2+4p^3+2^{2/3}(1-2p-2p^2)X+2^{1/3}(1+2p)X^2) \end{aligned}$$

as claimed.

12. Proof of Theorem 6. Set

$$(12.1)\qquad t = ((1-p)(1+p)(1+2p))^{1/2}$$

so

$$(12.2)\qquad t^2 = 1+2p-p^2-2p^3.$$

By Theorem 11 we have

$$(12.3)\qquad p(q^2)=\frac{1+p-p^2-t}{p^2},\qquad k(q^2)=\frac{(1+p-p^2+t)k}{2}.$$

Thus, by (12.2) and (12.3), we have

$$(12.4)\qquad p^2(q^2)=\frac{(2+4p-2p^2-4p^3+p^4)-2(1+p-p^2)t}{p^4}.$$

Then, by Theorem 4, we obtain

$$\begin{aligned} a(q^8) &= \big(1+p(q^2)-\tfrac{1}{2}p^2(q^2)\big)k(q^2) \\ &= \tfrac{1}{4}(1+p+p^2+3((1-p)(1+p)(1+2p))^{1/2})k \end{aligned}$$

as asserted.

13. The Eisenstein series $L(q)$. The Eisenstein series $L(q)$ is defined by

$$(13.1)\qquad L(q)=1-24\sum_{n=1}^{\infty}\sigma(n)q^n.$$

It is shown in [1, eqns. (3.84) and (3.87)] that

$$(13.2)\qquad L(q)-2L(q^2)=-(1+14p+24p^2+14p^3+p^4)k^2,$$

$$(13.3)\qquad L(q)-3L(q^3)=-2(1+8p+18p^2+8p^3+p^4)k^2,$$

with p and k as defined in (1.8) and (1.9). Applying the triplication principle (Theorem 10) to (13.2) and the duplication principle (Theorem 9) to (13.3), we obtain

$$(13.4)\qquad L(q^3)-2L(q^6)=-(1+2p+2p^3+p^4)k^2,$$

$$(13.5)\qquad L(q^2)-3L(q^6)=-2(1+2p+3p^2+2p^3+p^4)k^2,$$

in agreement with [1, eqns. (3.85) and (3.88)]. Applying the triplication principle to (13.4) and the duplication principle to (13.5), we have

$$\begin{aligned} (13.6)\qquad & L(q^9)-2L(q^{18}) \\ &= \tfrac{1}{27}(-11-10p+24p^2-10p^3-11p^4 \\ &\quad -2^{2/3}4(1-p^3)((1-p)(1+2p)(2+p))^{1/3} \\ &\quad -2^{1/3}4(1+4p+p^2)((1-p)(1+2p)(2+p))^{2/3})k^2, \end{aligned}$$

$$(13.7)\qquad L(q^4)-3L(q^{12})=\big(-2-4p+2p^3-\tfrac{1}{2}p^4\big)k^2.$$

These results will be needed in the following sections.

14. The sum $L_{1,2}(q)$. We define

$$L_{1,2}(q) := \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n. \tag{14.1}$$

In this section we evaluate $L_{1,2}(q)$ in terms of p and k. We begin by recalling (13.2). We have

$$L(q) - 2L(q^2) = -(1 + 14p + 24p^2 + 14p^3 + p^4)k^2. \tag{14.2}$$

Applying the change of sign principle (Theorem 11) to (14.2) we have

$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2. \tag{14.3}$$

Subtracting (14.2) from (14.3), we obtain

$$L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2. \tag{14.4}$$

Hence

$$\begin{aligned} L_{1,2}(q) = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n &= \frac{1}{2}\sum_{n=1}^{\infty}\sigma(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty}\sigma(n)(-q)^n \\ &= \frac{1}{48}(L(-q) - L(q)), \end{aligned}$$

so that by (14.4),

$$L_{1,2}(q) = \left(\tfrac{1}{2}p + \tfrac{3}{4}p^2 + \tfrac{3}{8}p^3 + \tfrac{1}{16}p^4\right)k^2. \tag{14.5}$$

