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Dynamical zeta functions and Kummer congruences

by

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Introduction. In this paper we establish a connection between the Artin–Mazur zeta function and Kummer's congruences. Some connections between Kummer's congruences and periodic points are pointed out in the paper by Everest, van der Poorten, Puri and Ward [4].

Inspired by the Hasse–Weil zeta function of an algebraic variety over a finite field, Artin and Mazur [2] defined the Artin–Mazur zeta function for an arbitrary map $T\colon X \to X$ of a topological space X:

$$Z(T;x) := \exp\biggl(\sum_{n=1}^{\infty} \frac{\operatorname{Fix} T^n}{n}\, x^n\biggr),$$

where $\operatorname{Fix} T^n$ is the number of isolated fixed points of T^n.

Manning [9] proved the rationality of the Artin–Mazur zeta function for diffeomorphisms of a smooth compact manifold satisfying Smale's axiom A.

Following [12], call a sequence $a = (a_n)_{n\ge 1}$ of non-negative integers *realizable* if there is a set X and a map $T\colon X \to X$ such that a_n is the number of fixed points of T^n.

In [12] it is proved that if (a_n) is realizable, then there exists a compact space X and a homeomorphism $T\colon X \to X$ such that $a_n = \operatorname{Fix} T^n$.

Puri and Ward [13] proved that a sequence $(a_n)_{n\ge 1}$ of non-negative integers is realizable if and only if for all $n \ge 1$, $\sum_{d\mid n} \mu(n/d)a_d$ is non-negative and divisible by n. Here $\mu(n)$ denotes the well known Möbius function (see [1]), defined by $\mu(n) = (-1)^k$ if n is a product of k different prime numbers, and $\mu(n) = 0$ if n is not squarefree.

The main point of this paper (see Theorem 3) is to prove the equivalence between two conditions for a given sequence (a_n) to be realizable:

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(a) The congruence $\sum_{d|n}\mu(n/d)a_d \equiv 0 \pmod{n}$ is true for every natural number n.

(b) For every prime number p prime to n and every non-negative integer α we have $a_{np^\alpha} \equiv a_{np^{\alpha-1}} \pmod{p^\alpha}$.

The advantage of our condition (b) is clear. For example in [13] Puri and Ward ask if the sequence $(|E_{2n}|)_{n\geq 1}$ is realizable, where E_{2n} are the Euler numbers. But they cannot use their condition (a) because of the cumbersome expression $\sum_{d|n}\mu(n/d)|E_{2d}|$; our condition (b) refers only to Euler numbers, and the well known Kummer congruences allow us to solve the problem.

As a matter of notation we shall delete the positivity condition, so we shall say that the sequence $(a_n)_{n=1}^\infty$ of integers is *pre-realizable* if for every natural number n, $\sum_{d|n}\mu(n/d)a_d$ is divisible by n.

In 1851 Kummer [8] discovered what we call Kummer's congruences for Bernoulli numbers (see the book by Nielsen [11]). Carlitz [3] extended these congruences to the generalized Bernoulli numbers of Leopoldt. Some restrictions of Carlitz's results have been removed by the work of Fresnel [5]. These congruences are important for the definition of the p-adic L-functions.

In Theorem 6 we establish a connection between Kummer congruences and our condition (b).

This theorem allows us to solve a problem posed by Gabcke [6]. This is connected with the Riemann–Siegel formula. In the investigation of the zeta function of Riemann it is important to compute the values $\zeta(1/2+it)$ of this function at points on the critical line with t very high. Riemann found a very convenient formula for these computations, yet he did not publish anything about this formula. In 1932 C. L. Siegel was able to recover it from Riemann's Nachlass. Now this formula is known as the Riemann–Siegel formula.

The terms of this formula involve certain numbers λ_n that can be defined by the recurrence relation

$$\lambda_0 = 1, \qquad (n+1)\lambda_{n+1} = \sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\lambda_{n-k}.$$

Here E_{2n} denotes the Euler numbers defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty}\frac{E_n}{n!}\,x^n.$$

Hence $E_{2n+1} = 0$ for $n \geq 0$ and

$$E_0 = 1, \qquad E_2 = -1, \qquad E_4 = 5, \qquad E_6 = -61, \qquad E_8 = 1\,385, \quad \ldots$$

Gabcke [6] observed that the first six numbers λ_n are integers and conjectured that this is so for all of them. Gabcke also considers analogous sequences (ϱ_n) and (μ_n). Although he does not mention it, the same motivation for his conjecture also suggests that these too are integer sequences.

