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Identities concerning Bernoulli and Euler polynomials

by

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1. Introduction. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, \ldots\}$. The well known Bernoulli numbers B_n $(n \in \mathbb{N})$ are rational numbers defined by

$$B_0 = 1, \qquad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n \in \mathbb{Z}^+).$$

Similarly, the Euler numbers E_n $(n \in \mathbb{N})$ are integers given by

$$E_0 = 1, \qquad \sum_{\substack{k=0 \\ 2 \mid n-k}}^{n} \binom{n}{k} E_k = 0 \quad (n \in \mathbb{Z}^+).$$

For $n \in \mathbb{N}$ the Bernoulli polynomial $B_n(x)$ and the Euler polynomial $E_n(x)$ are defined as follows:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \qquad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

Clearly $B_n(0) = B_n$ and $E_n(1/2) = E_n/2^n$. Here are some basic properties of the Bernoulli and Euler polynomials we will need later:

$$B_n(1-x) = (-1)^n B_n(x), \qquad \Delta(B_n(x)) = nx^{n-1},$$
$$E_n(1-x) = (-1)^n E_n(x), \qquad \Delta^*(E_n(x)) = 2x^n.$$

Here, the operators Δ and Δ^* are defined by $\Delta(f(x)) = f(x+1) - f(x)$ and $\Delta^*(f(x)) = f(x+1) + f(x)$. It is also known that $B'_{n+1}(x) = (n+1)B_n(x)$ and $E'_{n+1}(x) = (n+1)E_n(x)$.

For a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers, its dual sequence $\{a_n^*\}_{n\in\mathbb{N}}$ is given by $a_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$ $(n \in \mathbb{N})$. It is well known that $a_n^{**} = a_n$. In 2003 Z. W. Sun [S2] deduced some combinatorial identities in dual sequences. The sequences $\{(-1)^n B_n\}_{n\in\mathbb{N}}$ and $\{(-1)^n E_n(0)\}_{n\in\mathbb{N}}$ are both self-dual sequences (cf. [S2]); later we will make use of this fact.

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In 1978 H. Miki [Mi] discovered the following curious identity:

$$\sum_{k=2}^{n-2}\frac{B_kB_{n-k}}{k(n-k)}-\sum_{k=2}^{n-2}\binom{n}{k}\frac{B_kB_{n-k}}{k(n-k)}=2H_n\,\frac{B_n}{n}$$

for every $n=4,5,\dots$, where $H_n=1+1/2+\dots+1/n$. In 1997 Y. Matiyasevich [Ma] found another identity of this type:

$$(n+2)\sum_{k=2}^{n-2}B_kB_{n-k}-2\sum_{k=2}^{n-2}\binom{n+2}{k}B_kB_{n-k}=n(n+1)B_n$$

for any $n=4,5,\dots$. These two identities are of a deep nature. In fact, all known proofs of these identities given by other authors are complicated (cf. [Mi], [G] and [DS]); for example, the approach of G. V. Dunne and C. Schubert [DS] was even motivated by quantum field theory and string theory.

Recently the authors [PS] presented a new method to handle such identities. Though their approach only involves differences and derivatives of polynomials, they were able to use the powerful method to extend Miki's and Matiyasevich's identities to identities concerning $\sum_{k=0}^{n}B_k(x)B_{n-k}(y)$ and

$$\sum_{k=1}^{n-1}\frac{B_k(x)}{k}\cdot\frac{B_{n-k}(y)}{n-k}=\frac{1}{n}\sum_{k=1}^{n-1}\frac{B_k(x)}{k}\,B_{n-k}(y)+\frac{1}{n}\sum_{l=1}^{n-1}\frac{B_l(y)}{l}\,B_{n-l}(x)$$

(where n is a positive integer). They also handled similar sums related to Euler polynomials.

Let n be any positive integer. As usual, $\binom{z}{n}=z(z-1)\cdots(z-n+1)/n!$ (and $\binom{z}{0}=1$) even if $z\notin\mathbb{N}$. Observe that

$$\begin{aligned}\sum_{k=0}^{n}B_k(x)B_{n-k}(y)&=\sum_{k=0}^{n}(-1)^k\binom{-1}{k}B_k(x)B_{n-k}(y),\\ -\sum_{k=1}^{n}\frac{B_k(x)}{k}\,B_{n-k}(y)&=\sum_{k=1}^{n}(-1)^k\binom{-1}{k-1}\frac{B_k(x)}{k}\,B_{n-k}(y)\\ &=\lim_{t\to0}\frac{1}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}B_k(x)B_{n-k}(y).\end{aligned}$$

Inspired by this observation, we investigate here relations among the sums

$$\sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}P_k(x)Q_{n-k}(y)$$

with $P,Q\in\{B,E\}$.

Our central result is the following theorem.

THEOREM 1.1. *Let $n \in \mathbb{Z}^+$ and $x + y + z = 1$.*

(i) *If $r + s + t = n - 1$, then*

$$\begin{aligned}(1.1)\qquad &\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z)\\ &\quad - (-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\ &= \frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).\end{aligned}$$

(ii) *If $r + s + t = n$, then we have the symmetric relation*

$$(1.2)\qquad r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$(1.3)\qquad \begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y).$$

REMARK 1.1. It is interesting to compare (1.2) with the following property of determinants:

$$0 = \begin{vmatrix} r & s & t \\ r & s & t \\ z & x & y \end{vmatrix} = r\begin{vmatrix} s & t \\ x & y \end{vmatrix} + s\begin{vmatrix} t & r \\ y & z \end{vmatrix} + t\begin{vmatrix} r & s \\ z & x \end{vmatrix}.$$

In view of K. Dilcher's paper [D], the referee suggested that Theorem 1.1 might have a generalization involving sums of products of m Bernoulli or Euler polynomials. But we are unable to obtain a compact extension of Theorem 1.1 though we have made a serious attempt.