Applying the triplication principle (Theorem 10) to (14.5), we obtain

$$L_{1,2}(q^3) = \left(\tfrac{1}{8}p^3 + \tfrac{1}{16}p^4\right)k^2. \tag{14.6}$$

15. The sums $L_{1,3}(q)$ and $L_{2,3}(q)$. We define

$$L_{1,3}(q) := \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n)q^n, \qquad L_{2,3}(q) := \sum_{\substack{n=1 \\ n\equiv 2 \pmod 3}}^{\infty} \sigma(n)q^n. \tag{15.1}$$

These sums have been studied in [12]. In this section we evaluate them in terms of p and k. From [12, Theorem 1.4] we have

$$L_{1,3}(q) = (1 + (1-x)^{1/3} - 2(1-x)^{2/3})\,\frac{w^2}{27}, \tag{15.2}$$

$$L_{2,3}(q) = (1 - 2(1-x)^{1/3} + (1-x)^{2/3})\,\frac{w^2}{27}, \tag{15.3}$$

where, in the notation of [1], we have

$$x = x_3(q) = B = \frac{27p(1+p)^4}{2(1+4p+p^2)^3}, \tag{15.4}$$

$$w = w_3(x_3(q)) = X = (1+4p+p^2)k. \tag{15.5}$$

Now

$$1 - x = 1 - \frac{27p(1+p)^4}{2(1+4p+p^2)^3} = \frac{(1-p)^4(1+2p)(2+p)}{2(1+4p+p^2)^3}, \tag{15.6}$$

so

$$(1-x)^{1/3} = \frac{2^{-1/3}(1-p)((1-p)(1+2p)(2+p))^{1/3}}{1+4p+p^2}. \tag{15.7}$$

Hence

$$\begin{aligned} L_{1,3}(q) = \tfrac{1}{27}(1 &+ 8p + 18p^2 + 8p^3 + p^4 \\ &+ 2^{-1/3}(1+3p-3p^2-p^3)((1-p)(1+2p)(2+p))^{1/3} \\ &- 2^{1/3}(1-2p+p^2)((1-p)(1+2p)(2+p))^{2/3})k^2, \end{aligned} \tag{15.8}$$

$$\begin{aligned} L_{2,3}(q) = \tfrac{1}{27}(1 &+ 8p + 18p^2 + 8p^3 + p^4 \\ &- 2^{2/3}(1+3p-3p^2-p^3)((1-p)(1+2p)(2+p))^{1/3} \\ &+ 2^{-2/3}(1-2p+p^2)((1-p)(1+2p)(2+p))^{2/3})k^2. \end{aligned} \tag{15.9}$$

From (15.8) and (15.9) we deduce

$$\begin{aligned} L_{1,3}(q) + 2L_{2,3}(q) &= \tfrac{1}{9}(1+8p+18p^2+8p^3+p^4)k^2 \\ &\quad - \tfrac{1}{9}\cdot 2^{-1/3}(1+3p-3p^2-p^3)((1-p)(1+2p)(2+p))^{1/3}k^2 \end{aligned}$$

so that

$$\begin{aligned} \tfrac{1}{3}\cdot 2^{2/3}(1+3p-3p^2-p^3)&((1-p)(1+2p)(2+p))^{1/3}k^2 \\ &= \tfrac{2}{3}(1+4p+p^2)^2k^2 - 6L_{1,3}(q) - 12L_{2,3}(q). \end{aligned} \tag{15.10}$$

Applying the duplication principle (Theorem 9) to (15.9), we obtain

$$\begin{aligned} L_{2,3}(q^2) = \tfrac{1}{27}(1 &+ 2p + 3p^2 + 2p^3 + p^4 \\ &+ 2^{-4/3}(2+3p-3p^2-2p^3)((1-p)(1+2p)(2+p))^{1/3} \\ &- 2^{1/3}(1+p+p^2)((1-p)(1+2p)(2+p))^{2/3})k^2. \end{aligned} \tag{15.11}$$