We prove all these conjectures. These proofs were the first motivation of this paper.

The question of Puri and Ward about the realizability of $(|E_{2n}|)_{n\geq 1}$ is connected with Gabcke's problem. We shall prove that in fact this sequence is realizable.

Notations. When p is a prime number and m an integer we shall write $p^\alpha \parallel m$ or $\nu_p(m) = \alpha$ to indicate that p^α is the greatest power of p that divides m. The notation $n \perp m$ means that n and m are relatively prime.

1. Dynamical zeta function

THEOREM 1. *Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers and define the sequence $(b_n)_{n=1}^\infty$ by*

$$nb_n = \sum_{d\mid n} \mu(n/d) a_d. \tag{1}$$

Then we have the equality between formal power series

$$\prod_{n=1}^\infty (1-x^n)^{-b_n} = \exp\Big(\sum_{n=1}^\infty \frac{a_n}{n}\, x^n\Big).$$

Proof. By the well known Möbius inversion formula the relation (1) is equivalent to

$$a_n = \sum_{d\mid n} d b_d, \tag{2}$$

therefore we have the following equalities between formal power series:

$$\begin{aligned}\log \prod_{n=1}^\infty (1-x^n)^{-b_n} &= -\sum_{n=1}^\infty b_n \log(1-x^n) = \sum_{n=1}^\infty \sum_{k=1}^\infty b_n \frac{x^{nk}}{k} \\ &= \sum_{m=1}^\infty \frac{x^m}{m}\Big(\sum_{n\mid m} nb_n\Big) = \sum_{m=1}^\infty \frac{x^m}{m}\, a_m.\end{aligned}$$

And this is equivalent to the equality we want to prove. ∎

THEOREM 2. *Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers and define the sequence $(A_n)_{n=0}^\infty$ by the recurrence relation*

$$A_0 = 1, \qquad (n+1)A_{n+1} = \sum_{k=0}^{n} A_{n-k} a_{k+1}, \qquad n \geq 0.$$

Then we have the equality between formal power series

$$\sum_{n=0}^\infty A_n x^n = \exp\Big(\sum_{n=1}^\infty \frac{a_n}{n}\, x^n\Big).$$

Proof. First we have the equality between formal power series

$$\sum_{n=1}^{\infty} nA_n x^{n-1} = \Big(\sum_{n=0}^{\infty} A_n x^n\Big)\Big(\sum_{n=1}^{\infty} a_n x^{n-1}\Big)$$

because by the hypothesis the coefficient of x^n is equal in both members.

Since $A_0 = 1$, integrating formally gives us

$$\log\Big(\sum_{n=0}^{\infty} A_n x^n\Big) = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n.$$

That is equivalent to the relation we wanted to prove. ∎

The following theorem gives various equivalent conditions for a sequence $(a_n)_{n=1}^{\infty}$ of integers to be pre-realizable.

THEOREM 3. *Given a sequence $(a_n)_{n\ge 1}$ of integers, the following conditions are equivalent*:

(a) *The numbers $(b_n)_{n\ge 1}$ defined by*

$$nb_n = \sum_{d\mid n} \mu(n/d) a_d$$

are integers for every $n \in \mathbb{N}$.

(b) *The numbers $(A_n)_{n\ge 0}$ defined by*

$$A_0 = 1, \qquad (n+1)A_{n+1} = \sum_{k=0}^{n} A_{n-k} a_{k+1}, \qquad n \ge 0, \tag{3}$$

are integers for every $n \ge 0$.

(c) *For every prime number p and all natural numbers n, α with $p \perp n$ we have*

$$a_{np^\alpha} \equiv a_{np^{\alpha-1}} \pmod{p^\alpha}.$$

Proof. First we prove the equivalence of (a) and (b).

