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Well-rounded sublattices of planar lattices

by

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1. Introduction. A lattice in Euclidean space $\mathbb{R}^d$ is *well-rounded* if the non-zero lattice vectors of minimal length span $\mathbb{R}^d$. Well-rounded lattices are interesting for several reasons. First of all, the concept is put into a broader context by the notion of the *successive minima* of a lattice (more precisely, of a norm function on a lattice). By definition, a lattice is well-rounded if and only if all its d successive minima (norms of successively shortest linearly independent vectors) are equal to each other.

A first observation is that many important 'named' lattices in higher-dimensional space are well-rounded, such as the Leech lattice, the Barnes–Wall lattice(s), the Coxeter–Todd lattice, all irreducible root lattices, and many more [10]. There are essentially two reasons for this (which often apply both). First of all, distinct successive minima give rise to proper subspaces of $\mathbb{R}^d$ that are invariant under the orthogonal group (automorphism group) of the lattice. If this finite group acts irreducibly on $\mathbb{R}^d$, the lattice must be well-rounded. Secondly, a lattice which gives rise to a locally densest sphere packing (a so-called extreme lattice) is well-rounded. It is actually perfect by Voronoi's famous theorem (this part goes back to Korkin and Zolotarev); as is easily seen, perfection implies well-roundedness (cf. [20]).

However, these two observations are not at the core of the notion. They might give the impression that well-rounded lattices are very rare or special, which is not the case. In terms of Gram matrices or quadratic forms, the well-rounded ones lie in a subspace of codimension $d-1$ in the space of all symmetric matrices, similarly for the cone of positive definite Minkowski-reduced forms. Despite its codimension, this subspace is large enough so that certain questions about general forms can be reduced to well-rounded ones. A good illustration for this is Minkowski's proof

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of the fact that the geometric mean of all d successive minima of a lattice is bounded by the same quantity $\gamma_d(\mathrm{disc}(\varLambda))^{1/d}$ as the first minimum (see Section 2). Here, γ_d is the Hermite constant in dimension d, and for well-rounded lattices this estimate reduces to the definition of this constant. The proof is obtained by a certain deformation of the quadratic form (see [29]). A sharpened version of this technique asks for a diagonal matrix which transforms a given lattice into a well-rounded one. In general, its existence is unknown, but C. McMullen [21] recently proved a weaker version which suffices for applications to Minkowski's conjecture on the minimum of a (multiplicative) norm function on lattices. The method of proof is related to applications of well-rounded lattices to cohomology questions as described in the introduction of [17] (see also the references given there).

Having this kind of 'richness' of well-rounded lattices in mind, it is tempting to ask how frequent they are in terms of counting sublattices. So, the principal object of study in this paper is the function

$$a_{\varGamma}(n) := \mathrm{card}\{\varLambda \mid \varLambda \subseteq \varGamma \text{ is a well-rounded sublattice with } [\varGamma : \varLambda] = n\}, \tag{1.1}$$

where $\varGamma$ is an in principle arbitrary lattice, and $[\varGamma : \varLambda]$ denotes the index of $\varLambda$ in $\varGamma$. This question is of interest already in dimension 2 (where some of the general features described above reduce to rather obvious facts). Moreover, since the well-rounded sublattices are the objects of interest, and not so much the enveloping 'lattice of reference' $\varGamma$, it seems natural to focus mainly on the two most symmetric lattices, the hexagonal lattice and the square lattice. In this paper, we shall obtain complete and explicit results on the asymptotic number of well-rounded sublattices, as a function of the index, of the hexagonal lattice and of the square lattice. We also have results for general $\varGamma$ which are somewhat weaker, which seems unavoidable.

In special situations, lattice enumeration problems have a long history. The coefficients of the Dedekind zeta functions of an algebraic number field K of degree d over the rationals count the number of ideals of given index in the ring of integers $\mathbb{Z}_K$, which is considered as a lattice in a well-known way [7]. The perhaps most basic result on lattice enumeration, which is also one of the most frequently rediscovered ones, is the determination of the number $g(n)$ of all distinct sublattices of index n in a given lattice $\varGamma \subset \mathbb{R}^d$. The result follows easily from the Hermite normal form for integral matrices and reads

$$g_d(n) = g(n) = \sum_{m_1 \cdots m_d = n} m_1^0 \cdot m_2^1 \cdots m_d^{d-1} \tag{1.2}$$

with Dirichlet series generating function

$$D_g(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1) \tag{1.3}$$

(compare [25, p. 64], [27, p. 307], [19], [2]; for several different proofs, see [19, Theorem 15.1]). The result of (1.2) is insensitive to any geometric property of the lattice Γ, in the sense that it is actually a result for the free Abelian group of rank d and its subgroups. In [24, 14], extensions to more general classes of finitely generated groups are treated.

As for lattices, it is natural to refine the question by looking at classes of sublattices with particular properties (number-theoretic or geometric), possibly defined by an additional structure on the enveloping vector space. In addition to the classical case of the Dedekind zeta function mentioned above, we are aware of only few, scattered results. Quite a while ago, in [27, 9], modules in an order in a semisimple algebra over a number field were considered. Well-rounded lattices in dimension 2 have recently been analysed in [11, 13, 12, 17] (see also the references in [12]). Together with our earlier work on similar sublattices [3, 6] and on coincidence site sublattices (CSLs) [2, 31, 4, 33], these papers were our starting point.