From (15.8) and (15.11) we deduce

$$\begin{aligned} L_{1,3}(q) - 2L_{2,3}(q^2) = \tfrac{1}{27}(&-1 + 4p + 12p^2 + 4p^3 - p^4 \\ &- 2^{-1/3}(1-p^3)((1-p)(1+2p)(2+p))^{1/3} \\ &+ 2^{1/3}(1+4p+p^2)((1-p)(1+2p)(2+p))^{2/3})k^2. \end{aligned} \tag{15.12}$$

From (13.6) and (15.12) we obtain

$$(15.13)\quad 2^{2/3}(1-p^3)((1-p)(1+2p)(2+p))^{1/3}k^2 = \left(-\tfrac{5}{2}+p+12p^2+p^3-\tfrac{5}{2}p^4\right)k^2 - \tfrac{9}{2}L(q^9)+9L(q^{18})-18L_{1,3}(q)+36L_{2,3}(q^2),$$

$$(15.14)\quad 2^{1/3}(1+4p+p^2)((1-p)(1+2p)(2+p))^{2/3}k^2 = \left(-\tfrac{1}{4}-\tfrac{7}{2}p-6p^2-\tfrac{7}{2}p^3-\tfrac{1}{4}p^4\right)k^2 - \tfrac{9}{4}L(q^9)+\tfrac{9}{2}L(q^{18})+18L_{1,3}(q)-36L_{2,3}(q^2).$$

16. Proof of Theorem 12. By (1.12), Theorem 1 and (13.3), we have

$$\sum_{n=0}^{\infty} N(1,1;n)q^n = a(q)^2 = (1+8p+18p^2+8p^3+p^4)k^2 = -\tfrac{1}{2}(L(q)-3L(q^3)) = 1+\sum_{n=1}^{\infty}(12\sigma(n)-36\sigma(n/3))q^n.$$

Equating coefficients of q^n ($n\in\mathbb{N}$) we obtain $N(1,1;n)$.

17. Proof of Theorem 13. By (1.12), Theorem 1, Theorem 2, (13.2) and (13.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,2;n)q^n &= (1+4p+p^2)(1+p+p^2)k^2\\ &= \tfrac{1}{4}(1+14p+24p^2+14p^3+p^4)k^2+\tfrac{3}{4}(1+2p+2p^3+p^4)k^2\\ &= -\tfrac{1}{4}(L(q)-2L(q^2))-\tfrac{3}{4}(L(q^3)-2L(q^6))\\ &= 1+\sum_{n=1}^{\infty}(6\sigma(n)-12\sigma(n/2)+18\sigma(n/3)-36\sigma(n/6))q^n.\end{aligned}$$

Equating coefficients of q^n ($n\in\mathbb{N}$) we obtain $N(1,2;n)$.

18. Proof of Theorem 14. By (1.12), Theorems 1, 3, and the proof of Theorem 12, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,3;n)q^n &= \tfrac{1}{3}(1+4p+p^2)^2k^2\\ &\quad+\tfrac{1}{3}\cdot 2^{2/3}(1+3p-3p^2-p^3)((1-p)(1+2p)(2+p))^{1/3}k^2\\ &= (1+4p+p^2)^2k^2-6L_{1,3}(q)-12L_{2,3}(q)\\ &= a(q)^2-6L_{1,3}(q)-12L_{2,3}(q)\\ &= -\tfrac{1}{2}(L(q)-3L(q^3))-6L_{1,3}(q)-12L_{2,3}(q),\end{aligned}$$

that is,

$$\sum_{n=0}^{\infty} N(1,3;n)q^n = 1 + \sum_{n=1}^{\infty}(12\sigma(n) - 36\sigma(n/3))q^n \\ - 6 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n)q^n - 12 \sum_{\substack{n=1 \\ n\equiv 2 \pmod 3}}^{\infty} \sigma(n)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$ we obtain $N(1,3;n)$.