(a)⇒(b). By the definition of the (b_n) and Theorem 1 we have

$$\prod_{n=1}^{\infty} (1-x^n)^{-b_n} = \exp\Big(\sum_{n=1}^{\infty} \frac{a_n}{n} x^n\Big),$$

and by (a) the b_n are integers. By Theorem 2 the numbers $(A_n)_{n=0}^{\infty}$ satisfy the relation

$$\sum_{n=0}^{\infty} A_n x^n = \exp\Big(\sum_{n=1}^{\infty} \frac{a_n}{n} x^n\Big).$$

Thus we have

$$\sum_{n=0}^{\infty} A_n x^n = \prod_{n=1}^{\infty} (1-x^n)^{-b_n}.$$

Expanding this product, since the b_n are integers, we see that the A_n are also integers. Hence (a) implies (b).

(b)$\Rightarrow$(a). Now, by hypothesis the numbers $(A_n)_{n\geq 0}$ are integers. We can determine inductively a unique sequence of integers c_n such that

$$\sum_{n=0}^{\infty} A_n x^n = \prod_{n=1}^{\infty} (1-x^n)^{-c_n}.$$

In the first step observe that the coefficients of x in both members must be the same, hence $A_1 = c_1$. Then observe that

$$(1-x)^{c_1}\Big(\sum_{n=0}^{\infty} A_n x_n\Big) = 1 + \sum_{n=2}^{\infty} A_n^{(2)} x^n,$$

where the numbers $A_n^{(2)}$ are integers.

Assume by induction that we have determined integers c_j, for $j = 1, \dots,$ $n-1$, such that

$$\prod_{j=1}^{n-1} (1-x^j)^{c_j}\Big(\sum_{n=0}^{\infty} A_n x^n\Big) = 1 + \sum_{k=n}^{\infty} A_k^{(n)} x^k.$$

Then the $A_k^{(n)}$ are integers and we can define $c_n = A_n^{(n)}$ that satisfies the induction hypothesis. Now we have

$$\prod_{j=1}^{\infty} (1-x^j)^{c_j}\Big(\sum_{n=0}^{\infty} A_n x^n\Big) = 1.$$

By the hypothesis and Theorem 2,

$$\sum_{n=0}^{\infty} A_n x^n = \exp\Big(\sum_{n=1}^{\infty} \frac{a_n}{n}\, x^n\Big).$$

Therefore

$$\prod_{n=1}^{\infty} (1-x^n)^{-c_n} = \sum_{n=0}^{\infty} A_n x^n = \exp\Big(\sum_{n=1}^{\infty} \frac{a_n}{n}\, x^n\Big).$$

Now take logarithms in both members to obtain

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} \frac{x^{kn}}{k} = \sum_{n=1}^{\infty} \frac{a_n}{n}\, x^n.$$

Reasoning as in the proof of Theorem 1 we get

$$a_m = \sum_{n|m} n c_n.$$

Therefore by the Möbius inversion formula $c_n = b_n$, the numbers defined in (a), and by construction the c_n are integers. Thus we have proved (a).

(a)⇒(c). We know that condition (a) is equivalent to the existence of integers b_n that satisfy (2).

Assume that p is a prime number and n and α are natural numbers such that $p \perp n$. Then

$$a_{np^\alpha} = \sum_{d\mid np^\alpha} db_d = \sum_{k=0}^{\alpha}\sum_{d\mid n} dp^k b_{dp^k}.$$

Analogously

$$a_{np^{\alpha-1}} = \sum_{k=0}^{\alpha-1}\sum_{d\mid n} dp^k b_{dp^k}.$$

Therefore

$$a_{np^\alpha} - a_{np^{\alpha-1}} = \sum_{d\mid n} dp^\alpha b_{dp^\alpha} \equiv 0 \ (\operatorname{mod} p^\alpha).$$

(c)⇒(a). Let n be an integer. We have to show that

$$\sum_{d\mid n} \mu(n/d) a_d$$

is divisible by n. Let $p^\alpha \parallel n$ with $\alpha \ge 1$; then $n = p^\alpha m$ with $p \perp m$.