COROLLARY 1.1. *Let $n \in \mathbb{Z}^+$ and let α, x, y be parameters. Then*

$$\begin{aligned}(1.4)\qquad &\frac{\alpha+n+1}{2}\sum_{k=0}^{n-1}\binom{\alpha+k}{k}E_k(x)E_{n-1-k}(y)\\ &= \sum_{k=0}^{n}\binom{\alpha+n+1}{k}\left((-1)^{n-k}B_k(x) - \binom{\alpha+n-k}{n-k}B_k(y)\right)E_{n-k}(x-y),\end{aligned}$$

$$\begin{aligned}(1.5)\qquad &(\alpha+n+2)\sum_{k=0}^{n}\binom{\alpha+k}{k}B_k(x)B_{n-k}(y)\\ &= (\alpha+1)\sum_{k=0}^{n}\binom{\alpha+n+2}{k}(-1)^{n-k}B_k(x)B_{n-k}(x-y)\\ &\quad + \sum_{k=0}^{n}\binom{\alpha+n+2}{k}\binom{\alpha+n-k}{n-k}B_k(y)B_{n-k}(x-y).\end{aligned}$$

Proof. Let $x' = 1-x$ and $z' = x-y$. Then $x'+y+z' = 1$. Applying Theorem 1.1(i) with $r = \alpha+n+1$, $s = -1$ and $t = -\alpha-1$ we then get (1.4). (Note that $(-1)^k\binom{-z}{k} = \binom{z+k-1}{k}$.) By Theorem 1.1(ii),

$$\begin{aligned}(\alpha+n+2)&\begin{bmatrix} -1 & -\alpha-1 \\ 1-x & y \end{bmatrix}_n \\ &= \begin{bmatrix} -\alpha-1 & \alpha+n+2 \\ y & x-y \end{bmatrix}_n + (\alpha+1)\begin{bmatrix} \alpha+n+2 & -1 \\ x-y & 1-x \end{bmatrix}_n.\end{aligned}$$

This is an equivalent version of (1.5). ∎

REMARK 1.2. Formula (1.5) in the case $\alpha = x = y = 0$ yields Matiyasevich's identity since $B_{2l+1} = 0$ for $l = 1, 2, \ldots$.

COROLLARY 1.2. *Let $n > l \geq 0$ be integers. Then*

$$\begin{aligned}(1.6)\quad &\frac{n-l+1}{2}\sum_{k=\delta_{l,0}}^{n}\binom{n}{k}\binom{n}{k+l-1}E_{k+l-1}(x)E_{n-k}(y) \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{k+l}((-1)^{n-k}B_{k+l}(x) - B_{k+l}(y))E_{n-k}(x-y)\end{aligned}$$

(*where $\delta_{l,m}$ equals* 1 *or* 0 *according as $l = m$ or not*), *and*

$$\begin{aligned}(1.7)\quad &\frac{n-l}{n}\sum_{k=0}^{n-l}\binom{n}{k}\binom{n}{k+l}B_{k+l}(x)B_{n-k}(y) \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{k+n-1}{k+l}((-1)^{n-k}B_{k+l}(x) + B_{k+l}(y))B_{n-k}(x-y).\end{aligned}$$

In particular,

$$\begin{aligned}(1.8)\quad &\frac{(n+1)(n+1-l)}{8}\sum_{k=\delta_{l,0}}^{n}\binom{n}{k}\binom{n}{k+l-1}E_{k+l-1}(x)E_{n-k}(x) \\ &= \sum_{k=0}^{n-1}\binom{n+1}{k}\binom{k+n}{k+l}B_{k+l}(x)(2^{n-k+1}-1)B_{n-k+1},\end{aligned}$$

$$\begin{aligned}(1.9)\quad &\sum_{k=0}^{n-l}\binom{n}{k}\binom{n}{k+l}B_{k+l}(x)B_{n-k}(x) \\ &= \frac{2n}{n-l}\sum_{\substack{k=0 \\ k\neq n-1}}^{n}\binom{n}{k}\binom{k+n-1}{k+l}B_{k+l}(x)B_{n-k}.\end{aligned}$$

Proof. As $(l-n-1)+n+n = (n+l)-1$ and $(1-x)+y+(x-y) = 1$, by Theorem 1.1(i) we have

$$\sum_{k=0}^{n+l}(-1)^k\binom{l-n-1}{k}\binom{n}{n+l-k}B_k(1-x)E_{n+l-k}(x-y)$$
$$-(-1)^{n+l}\sum_{k=0}^{n+l}(-1)^k\binom{l-n-1}{k}\binom{n}{n+l-k}B_k(y)E_{n+l-k}(x-y)$$
$$=\frac{l-n-1}{2}\sum_{k=0}^{n-\delta_{l,0}}(-1)^k\binom{n}{k}\binom{n}{n+l-1-k}E_k(y)E_{n+l-1-k}(1-x)$$
$$=\frac{l-n-1}{2}\sum_{k=\delta_{l,0}}^{n}(-1)^{n-k}\binom{n}{k}\binom{n}{k+l-1}E_{n-k}(y)E_{k+l-1}(1-x),$$

which can be reduced to (1.6). (1.8) follows from (1.6) in the case $y=x$ since $((-1)^m-1)E_m(0)=4(2^{m+1}-1)B_{m+1}/(m+1)$ for $m=1,2,\ldots$. (It is known that $(m+1)E_m(x)=2(B_{m+1}(x)-2^{m+1}B_{m+1}(x/2))$; cf. [AS] and [S1].)