19. Proof of Theorem 15. By (1.12), Theorems 1 and 4, (13.7), (14.5) and (14.6), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,4;n)q^n &= -\tfrac{1}{2}\big(-2 - 4p + 2p^3 - \tfrac{1}{2}p^4\big)k^2 \\ &\quad + 6\big(\tfrac{1}{2}p + \tfrac{3}{4}p^2 + \tfrac{3}{8}p^3 + \tfrac{1}{16}p^4\big)k^2 - 18\big(\tfrac{1}{8}p^3 + \tfrac{1}{16}p^4\big)k^2 \\ &= -\tfrac{1}{2}(L(q^4) - 3L(q^{12})) + 6L_{1,2}(q) - 18L_{1,2}(q^3) \\ &= 1 + 12\sum_{n=1}^{\infty}\sigma(n/4)q^n - 36\sum_{n=1}^{\infty}\sigma(n/12)q^n \\ &\quad + 6 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)q^n - 18 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n/3)q^n.\end{aligned}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ we obtain $N(1,4;n)$.

20. Proof of Theorem 16. Appealing to (1.12), Theorems 1 and 5, (15.14) and (13.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,6;n)q^n &= \tfrac{1}{3}((1 + 4p + p^2)(1 + p + p^2) \\ &\quad + 2^{1/3}(1 + 4p + p^2)((1-p)(1+2p)(2+p))^{2/3})k^2 \\ &= \tfrac{1}{3}((1 + 5p + 6p^2 + 5p^3 + p^4)k^2 \\ &\quad + 2^{1/3}(1 + 4p + p^2)((1-p)(1+2p)(2+p))^{2/3}k^2) \\ &= \tfrac{1}{3}\big((1 + 5p + 6p^2 + 5p^3 + p^4)k^2 \\ &\quad + \big(-\tfrac{1}{4} - \tfrac{7}{2}p - 6p^2 - \tfrac{7}{2}p^3 - \tfrac{1}{4}p^4\big)k^2 \\ &\quad - \tfrac{9}{4}L(q^9) + \tfrac{9}{2}L(q^{18}) + 18L_{1,3}(q) - 36L_{2,3}(q^2)\big)\end{aligned}$$

$$
\begin{aligned}
&= \left(\tfrac{1}{4} + \tfrac{1}{2}p + \tfrac{1}{2}p^3 + \tfrac{1}{4}p^4\right)k^2 - \tfrac{3}{4}L(q^9) + \tfrac{3}{2}L(q^{18}) \\
&\quad + 6L_{1,3}(q) - 12L_{2,3}(q^2) \\
&= -\tfrac{1}{4}(L(q^3) - 2L(q^6)) - \tfrac{3}{4}L(q^9) + \tfrac{3}{2}L(q^{18}) \\
&\quad + 6L_{1,3}(q) - 12L_{2,3}(q^2) \\
&= 1 + \sum_{n=1}^{\infty}(6\sigma(n/3) - 12\sigma(n/6) + 18\sigma(n/9) - 36\sigma(n/18))q^n \\
&\quad + 6 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n)q^n - 12 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n/2)q^n.
\end{aligned}
$$

Equating coefficients of q^n $(n \in \mathbb{N})$ we obtain

$$
N(1,6;n) = \begin{cases} 6\sigma(n/3) - 12\sigma(n/6) + 18\sigma(n/9) - 36\sigma(n/18) & \text{if } n \equiv 0 \pmod 3, \\ 6\sigma(n) - 12\sigma(n/2) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}
$$

When $n \equiv 0 \pmod 3$ we have the elementary identities

$$\sigma(n) = 4\sigma(n/3) - 3\sigma(n/9), \tag{20.1}$$

$$\sigma(n/2) = 4\sigma(n/6) - 3\sigma(n/18), \tag{20.2}$$

so

$$
\begin{aligned}
&6\sigma(n/3) - 12\sigma(n/6) + 18\sigma(n/9) - 36\sigma(n/18) \\
&\qquad = -6\sigma(n) + 12\sigma(n/2) + 30\sigma(n/3) - 60\sigma(n/6),
\end{aligned}
$$

completing the proof.