Since $\mu(k) \neq 0$ only when k is squarefree, we get

$$\begin{aligned}\sum_{d\mid n} \mu(n/d) a_d &= \sum_{d\mid m} \mu(m/d) a_{dp^\alpha} - \sum_{d\mid m} \mu(m/d) a_{dp^{\alpha-1}} \\ &= \sum_{d\mid m} \mu(m/d)(a_{dp^\alpha} - a_{dp^{\alpha-1}}) \equiv 0 \ (\operatorname{mod} p^\alpha).\end{aligned}$$

The sum is divisible by every prime divisor of n, and so divisible by n. ■

2. Kummer congruences. In 1851 Kummer [8] proved the following theorem:

THEOREM 4 (Kummer). *Let p be a prime number. Assume that*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{k=0}^{\infty} c_k (e^{bx} - e^{ax})^k,$$

where a, b and the c_k are integral modulo p. Then the a_n are integers modulo p and for $e \ge 1$, $n \ge 1$, $m \ge 0$, and $p^{e-1}(p-1) \mid w$ we have

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} a_{m+sw} \equiv 0 \ (\operatorname{mod} (p^m, p^{ne})). \tag{4}$$

The congruences (4) are usually called the *Kummer congruences*. We shall say that the sequence (a_n) *satisfies the Kummer congruences* if (4) holds for every prime number p. By the Kummer theorem such sequences exist, but we are interested in some particular ones.

THEOREM 5. *The sequence $(E_{2n})_{n=1}^{\infty}$ satisfies the Kummer congruences.*

Proof. Since

$$\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2}{2 + (e^{x/2} - e^{-x/2})^2},$$

the Kummer theorem proves that $(E_n)_{n=1}^{\infty}$ satisfies (4) for every odd prime number p. Therefore $(E_{2n})_{n=1}^{\infty}$ satisfies these congruences for every odd prime number p. This reasoning can be found in Kummer [8].

The above procedure does not give the case $p = 2$, but Fresnel [5] has extended Kummer congruences for Euler numbers, as we see in the following lines.

Let $\chi : \mathbb{Z} \to \{-1, 0, 1\}$ be given by

$$\chi(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Then the generalized Bernoulli numbers associated to this character (see [5] for details) are related to the Euler numbers by

$$\frac{B^n(\chi)}{n} = -\frac{E_{n-1}}{2}.$$

In [5, p. 319], it is shown that, when $2^e \parallel w$ with $e \geq 1$,

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{B_{m+sw}(\chi)}{m+sw} \equiv 0 \pmod{(2^{n(e+2)}, 2^{m-1})}.$$

With a change of notation this is equivalent to

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{E_{2m+sw}}{2} \equiv 0 \pmod{(2^{n(e+2)}, 2^{2m})}.$$

Obviously this implies that for $2^{e-1} \mid w$ we have

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} E_{2(m+sw)} \equiv 0 \pmod{(2^{ne}, 2^m)}. \ \blacksquare$$

The above theorem is a model of many more interesting examples. In Carlitz [3], it is proved that if χ is a primitive character modulo f, and f is divisible by at least two distinct rational primes, then $B^n(\chi)/n$ is an algebraic integer and

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{B^{n+1+sw}(\chi)}{n+1+sw} \equiv 0 \pmod{(p^n, p^{en})}$$

if $p^{e-1}(p-1) \mid w$.

Thus the sequence $(a_n)_{n=1}^{\infty}$ with $a_n = B^{n+1}(\chi)/(n+1)$ satisfies the Kummer congruences if the character χ is real. When χ is complex the sequence defined by $a_n = \mathrm{Tr}(B^{n+1}(\chi)/(n+1))$ satisfies the Kummer congruences.

Sequences that satisfy the Kummer congruences are pre-realizable, as we will see in the following theorem.

THEOREM 6. *Let $(a_n)_{n=1}^{\infty}$ be a sequence that satisfies the Kummer congruences. Then*

$$a_{b+np^{\alpha}} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}}$$

for any natural numbers b, n, α and prime number p such that $p \perp n$. That is to say, if (a_n) satisfies the Kummer congruences then for every natural number b, the sequence $(a_{b+n})_{n=1}^{\infty}$ is pre-realizable.

Proof. By (4), with $n = 1$ we have

$$a_{m+p^{e-1}(p-1)} \equiv a_m \pmod{(p^m, p^e)}.$$

Therefore, for every natural number k, and assuming $m \geq e$,

$$a_{m+kp^{e-1}(p-1)} \equiv a_m \pmod{p^e}.$$

Now take $m = b + np^{\alpha-1}$, $k = n$ and $e = \alpha$. If $b + np^{\alpha-1} \geq \alpha$, we get

$$a_{b+np^{\alpha-1}+np^{\alpha-1}(p-1)} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}}.$$

Since $p^{\alpha-1} \geq \alpha$ for p prime and $\alpha \geq 1$, the condition is satisfied and we get

$$a_{b+np^{\alpha}} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}}. \blacksquare$$

3. Euler numbers as numbers of fixed points. We are now in a position to solve the problem posed by Puri and Ward [13], who ask if the sequence $(|E_{2n}|)_{n=1}^{\infty}$ is realizable. We shall show that this is indeed the case ([1]).