One benefit of Dirichlet series is the access to asymptotic results on the growth of a (non-negative) arithmetical function $f(n)$. Since f in general need not behave regularly, in particular need not be monotone, one usually considers the average growth of $f(n)$, that is, one studies the summatory function $F(x) = \sum_{n\le x} f(n)$. For the above counting function $g_d(n)$ for sublattices, the summatory function $G_d(x)$ satisfies

$$G_d(x) = cx^d + \Delta_d(x), \tag{1.4}$$

with $c = 1$ for $d = 1$ and $c = \frac{1}{d}\prod_{\ell=2}^{d} \zeta(\ell)$ otherwise, which follows from (1.3) by applying Delange's theorem (compare Theorem 7 in Appendix A). Clearly, $G_1(x) = [x]$, where $[\cdot]$ denotes the Gauss bracket, thus $\Delta_1(x) = \mathcal{O}(1)$. In dimension 2, $G_2 = \sigma_1(n) := \sum_{\ell|n} \ell$, so we have the well-known asymptotic behaviour of the divisor function, whose error term can be estimated as $\Delta_2(x) = \mathcal{O}(x\log(x))$ (see [1, Thm 3.4]).

One can ask for a more refined description of the asymptotic growth of an arithmetic function, consisting of a main term for the summatory function, a term of second order (a 'first order error term'), and an error term of a strictly smaller order of magnitude than the term of second order. For instance, for the number of divisors of n, it is known that

$$\sum_{n\le x} \sigma_0(n) = x\log(x) + (2\gamma - 1)x + \mathcal{O}\big(\sqrt{x}\big), \tag{1.5}$$

where γ is the Euler–Mascheroni constant (compare [1, 28]). So we have a term of second order which is linear in this case and thus of 'almost the same' growth as the main term, whereas the error term is much smaller.

The content of this paper can now be summarised as follows. In the short preparatory Section 2, we recall a few facts about reduced bases and Bravais classes of lattices in the plane, and state some auxiliary remarks about well-rounded (sub)lattices.

In Section 3, we begin with an explicit description of all well-rounded sublattices of the square lattice, the latter viewed as the ring $\mathbb{Z}[i]$ of Gaussian integers. After these preparations, the main result is Theorem 2, which gives a refined asymptotic description of the function $A_{\square}$, of the kind that we have explained above for the divisor function in (1.5); the constants for the main term and the term of second order are determined explicitly. The proof relies on classical methods from analytic number theory, including Delange's theorem and some elementary tools around Euler's summation formula and Dirichlet's hyperbola method. We describe the strategy and the main steps of the proof; some of the details, which are long and technical, have been transferred to a supplement to this paper. A weaker result, namely the explicit asymptotics without the second-order term, is stated in Theorem 1, which is fully proved here.

Section 4 provides the analogous analysis for the hexagonal lattice, realised as the ring $\mathbb{Z}[\rho]$ of Eisenstein integers with $\rho = e^{2\pi i/3}$; Theorems 3 and 4 are completely analogous to Theorems 1 and 2.

The general case of well-rounded sublattices of the plane is treated in Section 5, which is subdivided into two parts.

The first one starts with a criterion for the existence of well-rounded sublattices. The lattices that have a well-rounded sublattice include all 'rational' lattices, that is, lattices whose Gram matrix consists of rational numbers (or even rational integers), up to a common multiple. So these are exactly the lattices that correspond to integral quadratic forms in the classical sense. There is an interesting connection between well-rounded sublattices and CSLs, which is established in Lemma 1. In the rest of this part, it is shown in Theorem 5 that all non-rational lattices that contain well-rounded sublattices have essentially the same power-law growth (linear) of their average number $A_{\Gamma}(x)$.

The second part of Section 5 deals with the behaviour of $A_{\Gamma}(x)$ in the general rational case. The discussion is more complicated, but nevertheless we can show that the growth is proportional to that of $x\log(x)$, as in the square and hexagonal cases. Summarising, we see that three regimes exist as follows: A planar lattice can have many, some or no well-rounded sublattices; the first case is exactly the rational case, while the second case is explained by the existence of an essentially unique coincidence reflection.

Our paper is complemented by four appendices. In Appendix A, some classical results about Dirichlet series are collected in a way that suits our needs. In Appendix B, we explicitly record the asymptotic behaviour of the number of similar sublattices of the square and the hexagonal lattices, which are useful by-products of Sections 3 and 4. Appendix C summarises key properties of a special type of sublattices that we need, while Appendix D recalls some facts about Epstein's zeta functions.

2. Tools from the geometry of planar lattices. Let us collect some simple but useful facts from the geometric theory of lattices. We assume throughout this paper that we are in dimension $d = 2$, so we consider an arbitrary lattice $\varLambda$ in the Euclidean plane. Let $v \in \varLambda$ be a shortest non-zero vector, and $w \in \varLambda$ shortest among the lattice vectors linearly independent of v. Then v, w form a basis of $\varLambda$. (The reader may consult [7, Chapter 2, §7.7] for this and for related statements below.) Changing the sign of w if necessary, we may assume that the inner product satisfies $(v, w) \geq 0$. A basis of this kind is called a *reduced basis* of $\varLambda$. By definition, we have the chain of inequalities

$$|v| \leq |w| \leq |v - w| \leq |v + w|. \tag{2.1}$$

In terms of the quantities $a := |v|^2$, $c := |w|^2$, and $b := (v, w)$, which are the entries of the Gram matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ with respect to v, w, these conditions read

$$0 \leq 2b \leq a \leq c. \tag{2.2}$$

Conversely, if we start with any two linearly independent vectors v, w satisfying (2.1) or (2.2), then v, w form a reduced basis of the lattice they generate. Concerning the reduction conditions (2.1), there are six cases possible for the pair v, w:

(a) $|v| < |w| < |v - w| < |v + w|$, $(v, w) > 0$ (general type),

(b) $|v| < |w| < |v - w| = |v + w|$, $(v, w) = 0$ (rectangular type),

(c) $|v| < |w| = |v - w| < |v + w|$, $(v, w) > 0$ (centred rect. type),

(d) $|v| = |w| < |v - w| < |v + w|$, $(v, w) > 0$ (rhombic type),

(e) $|v| = |w| < |v - w| = |v + w|$, $(v, w) = 0$ (square type),

(f) $|v| = |w| = |v - w| < |v + w|$, $(v, w) > 0$ (hexagonal type).