In light of Theorem 1.1(ii),

$$(l-n)\begin{bmatrix} n & n\\ 1-x & y\end{bmatrix}_{n+l}+n\begin{bmatrix} n & l-n\\ y & x-y\end{bmatrix}_{n+l}+n\begin{bmatrix} l-n & n\\ x-y & 1-x\end{bmatrix}_{n+l}=0.$$

This is equivalent to (1.7). In the case $y=x$, (1.7) gives (1.9) because $((-1)^m+1)B_m=2B_m$ for $m=0,2,3,\ldots$. ■

Remark 1.3. Putting $l=0$ and $x=1/2$ in (1.8) and noting that $B_k(1/2)=(2^{1-k}-1)B_k$ (see, e.g., [AS] and [S1]), we get the following identity:

$$\frac{(n+1)^2}{8}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}E_kE_{n-1-k}$$
$$=-\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+k}{n}2^{n-k}(2^{k-1}-1)(2^{n-k+1}-1)B_kB_{n-k+1}$$

for any $n\in\mathbb{Z}^+$. Similarly, (1.9) in the case $l=x=0$ yields the following new identity:

$$\sum_{k=2}^{n-2}\binom{n}{k}^2B_kB_{n-k}-2\sum_{k=2}^{n-2}\binom{n}{k}\binom{n+k-1}{k}B_kB_{n-k}=2\binom{2n-1}{n-1}B_n$$

for every $n=4,5,\ldots$.

The following theorem can be deduced from Theorem 1.1.

THEOREM 1.2. *Let $l,m,n\in\mathbb{Z}^+$, $l\le\min\{m,n\}$ and $x+y+z=1$. Then*

$$\begin{aligned}(1.10)\qquad &(-1)^m\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{l-1}B_{n-l+k+1}(x)E_{m-k}(z)\\ &\quad+(-1)^{n-l}\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{l-1}B_{m-l+k+1}(y)E_{n-k}(z)\\ &=-\frac{l}{2}\sum_{k=0}^{l}(-1)^k\binom{m}{k}\binom{n}{l-k}E_{n-l+k}(x)E_{m-k}(y),\end{aligned}$$

$$\begin{aligned}(1.11)\qquad &\sum_{k=0}^{l}(-1)^k\binom{m}{k}\binom{n}{l-k}B_{m-k}(x)E_{n-l+k}(z)\\ &\quad-(-1)^m\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{l}B_{m-k}(y)E_{n-l+k}(z)\\ &=(-1)^{n-l-1}\frac{m}{2}\sum_{k=\delta_{l,m}}^{n}\binom{n}{k}\binom{m+k-1}{l}E_{n-k}(y)E_{m-l-1+k}(x).\end{aligned}$$

We also have

$$\begin{aligned}(1.12)\qquad &\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{n+k-1}{l-1}B_{n-l+k}(x)B_{m-k}(z)\\ &\quad+(-1)^l\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{m+k-1}{l-1}B_{m-l+k}(y)B_{n-k}(z)\\ &=\frac{l}{mn}\sum_{k=0}^{l}(-1)^k\binom{m}{k}\binom{n}{l-k}B_{n-l+k}(x)B_{m-k}(y).\end{aligned}$$

COROLLARY 1.3 (Woodcock [W]). *Let $m,n\in\mathbb{Z}^+$. Then*

$$\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}(-1)^kB_{m-k}B_{n-1+k}=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^kB_{n-k}B_{m-1+k}.$$

Proof. Simply set $x=y=0$ and $l=z=1$ in (1.12). ∎

From Theorem 1.1 we can also deduce the following result.

THEOREM 1.3. *Let $n\in\mathbb{Z}^+$, and let t,x,y,z be parameters with $x+y+z=1$. Then*

$$\begin{aligned}(1.13)\qquad &\frac{(-1)^n}{2}\sum_{k=0}^{n-1}\binom{t}{k}E_k(x)E_{n-1-k}(y)\\ &=\frac{1}{n-t}\sum_{k=0}^{n}\binom{n-t}{k}B_k(x)E_{n-k}(z)+\binom{t}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{E_k(z)}{t-k}B_{n-k}(y),\end{aligned}$$

$$\frac{n}{2}\binom{t}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}\,E_{n-1-k}(y)-(-1)^nE_n(z)\binom{t}{n}\sum_{k=0}^{n-1}\frac{1}{t-k} \tag{1.14}$$
$$=(-1)^n\sum_{k=1}^{n}\binom{t}{n-k}\frac{B_k(y)}{k}\,E_{n-k}(z)-\sum_{k=1}^{n}\binom{n-1-t}{n-k}\frac{B_k(x)}{k}\,E_{n-k}(z).$$

Also,

$$\frac{(-1)^{n-1}}{n}\binom{t-1}{n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_k(x)}{t-k}\,B_{n-k}(y)-\frac{B_n(z)}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k} \tag{1.15}$$
$$=\frac{1}{t}\sum_{k=1}^{n}\binom{t}{n-k}\frac{B_k(y)}{k}\,B_{n-k}(z)+\frac{(-1)^n}{n-t}\sum_{k=1}^{n}\binom{n-t}{n-k}\frac{B_k(x)}{k}\,B_{n-k}(z).$$

COROLLARY 1.4. *Let $n\in\mathbb{Z}^+$ and $x+y+z=1$. Then*

$$\sum_{k=0}^{n}\binom{n+1}{k}((-1)^nB_k(x)-B_k(y))E_{n-k}(z) \tag{1.16}$$
$$=\frac{n+1}{2}\sum_{l=0}^{n-1}(-1)^lE_l(x)E_{n-1-l}(y),$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{B_k(x)}{k}\,E_{n-k}(z)-\sum_{k=1}^{n}(-1)^k\,\frac{B_k(y)}{k}\,E_{n-k}(z) \tag{1.17}$$
$$=\frac{(-1)^n}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(y)E_{n-1-l}(x)-H_nE_n(z),$$

$$(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}B_{n-k}(x)B_k(y)+\sum_{k=0}^{n-1}\binom{n+1}{k}\frac{B_{n-k}(x)}{n-k}\,B_k(z) \tag{1.18}$$
$$=(n+1)\sum_{k=1}^{n}(-1)^k\,\frac{B_k(y)}{k}\,B_{n-k}(z)+(1-H_n)(n+1)B_n(z).$$