THEOREM 7. *There exists a map $T\colon X \to X$ such that*

$$|E_{2n}| = \mathrm{Fix}\, T^n.$$

Proof. First we show that $|E_{2n}|$ is a pre-realizable sequence. By Theorem 5 the sequence $(E_{2n})_{n=1}^{\infty}$ satisfies the Kummer congruences. Thus by Theorem 6, for $p \perp m$,

$$E_{2mp^{\alpha}} \equiv E_{2mp^{\alpha-1}} \pmod{p^{\alpha}}.$$

Therefore, since $|E_{2n}| = (-1)^n E_{2n}$ and $(-1)^{mp^{\alpha}} \equiv (-1)^{mp^{\alpha-1}} \pmod{p^{\alpha}}$ it follows that

$$|E_{2mp^{\alpha}}| \equiv |E_{2mp^{\alpha-1}}| \pmod{p^{\alpha}}.$$

([1]) After reading this paper, T. Ward tells me that his student P. Moss [10] has also independently proved it.

By Theorem 3 the numbers b_n defined by

$$nb_n = \sum_{d|n} \mu(n/d)|E_{2d}|$$

are integers.

Now we must show that the numbers b_n are non-negative. To this end we observe that

$$nb_n \geq |E_{2n}| - \sum_{d=1}^{n/2} |E_{2d}|.$$

Now we apply the well known formula

$$1 \leq \left(\frac{\pi}{2}\right)^{2d+1} \frac{|E_{2d}|}{(2d)!} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+1}} \leq 2.$$

Thus

$$\begin{aligned} nb_n &\geq (2n)!\left(\frac{2}{\pi}\right)^{2n+1} - 2\sum_{d=1}^{n/2}(2d)!\left(\frac{2}{\pi}\right)^{2d+1} \\ &\geq (2n)!\left\{\left(\frac{2}{\pi}\right)^{2n+1} - 2\,\frac{(n)!}{(2n)!}\sum_{d=1}^{\infty}\left(\frac{2}{\pi}\right)^{2d+1}\right\}. \end{aligned}$$

The value of the last sum can be computed to be $0.433\ldots$, therefore

$$nb_n \geq (2n)!\left\{\left(\frac{2}{\pi}\right)^{2n+1} - \frac{(n)!}{(2n)!}\right\}.$$

This is positive for $n \geq 2$, and we have $b_1 = 1 \geq 0$. ∎

The first values of the three sequences in this case are the following:

a_n		1	5	61	1385	50521	2702765	199360981	…
b_n		1	2	20	345	10104	450450	28480140	…
A_n	1	3	23	371	10515	461869	28969177	2454072147	…

4. Solution of Gabcke problem. By Theorem 3 the assertion of Gabcke—that the numbers λ_n are integers—is equivalent to saying that the sequence $a_k = 2^{4k-3}|E_{2k}|$ is pre-realizable. We shall show that it is in fact realizable.

If (a_n) and (a'_n) are realizable, then the sequence $(a_n a'_n)$ is also realizable. In fact given $T\colon X \to X$ and $T'\colon Y \to Y$ such that $a_n = \operatorname{Fix} T^n$ and $a'_n = \operatorname{Fix} T'^n$, it is easy to see that $T \times T'\colon X \times Y \to X \times Y$ satisfies $a_n a'_n = \operatorname{Fix}(T \times T')$.

Therefore by Theorem 7 we need to prove that the sequence 2^{4n-3} is realizable. This follows from the following theorem.

Theorem 8. *Let $a, b \in \mathbb{N}$ be such that $b \mid a$ and for every prime number $p \mid a$ we have $p \mid (a/b)$. Then the sequence a^n/b is realizable.*

Proof. By the result of Puri and Ward we must show that the sequence a^n/b is pre-realizable and that the corresponding b_n are non-negative integers.