21. Proof of Theorem 17. Appealing to (1.12), Theorems 2, 3 and (15.13), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} N(2,3;n)q^n &= \tfrac{1}{3}[(1+p+p^2)(1+4p+p^2)k^2 \\
&\quad + 2^{2/3}(1-p^3)((1-p)(1+2p)(2+p))^{1/3}k^2] \\
&= \tfrac{1}{3}\big[(1+5p+6p^2+5p^3+p^4)k^2 \\
&\quad + \left(-\tfrac{5}{2} + p + 12p^2 + p^3 - \tfrac{5}{2}p^4\right)k^2 \\
&\quad - \tfrac{9}{2}L(q^9) + 9L(q^{18}) - 18L_{1,3}(q) + 36L_{2,3}(q^2)\big] \\
&= \left(-\tfrac{1}{2} + 2p + 6p^2 + 2p^3 - \tfrac{1}{2}p^4\right)k^2 - \tfrac{3}{2}L(q^9) + 3L(q^{18}) \\
&\quad - 6L_{1,3}(q) + 12L_{2,3}(q^2).
\end{aligned}
$$

Next

$$\begin{aligned}&\left(-\tfrac{1}{2}+2p+6p^2+2p^3-\tfrac{1}{2}p^4\right)k^2\\&\qquad=\tfrac{1}{4}(1+14p+24p^2+14p^3+p^4)k^2-\tfrac{3}{4}(1+2p+2p^3+p^4)k^2\\&\qquad=-\tfrac{1}{4}(L(q)-2L(q^2))+\tfrac{3}{4}(L(q^3)-2L(q^6))\\&\qquad=-\tfrac{1}{4}L(q)+\tfrac{1}{2}L(q^2)+\tfrac{3}{4}L(q^3)-\tfrac{3}{2}L(q^6).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty}N(2,3;n)q^n &= -\tfrac{1}{4}L(q)+\tfrac{1}{2}L(q^2)+\tfrac{3}{4}L(q^3)-\tfrac{3}{2}L(q^6)-\tfrac{3}{2}L(q^9)\\&\quad+3L(q^{18})-6L_{1,3}(q)+12L_{2,3}(q^2)\\&=1+\sum_{n=1}^{\infty}(6\sigma(n)-12\sigma(n/2)-18\sigma(n/3)\\&\quad+36\sigma(n/6)+36\sigma(n/9)-72\sigma(n/18))q^n\\&\quad-6\sum_{\substack{n=1\\n\equiv 1\,(\mathrm{mod}\,3)}}^{\infty}\sigma(n)q^n+12\sum_{\substack{n=1\\n\equiv 1\,(\mathrm{mod}\,3)}}^{\infty}\sigma(n/2)q^n.\end{aligned}$$

Equating coefficients of q^n $(n\in\mathbb{N})$ we obtain

$$N(2,3;n)=\begin{cases}6\sigma(n)-12\sigma(n/2)-18\sigma(n/3)\\ \qquad+36\sigma(n/6)+36\sigma(n/9)-72\sigma(n/18) & \text{if } n\equiv 0\,(\mathrm{mod}\,3),\\ 0 & \text{if } n\equiv 1\,(\mathrm{mod}\,3),\\ 6\sigma(n)-12\sigma(n/2) & \text{if } n\equiv 2\,(\mathrm{mod}\,3).\end{cases}$$

When $n\equiv 0\,(\mathrm{mod}\,3)$ then from (20.1) and (20.2) we have

$$\sigma(n/9)=\tfrac{4}{3}\sigma(n/3)-\tfrac{1}{3}\sigma(n),\qquad \sigma(n/18)=\tfrac{4}{3}\sigma(n/6)-\tfrac{1}{3}\sigma(n/2),$$

so that

$$\begin{aligned}&6\sigma(n)-12\sigma(n/2)-18\sigma(n/3)+36\sigma(n/6)+36\sigma(n/9)-72\sigma(n/18)\\&\qquad=6\sigma(n)-12\sigma(n/2)-18\sigma(n/3)+36\sigma(n/6)+48\sigma(n/3)\\&\qquad\quad-12\sigma(n)-96\sigma(n/6)+24\sigma(n/2)\\&\qquad=-6\sigma(n)+12\sigma(n/2)+30\sigma(n/3)-60\sigma(n/6),\end{aligned}$$

completing the proof.

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