Proof. Setting $t=-1$ in Theorem 1.3 we immediately get (1.16)–(1.18). ■

COROLLARY 1.5. *Let $n\in\mathbb{Z}^+$ and $x+y+z=1$. Then*

$$\frac{1}{2}\sum_{k=1}^{n-1}(-1)^{k-1}\,\frac{E_k(x)}{k}\,E_{n-1-k}(y)+\frac{H_{n-1}E_{n-1}(y)}{2} \tag{1.19}$$
$$=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}\,B_{n-k}(y)+\frac{(-1)^n}{n}\sum_{k=1}^{n}\binom{n}{k}H_kE_k(z)B_{n-k}(x),$$

$$\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}\,E_{n-1-k}(y)+H_{n-1}\,\frac{E_n(z)+(-1)^nB_n(y)}{n} \tag{1.20}$$
$$=\sum_{k=1}^{n-1}(-1)^k\,\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}+\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}\,\frac{B_k(x)}{k}\,E_{n-k}(z).$$

We also have

$$\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k^2}\,(B_{n-k}(y)+(-1)^nB_{n-k}(z)) = \sum_{k=1}^{n-1}(-1)^{n-k}\,\frac{B_k(y)}{k}\cdot\frac{B_{n-k}(z)}{n-k}-H_{n-1}\,\frac{B_n(y)+(-1)^nB_n(z)}{n}. \tag{1.21}$$

REMARK 1.4. In the case $x=y=0$ and $z=1$, (1.21) yields Miki's identity.

The next section is devoted to proofs of Theorems 1.1 and 1.2. Theorem 1.3 and Corollary 1.5 will be proved in Section 3.

2. Proofs of Theorems 1.1 and 1.2

LEMMA 2.1. *Let $P(x),Q(x)\in\mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers.*

(i) *We have*

$$\Delta(P(x)Q(x))=P(x+1)\Delta(Q(x))+\Delta(P(x))Q(x), \tag{2.1}$$

$$\Delta^*(P(x)Q(x))=P(x+1)\Delta^*(Q(x))-\Delta(P(x))Q(x). \tag{2.2}$$

(ii) *If $\Delta(P(x))=\Delta(Q(x))$, then $P'(x)=Q'(x)$. If in turn $\Delta^*(P(x))=\Delta^*(Q(x))$, then $P(x)=Q(x)$.*

Proof. The first part can be verified easily. Part (ii) is Lemma 3.1 of [PS]. ∎

The following lemma has the same flavor as Theorem 1.1 of Sun [S2].

LEMMA 2.2. *Let $\{a_l\}_{l=0}^\infty$ be a sequence of complex numbers, and $\{a_l^*\}_{l=0}^\infty$ be its dual sequence. Set*

$$A_k(t)=\sum_{l=0}^{k}\binom{k}{l}(-1)^la_lt^{k-l},\qquad A_k^*(t)=\sum_{l=0}^{k}\binom{k}{l}(-1)^la_l^*t^{k-l} \tag{2.3}$$

for $k=0,1,2,\dots$. Let $n\in\mathbb{Z}^+$, $r+s+t=n-1$ and $x+y+z=1$. Then

$$\sum_{k=0}^{n}(-1)^k\binom{r}{k}x^{n-k}\left(\binom{s}{n-k}A_k(y)-(-1)^n\binom{t}{n-k}A_k^*(z)\right)=0. \tag{2.4}$$

Proof. By Remark 1.1 of Sun [S2],

$$(-1)^kA_k^*(z)=A_k(x+y)=\sum_{l=0}^{k}\binom{k}{l}x^{k-l}A_l(y).$$

Therefore

$$\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}x^{n-k}A_k^*(z)$$
$$\begin{aligned}&=\sum_{k=0}^{n}\binom{r}{k}\binom{t}{n-k}x^{n-k}\sum_{l=0}^{k}\binom{k}{l}x^{k-l}A_l(y)\\&=\sum_{l=0}^{n}x^{n-l}A_l(y)\sum_{k=l}^{n}\binom{r}{l}\binom{r-l}{k-l}\binom{t}{n-k}\\&=\sum_{l=0}^{n}\binom{r}{l}x^{n-l}A_l(y)c_l\end{aligned}$$

where

$$\begin{aligned}c_l&=\sum_{k=l}^{n}\binom{r-l}{k-l}\binom{t}{n-k}=\binom{r+t-l}{n-l}\quad\text{(by Vandermonde's identity)}\\&=(-1)^{n-l}\binom{l-r-t+n-l-1}{n-l}=(-1)^{n-l}\binom{s}{n-l}.\end{aligned}$$

Thus (2.4) follows. ∎

REMARK 2.1. If we let $a_l=(-1)^lB_l$ for $l=0,1,2,\ldots$, then $A_k(t)=A_k^*(t)=B_k(t)$. Also, $A_k(t)=A_k^*(t)=E_k(t)$ if $a_l=(-1)^lE_l(0)$ for $l=0,1,2,\ldots$.

Proof of Theorem 1.1. We fix y and view $z=1-x-y$ as a function in x.