It is well-known and easily shown that the entries a, b, c of the Gram matrix with respect to a reduced basis v, w only depend on the lattice, but not on the choice of v, w. Therefore, it is legitimate to talk about the *geometric type* of the lattice, which is one of the types (a) to (f) above. As a further consequence of this uniqueness property, the orthogonal group $\mathrm{O}(\varLambda)$ acts transitively (and thus sharply transitively) on the set of all (ordered) reduced bases of $\varLambda$. (By definition, $\mathrm{O}(\varLambda)$ is the set of orthogonal transformations of the enveloping vector space which maps the lattice into, and thus onto, itself.) $\mathrm{O}(\Lambda)$ is cyclic of order 2 for lattices of general type, a dihedral group of order 4 (generated by two perpendicular reflections) for the types (b), (c) and (d), a dihedral group of order 8 for the square lattice, and of order 12 for the hexagonal lattice.

Typically, one wants to classify lattices only up to similarity, which means that the Gram matrix may be multiplied with a positive constant. Clearly, a square or hexagonal lattice is unique up to similarity. Similarity classes of rhombic type depend on one parameter, the angle α formed by v and w, where $\pi/3 < \alpha < \pi/2$. The limiting cases $\alpha = \pi/3$ and $\alpha = \pi/2$ lead to the hexagonal, respectively square lattice.

A lattice Λ (in any dimension) is called *rational* if its similarity class contains a lattice with rational Gram matrix. The *discriminant* $\mathrm{disc}(\Lambda)$ of a lattice Λ is the determinant of any of its Gram matrices. (This is the square of the volume of a fundamental domain for the action of Λ by translations.)

Two lattices Γ, Λ (on the same space) are called *commensurate* (or commensurable) if their intersection $\Gamma \cap \Lambda$ has finite index in both. Equivalently, there exists a non-zero integer a such that $a\Gamma \subseteq \Lambda \subseteq a^{-1}\Gamma$. This in turn is equivalent to the condition that Γ and Λ generate the same space over the rationals, $\mathbb{Q}\Gamma = \mathbb{Q}\Lambda$. If Γ and Λ are commensurate, the ratio of their discriminants is a rational square.

A *coincidence isometry* for Λ is an isometry (an orthogonal transformation R of the underlying real space) such that Λ and $R\Lambda$ are commensurate. In earlier work [2], we have introduced the notation $\mathrm{OC}(\Lambda)$ for the set of all coincidence isometries for Λ. If $R \in \mathrm{OC}(\Lambda)$, it follows that $R\mathbb{Q}\Lambda = \mathbb{Q}R\Lambda = \mathbb{Q}\Lambda$ (see above), i.e. R induces an orthogonal transformation of the rational space $\mathbb{Q}\Lambda$. Conversely, any such orthogonal transformation maps Λ onto a lattice of full rank in the same rational space, which, by the above remarks, is commensurate with Λ. Altogether, $\mathrm{OC}(\Lambda)$ is equal to the rational orthogonal group $\mathrm{O}(\mathbb{Q}\Lambda)$ (in particular, it is a group). If Γ and Λ are commensurate, their groups of coincidence isometries coincide,

$$\mathrm{OC}(\Gamma) = \mathrm{O}(\mathbb{Q}\Gamma) = \mathrm{O}(\mathbb{Q}\Lambda) = \mathrm{OC}(\Lambda).$$

A *coincidence site lattice* (CSL) for Λ is a sublattice of the form $\Lambda \cap R\Lambda$ with $R \in \mathrm{OC}(\Lambda)$; see [2] for further motivation concerning this notion.

Geometric types as introduced above are closely related, but not identical, with the so-called *Bravais types* of lattices, which are defined in any dimension. Two lattices Γ and Λ are *Bravais equivalent* if there exists a linear transformation which maps Γ onto Λ and also conjugates $\mathrm{O}(\Gamma)$ into $\mathrm{O}(\Lambda)$. The Bravais type (or Bravais class) of a lattice depends only on its geometric type; the centred rectangular and the rhombic lattices belong to the same Bravais type (so we call them rhombic-cr lattices). Otherwise, geometric types and Bravais types (or rather the respective equivalence classes of lattices) coincide.

Let us return to well-rounded lattices. Clearly, a planar lattice is well-rounded if and only if it is of rhombic, square or hexagonal type. Any rhombic-cr lattice contains a rectangular sublattice of index 2. In fact, if v and w form a reduced basis, then $v-w$ and $v+w$ are orthogonal, and form a reduced basis of the desired sublattice. Conversely, if v, w is a reduced basis of a rectangular lattice, and if we further assume that $|w^2| = c < 3a = 3|v|^2$, then $v+w$ and $-v+w$ form a reduced basis of a rhombic sublattice of index 2. (If $c = 3a$, this sublattice is hexagonal, whereas for $c > 3a$, we have $|2v| < |\pm v+w|$, and thus the vectors are not shortest any more; in this case, the sublattice is centred rectangular.)