First let $p \perp a$ be a prime number, and let $n \perp p$ and α be natural numbers. We must show that

$$a^{np^\alpha}/b \equiv a^{np^{\alpha-1}}/b \pmod{p^\alpha}.$$

Since $b \perp p$, this is equivalent to

$$a^{np^\alpha} \equiv a^{np^{\alpha-1}} \pmod{p^\alpha}.$$

Now for $\alpha = 1$ this is Fermat's little theorem, and for a general α it follows, by induction, from the fact that for $\alpha \geq 1$, if $a \equiv b \pmod{p^\alpha}$ then $a^p \equiv b^p \pmod{p^{\alpha+1}}$.

Now if $p \mid a$, assume that $p^r \parallel a$ and $p^s \parallel b$. By hypothesis we have $r \geq s+1$. We have to show that

$$a^{np^\alpha}/b \equiv a^{np^{\alpha-1}}/b \pmod{p^\alpha},$$

where $p \perp n$ and $\alpha \geq 1$. But the two numbers are divisible by $p^{rnp^{\alpha-1}-s}$. All we have to show is that $rnp^{\alpha-1} \geq s+\alpha$. We can assume that $n = 1$. For $\alpha = 1$ this is $r \geq s+1$, true by hypothesis. For $\alpha \geq 2$ we have

$$rp^{\alpha-1} = r(p^{\alpha-1}-1) + r \geq (\alpha-1) + (s+1).$$

Now we define the numbers b_n by

$$nb_n = \sum_{d \mid n} \mu(n/d) a^d/b.$$

By the previous reasoning we know the b_n are integers. If $a = 1$, it is easy to see that $b_1 = 1$ and $b_n = 0$ for $n > 1$. If $a \geq 2$ then

$$nb_n \geq \frac{1}{b}\Big(a^n - \sum_{d=1}^{n/2} a^d\Big).$$

This is easily seen to be non-negative. ■

Corollary 1. *The sequence $(a_n)_{n=1}^\infty$, where $a_n = 2^{4n-3}$, is realizable.*

The three sequences associated to this realizable sequence are

a_n		2	32	512	8192	131072	2097152	33554432	...
b_n		2	15	170	2040	26214	349435	4793490	...
A_n	1	2	18	204	2550	33660	460020	6440280	...

Now we are in a position to prove Gabcke's conjecture.

THEOREM 9. *Let λ_n be the numbers defined by*

$$\lambda_0 = 1, \qquad (n+1)\lambda_n = \sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\lambda_{n-k} \qquad (n \ge 0), \tag{5}$$

ϱ_n those defined by

$$\varrho_0 = -1, \qquad (n+1)\varrho_n = -\sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\varrho_{n-k} \qquad (n \ge 0), \tag{6}$$

and finally let $\mu_n = (\lambda_n + \varrho_n)/2$. All those numbers are integers.

Proof. By Theorem 7 and Corollary 1 the sequences $(|E_{2n}|)_{n=1}^{\infty}$ and $(2^{4n-3})_{n=1}^{\infty}$ are realizable. Since the product of two realizable sequences is realizable, the sequence $(2^{4n-3}|E_{2n}|)_{n=1}^{\infty}$ is realizable. Therefore it satisfies condition (a) of Theorem 3. So it satisfies condition (b), which says precisely that the numbers λ_n are integers.

Now condition (c) of the same theorem states that for every prime number p and natural numbers $n \perp p$ and α, the numbers $a_n = 2^{4n-3}|E_{2n}|$ satisfy

$$a_{np^\alpha} \equiv a_{np^{\alpha-1}} \pmod{p^\alpha}.$$

Thus the same congruences are satisfied by the numbers $a'_n = -a_n$. Once again Theorem 3 says that the numbers a'_n satisfy condition (b). This is the same as saying that the numbers A'_n defined by

$$A'_0 = 1, \qquad (n+1)A'_n = -\sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|A'_{n-k} \qquad (n \ge 0) \tag{7}$$

are integers. But it is easily seen that $\varrho_n = -A'_n$.

The assertion about the numbers μ_n follows from the fact that $\lambda_n \equiv \varrho_n \pmod 2$, to be proved in Theorem 11. ∎

The following theorem is well known. I give a proof for completeness.