(i) Set

$$P(x)=\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z).$$

Then, by Lemma 2.1, $\Delta^*(P(x))$ coincides with

$$\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}\Delta^*(B_k(x)E_{n-k}(z))$$
$$\begin{aligned}&=\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}(B_k(x+1)2(z-1)^{n-k}-kx^{k-1}E_{n-k}(z))\\&=2\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}(z-1)^{n-k}B_k(x+1)+r\Sigma\end{aligned}$$

where

$$\begin{aligned}\Sigma&=\sum_{k=1}^{n}(-1)^{k-1}\binom{r-1}{k-1}\binom{s}{n-k}x^{k-1}E_{n-k}(z)\\&=(-1)^{n-1}\sum_{l=0}^{n-1}(-1)^l\binom{r-1}{n-1-l}\binom{s}{l}x^{n-1-l}E_l(z).\end{aligned}$$

Applying Lemma 2.2 and Remark 2.1 we obtain

$$\begin{aligned}\Delta^*(P(x)) = {} & 2(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}(z-1)^{n-k}B_k(y)\\ & + r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}x^{n-1-l}E_l(y).\end{aligned}$$

It follows that $\Delta^*(P(x)) = \Delta^*(Q(x))$ where

$$\begin{aligned}Q(x) = {} & (-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\ & + \frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).\end{aligned}$$

Thus $P(x) = Q(x)$ by Lemma 2.1. This is equivalent to the desired equality (1.1).

(ii) Set

$$P_n(x) = \begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = \sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)B_{n-k}(z).$$

By Lemma 2.1,

$$\begin{aligned}\Delta(B_k(x)B_{n-k}(z)) &= \Delta(B_k(x))B_{n-k}(z) + B_k(x+1)\Delta(B_{n-k}(z))\\ &= kx^{k-1}B_{n-k}(z) - (n-k)B_k(x+1)(z-1)^{n-k-1}\end{aligned}$$

for every $k = 0, 1, \ldots, n$. Thus

$$\Delta(P_n(x)) = rR(x) - s\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s-1}{n-k-1}B_k(x+1)(z-1)^{n-k-1}$$

where

$$\begin{aligned}R(x) &= \sum_{k=1}^{n}(-1)^k\binom{r-1}{k-1}\binom{s}{n-k}x^{k-1}B_{n-k}(z)\\ &= (-1)^n\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{r-1}{n-1-l}x^{n-1-l}B_l(z).\end{aligned}$$

Applying Lemma 2.2 and Remark 2.1 we obtain

$$\begin{aligned}\Delta(P_n(x)) = {} & -r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t-1}{n-1-l}x^{n-1-l}B_l(y)\\ & - s(-1)^{n-1}\sum_{l=0}^{n-1}(-1)^l\binom{r}{l}\binom{t-1}{n-1-l}(z-1)^{n-1-l}B_l(y).\end{aligned}$$

It follows that $\Delta(P_n(x)) = \Delta(Q_n(x))$ where

$$\begin{aligned} Q_n(x) = & -\frac{r}{t}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-l}B_{n-l}(x)B_l(y) \\ & -(-1)^n\,\frac{s}{t}\,\sum_{l=0}^{n-1}(-1)^l\binom{r}{l}\binom{t}{n-l}B_{n-l}(z)B_l(y) \\ = & -\frac{r}{t}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-l}B_{n-l}(x)B_l(y) \\ & -\frac{s}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}\binom{r}{n-k}B_k(z)B_{n-k}(y). \end{aligned}$$

Thus $P'_n(x) = Q'_n(x)$ by Lemma 2.1.

Observe that $P'_n(x)$ coincides with

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}kB_{k-1}(x)B_{n-k}(z) \\ &\quad -\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s}{n-k}(n-k)B_k(x)B_{n-k-1}(z) \\ &=\sum_{k=0}^{n-1}(-1)^{k+1}\binom{r}{k+1}\binom{s}{n-1-k}(k+1)B_k(x)B_{n-1-k}(z) \\ &\quad -\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s}{n-k}(n-k)B_k(x)B_{n-1-k}(z) \\ &=\sum_{k=0}^{n-1}(-1)^{k-1}\binom{r}{k}\binom{s}{n-1-k}(r-k+(s-n+k+1))B_k(x)B_{n-1-k}(z) \\ &=(t-1)\begin{bmatrix} r & s \\ z & x \end{bmatrix}_{n-1} \end{aligned}$$

and

$$\begin{aligned} Q'_n(x) = & -r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t-1}{n-l-1}B_{n-l-1}(x)B_l(y) \\ & +s\sum_{k=1}^{n}(-1)^k\binom{t-1}{k-1}\binom{r}{n-k}B_{k-1}(z)B_{n-k}(y) \\ = & -r\begin{bmatrix} s & t-1 \\ x & y \end{bmatrix}_{n-1} - s\begin{bmatrix} t-1 & r \\ y & z \end{bmatrix}_{n-1}. \end{aligned}$$

Thus the equality $P'_n(x) = Q'_n(x)$ gives

$$r\begin{bmatrix} s & t' \\ x & y \end{bmatrix}_{n-1} + s\begin{bmatrix} t' & r \\ y & z \end{bmatrix}_{n-1} + t'\begin{bmatrix} r & s \\ z & x \end{bmatrix}_{n-1} = 0$$

where $t' = t - 1 = n - 1 - (r + s)$. Replacing $n - 1$ by n we then obtain the required identity (1.2). This concludes the proof. ∎

Proof of Theorem 1.2. Clearly $\overline{n} = m + n - l \in \mathbb{Z}^+$. By Theorem 1.1(i),

$$\begin{aligned}\sum_{k=0}^{\overline{n}+1}(-1)^k\binom{-l}{k}\binom{m}{\overline{n}+1-k}B_k(x)E_{\overline{n}+1-k}(z)& \\ -(-1)^{\overline{n}+1}\sum_{k=0}^{\overline{n}+1}(-1)^k\binom{-l}{k}&\binom{n}{\overline{n}+1-k}B_k(y)E_{\overline{n}+1-k}(z) \\ =\frac{-l}{2}\sum_{k=0}^{\overline{n}}(-1)^k\binom{m}{k}&\binom{n}{\overline{n}-k}E_k(y)E_{\overline{n}-k}(x).\end{aligned}$$