Similarly, a hexagonal lattice contains a rectangular sublattice of index 2, or more precisely, it contains exactly three rectangular sublattices of index 2 for symmetry reasons. Analogously, the square lattice contains precisely one square sublattice of index 2.

3. Well-rounded sublattices of $\mathbb{Z}[i]$. We use the Gaussian integers as a representation of the square lattice. Note that there is no hexagonal sublattice of $\mathbb{Z}[i]$ (consider the discriminant). Hence, all well-rounded sublattices are either rhombic or square lattices, which we treat separately, in line with the geometric classification explained above.

A fundamental quantity that will appear frequently below is the Dirichlet series generating function for the number of similar sublattices of $\mathbb{Z}[i]$ (cf. [3, 6]), which is equal to the Dedekind zeta function of the quadratic field $\mathbb{Q}(i)$,

$$\Phi_{\square}(s) = \zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(s,\chi_{-4}). \tag{3.1}$$

Here, $\zeta(s)$ is Riemann's zeta function, and $L(s,\chi_{-4})$ is the L-series corresponding to the Dirichlet character χ_{-4} defined by

$$\chi_{-4}(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \bmod 4, \\ -1 & \text{if } n \equiv 3 \bmod 4; \end{cases}$$

see [2, 6, 30] and Appendix A.

Before dealing with the well-rounded sublattices, let us consider all rhombic-cr and square sublattices of $\mathbb{Z}[i]$ (recall that the term 'rhombic-cr' means rhombic or centred rectangular). Let $z_1, z_2 \in \mathbb{Z}[i]$ be any two elements of equal norm. The sublattice $\Gamma = \langle z_1, z_2\rangle_{\mathbb{Z}}$ is of rhombic or centred rectangular or square type, and every rhombic-cr or square sublattice is obtained in this way (see Section 2). We can write $z_1 + z_2$ and $z_1 - z_2$ as $z_1 + z_2 = pz$ and $z_1 - z_2 = iqz$ where p, q are integers and

z is primitive, which means that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are relatively prime. Without loss of generality, we may assume that p and q are positive (interchange z_1 and z_2 if necessary). Thus $\Gamma = \langle z_1, z_2\rangle_{\mathbb{Z}} = \big\langle \frac{p+iq}{2}z, \frac{p-iq}{2}z\big\rangle_{\mathbb{Z}}$ is a sublattice of $\mathbb{Z}[i]$ of index $\frac{1}{2}pq|z|^2$. The lattice Γ is a square lattice if and only if $p = q$. Determining the number of rhombic-cr and square sublattices is thus equivalent to finding all rectangular and square sublattices of $\mathbb{Z}[i]$ with the additional constraint that $(p + qi)z$ is divisible by 2.

We distinguish two cases (note that z is primitive, hence, in particular, not divisible by 2, and thus p and q must have the same parity), which we call 'rectangular case' and 'rhombic case' for reasons that will become clear later.

(1) 'Rectangular case': z is not divisible by $1+i$, hence p and q must be even. We write $p = 2p'$, $q = 2q'$. The index is even since it is given by $2p'q'|z|^2$. Note that p', q' may take any positive integral value, even or odd.

(2) 'Rhombic case': z is divisible by $1 + i$. We write $z = (1 + i)w$.

   (a) If p and q are both even, we again write $p = 2p'$, $q = 2q'$. The index is divisible by 4 since it is given by $4p'q'|w|^2$. Note that p', q' may take any positive integral value, even or odd.

   (b) If p and q are both odd, the index is odd and given by $pq|w|^2$.

For fixed z, interchanging $p \neq q$ gives a rhombic-cr (and rectangular) lattice which is rotated through an angle $\pi/2$, hence we count no lattice twice if we let p, q run over all positive integers.

Let $\Phi_{\mathrm{even}}(s)$ be the Dirichlet series for the number of rhombic-cr and square sublattices of even index. This comprises cases (1) and (2a). As p', q' run over all positive integers, they each contribute a factor of $\zeta(s)$, and since z is primitive, this gives the factor $\Phi^{\mathsf{pr}}_{\square}(s)$, which is the Dirichlet series generating function of primitive similar sublattices of $\mathbb{Z}[i]$. The additional factor of 2 in the index formula gives a contribution of 2^{-s}, and combining all these factors finally yields

$$\Phi_{\mathrm{even}}(s) = \frac{1}{2^s}\zeta(s)^2\Phi^{\mathsf{pr}}_{\square}(s). \tag{3.2}$$

It remains to calculate the number of rhombic-cr and square sublattices of odd index, with generating function $\Phi_{\mathrm{odd}}(s)$. Here, p and q run over all odd positive integers and hence each contribute a factor of $(1-2^{-s})\zeta(s)$, whereas w runs over all primitive w with $|w|^2$ odd, and hence gives the contribution $\frac{1}{1+2^{-s}}\Phi^{\mathsf{pr}}_{\square}(s)$, so that we have

$$\Phi_{\mathrm{odd}}(s) = \frac{(1-2^{-s})^2}{1+2^{-s}}\zeta(s)^2\Phi^{\mathsf{pr}}_{\square}(s). \tag{3.3}$$

In total, the generating function $\Phi_{\Diamond+\Box}(s)$ for the number of all rhombic-cr and square sublattices is given by

$$\Phi_{\Diamond+\Box}(s) = \Phi_{\text{even}}(s) + \Phi_{\text{odd}}(s) = \frac{1-2^{-s}+2^{-2s+1}}{1+2^{-s}}\,\zeta(s)^2\,\Phi_{\Box}^{\mathsf{pr}}(s). \tag{3.4}$$