THEOREM 10. *Let $s(n)$ be the sum of the digits of the binary representation of n. Then*

$$s(n) = n - \sum_{j=1}^{\infty} \left\lfloor \frac{n}{2^j} \right\rfloor.$$

Proof. Let the binary representation of n be of type $\cdots 0\overbrace{11\cdots 1}^{k \text{ times}}$, with $k \ge 0$; then $n+1 = \cdots 1\overbrace{00\cdots 0}^{k \text{ times}}$. Therefore

$$s(n) - k = s(n+1) - 1.$$

Also $k = \nu_2(n+1)$, the exponent of 2 in the prime factorization of $n+1$.

Thus we have proved that for every integer $n \geq 0$,

$$s(n+1) + \nu_2(n+1) = s(n) + 1. \tag{8}$$

We add these equalities for $n = 0, 1, \ldots, n-1$ to get

$$s(n) + \sum_{k=1}^{n} \nu_2(k) = n.$$

It is easily checked that

$$\sum_{k=1}^{n} \nu_2(k) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{2^j} \right\rfloor. \ \blacksquare$$

THEOREM 11. *The numbers* λ_n *and* ϱ_n *defined by* (5) *and* (6) *satisfy*

$$\nu_2(\lambda_n) = \nu_2(\varrho_n) = s(n).$$

Proof. First consider the sequence λ_n. Clearly the conclusion is true for the first λ_n which are

$$\lambda_0 = 1, \quad \lambda_1 = 2, \quad \lambda_3 = 82, \quad \lambda_4 = 10572.$$

Since the Euler numbers E_{2k} are odd, from the definition of λ_n it follows that

$$\nu_2(n+1) + \nu_2(\lambda_{n+1}) = \nu_2\Big(\sum_{k=0}^{n} 2^{4k+1} |E_{2k+2}| \lambda_{n-k}\Big). \tag{9}$$

By induction the terms of this sum are exactly divided by the powers of 2 with exponents

$$1 + s(n), \quad 5 + s(n-1), \quad 9 + s(n-2), \quad \ldots, \quad (4n+1) + s(0).$$

This is a strictly increasing sequence, since

$$s(n) - s(n-1) = 1 - \nu_2(n) < 4.$$

Hence from (9) we get

$$\nu_2(n+1) + \nu_2(\lambda_{n+1}) = 1 + s(n).$$

By (8),

$$\nu_2(\lambda_{n+1}) = s(n) - \nu_2(n+1) + 1 = s(n+1).$$

The same proof applies to the sequence (ϱ_n). $\blacksquare$

5. Examples. Here we give some examples of numbers satisfying the equivalent conditions of Theorem 3. First consider the case of Gabcke's numbers $A_n = \lambda_n$. The first terms of the associated sequences are given in the following table.

a_n		2	160	31232	11345920	947622146676	...
b_n		2	79	10410	2836440	1324377702	...
λ_n	1	2	82	10572	2860662	1330910844	...

We can take an arbitrary sequence (b_n) of integers and obtain sequences (a_n) and (A_n) that automatically satisfy our theorems. We give two simple examples.

With $b_n = 1$ for every n, we get $a_n = \sigma(n)$.

a_n		1	3	4	7	6	12	8	15	13	18	12	28	...
b_n		1	1	1	1	1	1	1	1	1	1	1	1	...
A_n	1	1	2	3	5	7	11	15	22	30	42	56	77	...

With $b_n = -24$, the numbers A_n are given by Ramanujan's τ function, $A_n = \tau(n+1)$.

a_n		-24	-72	-96	-168	-144	-288	-192	-360	...
b_n		-24	-24	-24	-24	-24	-24	-24	-24	...
A_n	1	-24	252	-1472	4830	-6048	-16744	84480	-113643	...

Finally, let (T_n) be the tangent numbers of [7]:

$$T_{2n} = 0, \qquad T_{2n+1} = (-1)^n \frac{4^{n+1}(4^{n+1}-1)B_{2n+2}}{2n+2}.$$

It can be proved that $a_n = (-1)^n T_{2n+1}$ satisfies the Kummer congruences. It follows that the sequence (T_{2n+1}) is realizable; in this case the three sequences are

a_n		2	16	272	7936	353792	22368256	1903757312	...
b_n		2	7	90	1980	70758	3727995	271965330	...
A_n	1	2	10	108	2214	75708	3895236	280356120	...

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