That is,

$$\begin{aligned}\sum_{k=0}^{m}(-1)^{\overline{n}+1-k}\binom{-l}{\overline{n}+1-k}\binom{m}{k}B_{\overline{n}+1-k}(x)E_k(z)& \\ -\sum_{k=0}^{n}(-1)^k\binom{-l}{\overline{n}+1-k}&\binom{n}{k}B_{\overline{n}+1-k}(y)E_k(z) \\ =\frac{-l}{2}\sum_{k=0}^{m}(-1)^{m-k}&\binom{m}{k}\binom{n}{n-l+k}E_{m-k}(y)E_{n-l+k}(x) \\ =(-1)^{m-1}\frac{l}{2}\sum_{k=0}^{l}&(-1)^k\binom{m}{k}\binom{n}{l-k}E_{m-k}(y)E_{n-l+k}(x).\end{aligned}$$

Therefore (1.10) follows. By Theorem 1.1(i) we also have

$$\begin{aligned}\sum_{k=0}^{\overline{n}}(-1)^k\binom{m}{k}\binom{n}{\overline{n}-k}B_k(x)E_{\overline{n}-k}(z)& \\ -(-1)^{\overline{n}}\sum_{k=0}^{\overline{n}}(-1)^k\binom{m}{k}&\binom{-l-1}{\overline{n}-k}B_k(y)E_{\overline{n}-k}(z) \\ =\frac{m}{2}\sum_{k=0}^{\overline{n}-1}(-1)^k\binom{n}{k}&\binom{-l-1}{\overline{n}-1-k}E_k(y)E_{\overline{n}-1-k}(x) \\ =\frac{m}{2}\sum_{k=0}^{n-\delta_{l,m}}(-1)^k\binom{n}{k}&\binom{-l-1}{m+n-l-1-k}E_k(y)E_{m+n-l-1-k}(x) \\ =\frac{m}{2}\sum_{k=\delta_{l,m}}^{n}(-1)^{n-k}\binom{n}{k}&\binom{-l-1}{m-l-1+k}E_{n-k}(y)E_{m-l-1+k}(x),\end{aligned}$$

which gives (1.11) after a few trivial steps.

In light of Theorem 1.1(ii),

$$l\begin{bmatrix} m & n \\ x & y \end{bmatrix}_{\overline{n}} = m\begin{bmatrix} n & -l \\ y & z \end{bmatrix}_{\overline{n}} + n\begin{bmatrix} -l & m \\ z & x \end{bmatrix}_{\overline{n}}.$$

That is,

$$\begin{aligned} l\sum_{k=0}^{m}&(-1)^{m-k}\binom{m}{k}\binom{n}{n-l+k}B_{n-l+k}(x)B_{m-k}(y) \\ &= m\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{-l}{m-l+k}B_{m-l+k}(y)B_{n-k}(z) \\ &\quad + n\sum_{k=0}^{m}(-1)^{\overline{n}-k}\binom{-l}{\overline{n}-k}\binom{m}{k}B_k(z)B_{\overline{n}-k}(x). \end{aligned}$$

This is equivalent to (1.12). We are done. ■

3. Proofs of Theorem 1.3 and Corollary 1.5

LEMMA 3.1. *Let n be a nonnegative integer and s be a parameter. Then*

$$\lim_{t\to 0}\frac{1}{t}\left(\binom{s+t}{n} - \binom{s}{n}\right) = \binom{s}{n}\sum_{0\le l<n}\frac{1}{s-l}. \tag{3.1}$$

In particular,

$$\lim_{t\to 0}\frac{1}{t}\left(\binom{t-1}{n} - (-1)^n\right) = (-1)^{n-1}H_n. \tag{3.2}$$

Proof. Observe that

$$\binom{s+t}{n} = \binom{s}{n}\prod_{0\le l<n}\frac{s+t-l}{s-l} = \binom{s}{n}\prod_{0\le l<n}\left(1+\frac{t}{s-l}\right).$$

So (3.1) follows. In the case $s=-1$, (3.1) turns out to be (3.2). ■

Proof of Theorem 1.3. Formula (1.1) in the case $s=-1$ yields

$$\begin{aligned} (-1)^n\sum_{k=0}^{n}&\binom{n-t}{k}B_k(x)E_{n-k}(z) \\ &- (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n-t}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z) \\ &= \frac{n-t}{2}\sum_{l=0}^{n-1}\binom{t}{n-1-l}E_{n-1-l}(x)E_l(y). \end{aligned}$$

For each $k=0,1,\ldots,n$ we clearly have

$$\binom{n-t}{k}\binom{t}{n-k}=\binom{n}{k}\binom{t}{n}\frac{(n-t)(n-t-1)\cdots(n-t-k+1)}{(t-n+k)\cdots(t-n+1)}$$
$$=(-1)^k\binom{n}{k}\binom{t}{n}\frac{t-n}{t-n+k}.$$

Therefore

$$\frac{(-1)^n}{2}\sum_{k=0}^{n-1}\binom{t}{k}E_k(x)E_{n-1-k}(y)-\frac{1}{n-t}\sum_{k=0}^{n}\binom{n-t}{k}B_k(x)E_{n-k}(z)$$
$$=\binom{t}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{B_k(y)}{t+k-n}E_{n-k}(z)=\binom{t}{n}\sum_{l=0}^{n}\binom{n}{l}\frac{E_l(z)}{t-l}B_{n-l}(y).$$

This proves (1.13).