Via standard arguments involving Möbius inversion (see [6] and references therein), the number of *primitive* rhombic-cr and square sublattices together is given by

$$\Phi_{\Diamond+\Box}^{\mathsf{pr}}(s) = \frac{1}{\zeta(2s)}\,\Phi_{\Diamond+\Box}(s) = \frac{1-2^{-s}+2^{-2s+1}}{1+2^{-s}}\,\frac{\zeta(s)^2}{\zeta(2s)}\Phi_{\Box}^{\mathsf{pr}}(s). \tag{3.5}$$

Putting all this together, we obtain the generating functions $\Phi_{\Box}^{\mathsf{pr}}$, $\Phi_{\Diamond}^{\mathsf{pr}}$ and $\Phi_{\rectangle}^{\mathsf{pr}}$ for the number of primitive square, rhombic-cr and rectangular sublattices, respectively, as

$$\Phi_{\Box}^{\mathsf{pr}}(s) = (1+2^{-s})\prod_{p\equiv 1 \bmod 4}\frac{1+p^{-s}}{1-p^{-s}} = \frac{\zeta(s)L(s,\chi_{-4})}{\zeta(2s)}, \tag{3.6}$$

$$\Phi_{\Diamond}^{\mathsf{pr}}(s) = \left(\frac{1-2^{-s}+2^{-2s+1}}{1+2^{-s}}\,\frac{\zeta(s)^2}{\zeta(2s)} - 1\right)\Phi_{\Box}^{\mathsf{pr}}(s), \tag{3.7}$$

$$\Phi_{\rectangle}^{\mathsf{pr}}(s) = \left(\frac{\zeta(s)^2}{\zeta(2s)} - 1\right)\Phi_{\Box}^{\mathsf{pr}}(s), \tag{3.8}$$

with the L-series and the character χ_{-4} as above (see Appendix A for details and notation). Note that the last equation follows from the fact that the generating function for all rectangular lattices including the square lattices is given by $\zeta(s)^2\Phi_{\Box}^{\mathsf{pr}}(s)$.

Let us return to the well-rounded sublattices. Since z_1 and z_2 are shortest (non-zero) vectors, we have $|z_1 \pm z_2|^2 \geq |z_1|^2 = |z_2|^2$, which is equivalent to $\min(p^2,q^2) \geq (p^2+q^2)/4$, which in turn is equivalent to $3p^2 \geq q^2 \geq \frac{1}{3}p^2$. Note that this condition is also sufficient. Hence, we have to apply this extra condition to our considerations from above. We distinguish two cases:

(1) p and q are both even, $\sqrt{3}\,p \geq q \geq \frac{1}{\sqrt{3}}p$, and z may or may not be divisible by $1+i$. We write $p = 2p'$, $q = 2q'$, for which we likewise have $\sqrt{3}\,p' \geq q' \geq \frac{1}{\sqrt{3}}p'$. The index is even since it is given by $2p'q'|z|^2$. Here, p' and q' may take any positive integral values, even or odd, which satisfy $\sqrt{3}\,p' \geq q' \geq \frac{1}{\sqrt{3}}p'$. This corresponds to $\mathcal{E}, \mathcal{E}'$ in [11, (29) and (31)].

(2) p and q are both odd, $\sqrt{3}\,p \geq q \geq \frac{1}{\sqrt{3}}p$, and z is divisible by $1+i$. We write $z = (1+i)w$. The index is odd and given by $pq|w|^2$. This corresponds to $\mathcal{O}, \mathcal{O}'$ in [11, (30) and (32)].

The set of all possible indices of well-rounded sublattices is thus given by the union (we may interchange p and q if necessary)

$$\{2pq|z|^2 \mid q \le p \le \sqrt{3}\,q,\ z \in \mathbb{Z}[i]\} \cup \{pq|z|^2 \mid q \le p \le \sqrt{3}\,q,\ z \in \mathbb{Z}[i],\ 2 \nmid pq|z|^2\}. \tag{3.9}$$

It is a proper subset of Fukshansky's [11, Thms. 1.2, 3.6] index set

$$\mathcal{D} := \{pq|z|^2 \mid q \le p \le \sqrt{3}\,q,\ z \in \mathbb{Z}[i]\} \tag{3.10}$$

since $6 = 2 \cdot 3 \cdot |1|^2 \in \mathcal{D}$, but 6 is not contained in the set (3.9).

The Dirichlet series generating function for the well-rounded sublattices may now be calculated as above by taking the condition $\sqrt{3}\,p \ge q \ge \frac{1}{\sqrt{3}}p$ into account, so that the generating Dirichlet series for the well-rounded sublattices of even index is given by

$$\frac{1}{2^s} \sum_{p\in\mathbb{N}} \sum_{\frac{1}{\sqrt{3}}p<q<\sqrt{3}\,p} \frac{1}{p^s q^s} \Phi_{\square}^{\mathsf{pr}}(s). \tag{3.11}$$

Clearly, this sum is symmetric in p and q, and comprises the similar sublattices. In fact, if we exclude the square sublattices (those lattices with $p = q$) from (3.11) and note that $\sum_{p\in\mathbb{N}} \sum_{\frac{1}{\sqrt{3}}p<q<p} = \sum_{q\in\mathbb{N}} \sum_{q<p<\sqrt{3}\,q}$, we obtain the generating function for the rhombic lattices with even index as

$$\Phi_{\mathsf{wr},\mathrm{even}}(s) = \frac{2}{2^s} \sum_{p\in\mathbb{N}} \sum_{p<q<\sqrt{3}\,p} \frac{1}{p^s q^s} \Phi_{\square}^{\mathsf{pr}}(s). \tag{3.12}$$