Now we come to prove (1.14) and view $s=n-1-r-t$ as a function in r. In light of (1.1),

$$\frac{1}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x)$$
$$=\frac{1}{r}\sum_{k=0}^{n}(-1)^k\binom{r}{k}E_{n-k}(z)\left(\binom{s}{n-k}B_k(x)-(-1)^n\binom{t}{n-k}B_k(y)\right)$$
$$=\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{r-1}{k-1}E_{n-k}(z)\left(\binom{s}{n-k}B_k(x)-(-1)^n\binom{t}{n-k}B_k(y)\right)$$
$$+(-1)^nE_n(z)\,\frac{(-1)^n\binom{s}{n}-\binom{t}{n}}{r}.$$

By Lemma 3.1,

$$\lim_{r\to 0}\frac{1}{r}\left((-1)^n\binom{s}{n}-\binom{t}{n}\right)=\lim_{r\to 0}\frac{1}{r}\left(\binom{r+t}{n}-\binom{t}{n}\right)=\binom{t}{n}\sum_{l=0}^{n-1}\frac{1}{t-l}.$$

As in the proof of (1.13), we also have

$$(-1)^l\binom{n-1-t}{l}\binom{t}{n-1-l}=\binom{n-1}{l}\binom{t}{n-1}\frac{t-(n-1)}{t-(n-1)+l}$$
$$=\frac{n}{t+l-(n-1)}\binom{t}{n}\binom{n-1}{l}$$

for every $l=0,1,\ldots,n-1$. Thus, by letting $r\to 0$ we deduce from the above that

$$\frac{n}{2}\binom{t}{n}\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{E_l(y)E_{n-1-l}(x)}{t+l-n+1}-(-1)^nE_n(z)\binom{t}{n}\sum_{l=0}^{n-1}\frac{1}{t-l}$$
$$=-\sum_{k=1}^{n}E_{n-k}(z)\left(\binom{n-1-t}{n-k}\frac{B_k(x)}{k}-(-1)^n\binom{t}{n-k}\frac{B_k(y)}{k}\right),$$

which is equivalent to (1.14).

Now we turn to proving (1.15). Let us view $s=n-r-t$ as a function in r. Then

$$\lim_{r\to 0}\begin{bmatrix} s & t\\ x & y\end{bmatrix}_n=\begin{bmatrix} n-t & t\\ x & y\end{bmatrix}_n=\sum_{k=0}^{n}\binom{n}{k}\binom{t}{n}\frac{t-n}{t-n+k}\,B_{n-k}(x)B_k(y)$$
$$=(t-n)\binom{t}{n}\sum_{l=0}^{n}\binom{n}{l}\frac{B_l(x)}{t-l}\,B_{n-l}(y).$$

On the other hand,

$$\lim_{r\to 0}\frac{1}{r}\left(s\begin{bmatrix} t & r\\ y & z\end{bmatrix}_n+t\begin{bmatrix} r & s\\ z & x\end{bmatrix}_n\right)$$
$$=(n-t)(-1)^{n-1}\sum_{k=0}^{n-1}\binom{t}{k}\frac{B_{n-k}(y)}{n-k}\,B_k(z)$$
$$-t\sum_{k=1}^{n}\binom{n-t}{n-k}\frac{B_k(x)}{k}\,B_{n-k}(z)+(-1)^nB_n(z)R$$

where

$$R=\lim_{r\to 0}\frac{1}{r}\left((n-t-r)\binom{t}{n}+(-1)^nt\binom{n-t-r}{n}\right)$$
$$=\lim_{r\to 0}\frac{1}{r}\left(t\binom{r+t-1}{n}-(t-n)\binom{t}{n}\right)-\binom{t}{n}$$
$$=\lim_{r\to 0}\frac{t}{r}\left(\binom{r+t-1}{n}-\binom{t-1}{n}\right)-\binom{t}{n}$$
$$=t\binom{t-1}{n}\sum_{l=0}^{n-1}\frac{1}{t-1-l}-\binom{t}{n}=t\binom{t-1}{n}\sum_{k=1}^{n-1}\frac{1}{t-k}.$$

Applying (1.2) we then get (1.15) from the above.

The proof of Theorem 1.3 is now complete. ∎

Proof of Corollary 1.5. We can easily get (1.21) by calculating the limit of the left-hand side of (1.15) minus the right-hand side of (1.15) as t tends to 0. Thus it remains to show (1.19) and (1.20).

Equation (1.13) can be rewritten in the form

$$\begin{aligned}\frac{(-1)^n}{2}\,t\sum_{k=1}^{n-1}\binom{t-1}{k-1}\frac{E_k(x)}{k}\,E_{n-1-k}(y)+\frac{(-1)^n}{2}\,E_{n-1}(y)\\ =\sum_{k=0}^{n}\left(\frac{\binom{n-t}{k}}{n-t}-\frac{\binom{n}{k}}{n}\right)B_k(x)E_{n-k}(z)+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)\\ +\frac{t}{n}\binom{t-1}{n-1}\left(\frac{B_n(y)}{t}+\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{t-k}\,B_{n-k}(y)\right).\end{aligned}$$

Letting $t\to 0$ we get

$$\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)+(-1)^{n-1}\,\frac{B_n(y)}{n}=\frac{(-1)^n}{2}\,E_{n-1}(y). \tag{3.3}$$

Thus

$$\begin{aligned}&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{t-1}{k-1}\frac{E_k(x)}{k}\,E_{n-1-k}(y)\\ &=\sum_{k=0}^{n}\frac{1}{t}\left(\frac{\binom{n-t}{k}}{n-t}-\frac{\binom{n}{k}}{n}\right)B_k(x)E_{n-k}(z)+\frac{B_n(y)}{nt}\left(\binom{t-1}{n-1}-(-1)^{n-1}\right)\\ &\quad+\frac{1}{n}\binom{t-1}{n-1}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{t-k}\,B_{n-k}(y).\end{aligned}$$