The case of odd indices is slightly more cumbersome. Here, we have to replace the factor $(1 - 2^{-s})^2 \zeta(s)^2$ by the corresponding sum over all odd integers with $p < q < \sqrt{3}\,p$. Writing $p = 2k + 1$ and $q = 2\ell + 1$ turns our condition into $k < \ell < \sqrt{3}\,k + \frac{\sqrt{3}-1}{2}$. Since this inequality has no integral solution for $k = 0$, we may start our sum with $k = 1$, and finally arrive at

$$\Phi_{\mathsf{wr},\mathrm{odd}}(s) = \frac{2}{1 + 2^{-s}} \Phi_{\square}^{\mathsf{pr}}(s) \sum_{k\in\mathbb{N}} \sum_{k<\ell<\sqrt{3}\,k+(\sqrt{3}-1)/2} \frac{1}{(2k+1)^s (2\ell+1)^s}. \tag{3.13}$$

Now, $\Phi_{\mathsf{wr},\mathrm{even}}(s) + \Phi_{\mathsf{wr},\mathrm{odd}}(s) + \Phi_{\square}(s)$ gives the Dirichlet series generating function $\Phi_{\square,\mathsf{wr}}(s)$ for the arithmetic function $a_{\square}(n)$ counting the well-rounded sublattices of $\mathbb{Z}[i]$ of index n. To get a better understanding of it, we 'sandwich' it, on the half-axis $s > 1$, between two explicitly known meromorphic functions. All these Dirichlet series satisfy the conditions of Theorem 7 (see Appendix A). This gives a result on the asymptotic growth and its error as follows.

THEOREM 1. *Let $a_\square(n)$ be the number of well-rounded sublattices of index n in the square lattice, and $\Phi_{\square,\mathsf{wr}}(s) = \sum_{n=1}^{\infty} a_\square(n) n^{-s}$ the corresponding Dirichlet series generating function. The latter is given by*

$$\Phi_{\square,\mathsf{wr}}(s) = \Phi_\square(s) + \Phi_{\mathsf{wr},\mathrm{even}}(s) + \Phi_{\mathsf{wr},\mathrm{odd}}(s)$$

via (3.1), (3.12) *and* (3.13). *The generating function $\Phi_{\square,\mathsf{wr}}$ is meromorphic in the half-plane $\{\mathrm{Re}(s) > 1/2\}$, with a pole of order 2 at $s = 1$, and no other pole in $\{\mathrm{Re}(s) \geq 1\}$.*

If $s > 1$, we have the inequality

$$D_\square(s) - \Phi_\square(s) < \Phi_{\square,\mathsf{wr}}(s) < D_\square(s) + \Phi_\square(s),$$

with $\Phi_\square(s)$ from (3.1) *and*

$$D_\square(s) = \frac{2+2^s}{1+2^s}\,\frac{1-\sqrt{3}^{\,1-s}}{s-1}\,\frac{L(s,\chi_{-4})}{\zeta(2s)}\zeta(s)\zeta(2s-1).$$

As a consequence, the summatory function $A_\square(x) = \sum_{n \leq x} a_\square(n)$ has asymptotic behaviour

$$A_\square(x) = \frac{\log(3)}{2\pi} x \log(x) + \mathcal{O}(x\log(x)) \qquad \text{as } x \to \infty.$$

Proof. Clearly, $\Phi_{\square,\mathsf{wr}}(s)$ is the sum of $\Phi_\square(s)$ and the two contributions from (3.12) and (3.13). For real $s > 1$, the latter can both be bounded from below and above by an application of Lemma 4 from Appendix A with $\alpha = \sqrt{3}$, the former with parameters $\beta = \gamma = 0$ and the latter (after pulling out a factor of 2^s in the denominator) with $\beta = (\sqrt{3}-1)/2$ and $\gamma = 1/2$. A straightforward calculation leads to the explicit expression for the function $D_\square(s)$, as well as to the inequality stated.

It follows from the explicit expression for $D_\square(s)$ that it is a meromorphic function in the whole plane. Using the Euler summation formula, we see that $(\Phi_{\square,\mathsf{wr}}(s) - D_\square(s))/\Phi_\square^{\mathsf{pr}}(s)$ is an analytic function for $\mathrm{Re}(s) > 1/2$, guaranteeing that $\Phi_{\square,\mathsf{wr}}(s)$ is meromorphic in the half-plane $\{\mathrm{Re}(s) > 1/2\}$.

The rightmost singularity of $\zeta(s)\zeta(2s-1)$ is $s = 1$, with a pole of the form $1/(2(s-1)^2)$, while the entire factor of $D_\square(s)$ in front of it is analytic near $s = 1$ (as well as on the line $\{\mathrm{Re}(s) = 1\}$). An application of Theorem 7 from Appendix A now leads to the claimed asymptotics. ∎

The difference of the bounds in Theorem 1 is $2\Phi_\square(s)$, which is a Dirichlet series that itself allows an application of Theorem 7. The corresponding summatory function has the asymptotic behaviour of $cx + \mathcal{O}(x)$, which suggests that the error term of $A_\square(x)$ might be improved in this direction. However, it seems difficult to extract good error terms from Delange's theorem; compare the example in [8, Sec. 1.8]. Since numerical calculations support the above suggestion, we employed direct methods such as Dirichlet's hyperbola method ([1, Sec. 3.5] or [28, Sec. I.3]). A lengthy calculation (see [32] for the details) finally leads to the following result.

Theorem 2. *Let $a_{\square}(n)$ be the number of well-rounded sublattices of index n in the square lattice. Then the summatory function*

$$A_{\square}(x) = \sum_{n \le x} a_{\square}(n)$$

has asymptotic behaviour

$$\begin{aligned} A_{\square}(x) &= \frac{\log(3)}{3} \frac{L(1,\chi_{-4})}{\zeta(2)} x(\log(x) - 1) + c_{\square} x + \mathcal{O}(x^{3/4}\log(x)) \\ &= \frac{\log(3)}{2\pi} x \log(x) + \left(c_{\square} - \frac{\log(3)}{2\pi}\right) x + \mathcal{O}(x^{3/4}\log(x)) \end{aligned}$$

where, with γ denoting the Euler–Mascheroni constant,

$$\begin{aligned} c_{\square} := \frac{L(1,\chi_{-4})}{\zeta(2)} \Bigg(& \zeta(2) + \frac{\log(3)}{3}\left(\frac{L'(1,\chi_{-4})}{L(1,\chi_{-4})} + \gamma - 2\frac{\zeta'(2)}{\zeta(2)}\right) \\ & + \frac{\log(3)}{3}\left(2\gamma - \frac{\log(3)}{4} - \frac{\log(2)}{6}\right) - \sum_{p=1}^{\infty} \frac{1}{p}\Bigg(\frac{\log(3)}{2} - \sum_{p<q<p\sqrt{3}} \frac{1}{q}\Bigg) \\ & - \frac{4}{3}\sum_{k=0}^{\infty} \frac{1}{2k+1}\Bigg(\frac{1}{4}\log(3) - \sum_{k<\ell<k\sqrt{3}+(\sqrt{3}-1)/2} \frac{1}{2\ell+1}\Bigg)\Bigg) \\ \approx 0.6272237 & \end{aligned}$$

is the coefficient of $(s-1)^{-1}$ in the Laurent series of $\sum_{n\ge1} a_{\square}(n) n^{-s}$ around $s=1$.

Note that $L'(1,\chi_{-4})$ can be computed efficiently via

$$\frac{L'(1,\chi_{-4})}{L(1,\chi_{-4})} = \log\left(M(1,\sqrt{2})^2 \frac{e^{\gamma}}{2}\right) = \log\left(\Gamma\left(\frac{3}{4}\right)^4 \frac{e^{\gamma}}{\pi}\right) \approx 0.2456096, \tag{3.14}$$

where $M(x,y)$ is the arithmetic-geometric mean of x and y, and Γ denotes the gamma function (see [22] and references therein).

Sketch of proof of Theorem 2. Observe that $\Phi_{\square,\mathsf{wr}}(s) = \sum_{n=1}^{\infty} a_{\square}(n) n^{-s}$ is a sum of three Dirichlet series, each of which is itself a product of several Dirichlet series. Hence, each contribution to $a_{\square}(n)$ is a Dirichlet convolution of arithmetic functions. The asymptotic behaviour can thus be calculated by elementary methods as described in [1, Sec. 3.5], making use of Euler's summation formula (A.6) wherever appropriate. To be more specific, let

$$\Phi_{\mathsf{wr},\text{even}}(s) = \sum_{n\in\mathbb{N}} \frac{a_{\text{even}}(n)}{n^s}, \tag{3.15}$$

which is a product of the Dirichlet series

$$\frac{2}{2^s}\frac{1}{\zeta(2s)} = \sum_{n\in\mathbb{N}} \frac{c(n)}{n^s}, \qquad \sum_{p\in\mathbb{N}} \sum_{p<q<\sqrt{3}\,p} \frac{1}{p^s q^s} = \sum_{n\in\mathbb{N}} \frac{w(n)}{n^s},$$

$$\Phi_{\Box}(s) = \sum_{n\in\mathbb{N}} \frac{b(n)}{n^s}.$$

Hence $a_{\text{even}} = c*w*b$ is the Dirichlet convolution of c, w, b. The summatory function of a Dirichlet convolution $f*g$ can now be calculated via the classical formulas (cf. [1] and [28, Sec. I.3.2])

$$\sum_{n\le x}(f*g)(n) = \sum_{m\le x}\sum_{d\le x/m} f(m)g(d) \tag{3.16}$$

$$= \sum_{m\le\sqrt{x}}\sum_{m<d\le x/m} \big(f(m)g(d)+f(d)g(m)\big) + \sum_{m\le\sqrt{x}} f(m)g(m), \tag{3.17}$$

where the latter is used for the convolutions $w*b$ and $b = \chi_{-4}*1$. ∎

4. Well-rounded sublattices of $\mathbb{Z}[\rho]$. Next, we consider the hexagonal lattice $\mathbb{Z}[\rho]$ with $\rho = \frac{1+i\sqrt{3}}{2}$. As an arithmetic object, it is the ring of Eisenstein integers, the maximal order of the quadratic field $\mathbb{Q}(i\sqrt{3}\,)$. The Dirichlet series generating function for the number of similar sublattices of $\mathbb{Z}[\rho]$ is

$$\Phi_{\triangle}(s) = \zeta_{\mathbb{Q}(\rho)}(s) = L(s,\chi_{-3})\zeta(s), \tag{4.1}$$

with the character

$$\chi_{-3}(n) = \begin{cases} 0 & \text{if } n\equiv 0 \bmod 3,\\ 1 & \text{if } n\equiv 1 \bmod 3,\\ -1 & \text{if } n\equiv 2 \bmod 3; \end{cases}$$

see [6, 30] and Appendix A.