Letting $t\to 0$ we then have

$$\begin{aligned}\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\,E_k(x)E_{n-1-k}(y)+\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}\,B_{n-k}(y)\\ =\sum_{k=0}^{n}\lim_{t\to 0}\frac{n\binom{n-t}{k}-(n-t)\binom{n}{k}}{tn(n-t)}\,B_k(x)E_{n-k}(z)+\frac{B_n(y)}{n}\,(-1)^nH_{n-1}.\end{aligned}$$

Observe that

$$\begin{aligned}&\lim_{t\to 0}\frac{n\binom{n-t}{k}-(n-t)\binom{n}{k}}{t(n-t)}=\lim_{t\to 0}\left(\frac{\binom{n}{k}}{n-t}-\frac{n}{n-t}\cdot\frac{\binom{n-t}{k}-\binom{n}{k}}{-t}\right)\\ &=\frac{1}{n}\binom{n}{k}-\binom{n}{k}\sum_{l=0}^{k-1}\frac{1}{n-l}=-\binom{n}{k}\sum_{0<l<k}\frac{1}{n-l}=\binom{n}{k}(H_{n-k}-H_{n-1}).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\,E_k(x)E_{n-1-k}(y)+\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}\,B_{n-k}(y)\\ =(-1)^nH_{n-1}\,\frac{B_n(y)}{n}+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(H_{n-k}-H_{n-1})B_k(x)E_{n-k}(z)\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n H_{n-1}\,\frac{B_n(y)}{n} - \frac{H_{n-1}}{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)\\
&\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}H_l E_l(z)B_{n-l}(x)\\
&= -H_{n-1}\,\frac{(-1)^n}{2}\,E_{n-1}(y) + \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}H_k E_k(z)B_{n-k}(x).
\end{aligned}
$$

This proves (1.19).

We can reformulate (1.14) as follows:

$$
\begin{aligned}
&\frac{t}{2}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}\,E_{n-1-k}(y) + \frac{1}{2}\binom{t-1}{n-1}E_{n-1}(y)\\
&\qquad - (-1)^n E_n(z)\,\frac{t}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k} - (-1)^n\,\frac{E_n(z)}{n}\binom{t-1}{n-1}\\
&\quad = (-1)^n t\sum_{k=1}^{n-1}\binom{t-1}{n-k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k} + (-1)^n\,\frac{B_n(y)}{n}\\
&\qquad - \sum_{k=1}^{n}\left(\binom{n-1-t}{n-k} - \binom{n-1}{n-k}\right)\frac{B_k(x)}{k}\,E_{n-k}(z)\\
&\qquad - \sum_{k=1}^{n}\binom{n-1}{n-k}\frac{B_k(x)}{k}\,E_{n-k}(z).
\end{aligned}
$$

In view of (3.3),

$$
\begin{aligned}
\sum_{k=1}^{n}\binom{n-1}{n-k}\frac{B_k(x)}{k}\,E_{n-k}(z) &= \sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k}\,E_{n-k}(z)\\
&= (-1)^n\left(\frac{B_n(y)}{n} + \frac{E_{n-1}(y)}{2}\right) - \frac{E_n(z)}{n}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}\,E_{n-1-k}(y) - (-1)^n\,\frac{E_n(z)}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k}\\
&\qquad = (-1)^n\sum_{k=1}^{n-1}\binom{t-1}{n-k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\
&\qquad\quad + \sum_{k=1}^{n-1}\frac{\binom{n-1-t}{n-k} - \binom{n-1}{n-k}}{-t}\cdot\frac{B_k(x)}{k}\,E_{n-k}(z)\\
&\qquad\quad - \left(\frac{E_{n-1}(y)}{2} + (-1)^{n-1}\,\frac{E_n(z)}{n}\right)\frac{\binom{t-1}{n-1} - (-1)^{n-1}}{t}.
\end{aligned}
$$

Letting $t \to 0$ we obtain

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}\,E_{n-1-k}(y)-\frac{E_n(z)}{n}\,H_{n-1}\\
&\qquad=\sum_{k=1}^{n-1}(-1)^{k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\
&\qquad\quad+\sum_{k=1}^{n-1}\binom{n-1}{n-k}\bigg(\sum_{l=0}^{n-k-1}\frac{1}{n-1-l}\bigg)\frac{B_k(x)}{k}\,E_{n-k}(z)\\
&\qquad\quad+H_{n-1}\bigg(\frac{(-1)^{n-1}}{2}\,E_{n-1}(y)+\frac{E_n(z)}{n}\bigg).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}\,E_{n-1-k}(y)+\sum_{k=1}^{n-1}(-1)^k\,\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\
&\qquad=\sum_{k=1}^{n}\binom{n-1}{n-k}(H_{n-1}-H_{k-1})\,\frac{B_k(x)}{k}\,E_{n-k}(z)\\
&\qquad\quad+H_{n-1}\bigg(\frac{(-1)^{n-1}}{2}\,E_{n-1}(y)+2\,\frac{E_n(z)}{n}\bigg)\\
&\qquad=-\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}\,\frac{B_k(x)}{k}\,E_{n-k}(z)+H_{n-1}R
\end{aligned}$$

where

$$\begin{aligned}
R&=\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k}\,E_{n-k}(z)+\frac{(-1)^{n-1}}{2}\,E_{n-1}(y)+\frac{2}{n}\,E_n(z)\\
&=\frac{1}{n}\left(E_n(z)+(-1)^nB_n(y)\right)\qquad\text{(by (3.3))}.
\end{aligned}$$

This proves (1.20). We are done. ∎

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