Let $\{z_1, z_2\}$ be a reduced basis of a well-rounded sublattice of $\mathbb{Z}[\rho]$. The orthogonality of z_1+z_2 and z_1-z_2 implies that $\frac{z_1+z_2}{z_1-z_2} = i\sqrt{3}\,r$ with $r\in\mathbb{Q}$. This shows that square lattices cannot occur here since this would require $|z_1+z_2|^2 = |z_1-z_2|^2$, which is impossible. Thus, the well-rounded sublattices of $\mathbb{Z}[\rho]$ are rhombic-cr or hexagonal lattices. However, at least one of z_1+z_2 and z_1-z_2 is divisible by $i\sqrt{3} = \rho-\bar\rho$, and without loss of generality we may assume that $i\sqrt{3}$ divides z_1-z_2. Hence, there exist $p,q\in\mathbb{Z}$ together with a primitive $z\in\mathbb{Z}[\rho]$ such that $z_1+z_2 = pz$ and $z_1-z_2 = i\sqrt{3}\,qz$. Here, *primitive* means that $n=1$ is the only integer $n\in\mathbb{N}$ that divides z. We may again choose p and q positive, and thus

$$\Gamma = \langle z_1, z_2 \rangle_{\mathbb{Z}} = \left\langle \frac{p + i\sqrt{3}\,q}{2} z, \frac{p - i\sqrt{3}\,q}{2} z \right\rangle_{\mathbb{Z}} \\ = \left\langle \left(\frac{p-q}{2} + \rho q \right) z, \left(\frac{p+q}{2} - \rho q \right) z \right\rangle_{\mathbb{Z}} \tag{4.2}$$

is a sublattice of index $pq|z|^2$. In particular, Γ is the hexagonal lattice if and only if $p = q$ or $p = 3q$. Note that (4.2) shows that p and q have the same parity.

Well-rounded sublattices have to satisfy the additional inequality constraints $|z_1 \pm z_2|^2 \geq |z_1|^2 = |z_2|^2$, which, in this case, are equivalent to $q \leq p \leq 3q$. The set of possible indices of well-rounded sublattices is thus given by

$$\{4pq|z|^2 \mid q \leq p \leq 3q,\ z \in \mathbb{Z}[\rho]\} \\ \cup \{pq|z|^2 \mid q \leq p \leq 3q,\ z \in \mathbb{Z}[\rho],\ 2 \nmid pq\}. \tag{4.3}$$

An alternative parametrisation of this set can be found in [13, Cor. 4.9]. The equivalence of these formulations can easily be checked by recalling that the (rational) primes represented by the norm form $m^2 - mn + n^2$ of $\mathbb{Z}[\rho]$ are precisely 3 and all primes $p \equiv 1 \pmod 3$.

Counting the number of distinct well-rounded sublattices of a given index works essentially as in the square lattice case. However, we have to avoid counting the same lattice twice. Let z be divisible by $i\sqrt{3}$, so that $z = i\sqrt{3}\,w$. Then

$$z_1 = \frac{p + i\sqrt{3}\,q}{2} z = -\frac{3q - i\sqrt{3}\,p}{2} w, \tag{4.4}$$

$$z_2 = \frac{p - i\sqrt{3}\,q}{2} z = \frac{3q + i\sqrt{3}\,p}{2} w \tag{4.5}$$

shows that the tuples (p, q, z) and $(3q, p, w)$ correspond to the same sublattice. Thus, we only sum over primitive z that are not divisible by $i\sqrt{3}$.

Since we already know from [3] the generating function (4.1) for the similar sublattices, we concentrate on the rhombic sublattices here (excluding hexagonal sublattices, as before). The summation over all primitive $z \in \mathbb{Z}[\rho]$ not divisible by $i\sqrt{3}$ gives the contribution $\frac{1}{1+3^{-s}} \Phi^{\mathsf{pr}}_{\triangle}(s)$. The generating function of all rhombic sublattices of even index then reads

$$\Phi_{\triangle,\mathsf{wr},\mathrm{even}}(s) = \frac{3}{4^s(1+3^{-s})} \sum_{p\in\mathbb{N}} \sum_{p<q<3p} \frac{1}{p^s q^s}\, \Phi^{\mathsf{pr}}_{\triangle}(s), \tag{4.6}$$

where the factor of 3 reflects that each sublattice occurs in three different orientations.

For odd indices, we substitute again $p = 2k + 1$ and $q = 2\ell + 1$, so that our constraints read $k < \ell < 3k + 1$. This leads to the following expression for the generating function of all rhombic sublattices of odd index:

$$\Phi_{\triangle,\mathsf{wr},\mathrm{odd}}(s) = \frac{3}{1+3^{-s}} \sum_{k\in\mathbb{N}} \sum_{k<\ell<3k+1} \frac{1}{(2k+1)^s(2\ell+1)^s}\, \Phi_{\triangle}^{\mathsf{pr}}(s). \tag{4.7}$$

Now, we can apply the same strategy as in the square lattice case.

THEOREM 3. *Let $a_\triangle(n)$ be the number of well-rounded sublattices of index n in the hexagonal lattice, and $\Phi_{\triangle,\mathsf{wr}}(s) = \sum_{n=1}^{\infty} a_\triangle(n)n^{-s}$ the corresponding Dirichlet series generating function, given by*

$$\Phi_{\triangle,\mathsf{wr}}(s) = \Phi_\triangle(s) + \Phi_{\triangle,\mathsf{wr},\mathrm{even}}(s) + \Phi_{\triangle,\mathsf{wr},\mathrm{odd}}(s),$$

with the summands from (4.1), (4.6) *and* (4.7).

If $s > 1$, then

$$D_\triangle(s) - E_\triangle(s) < \Phi_{\triangle,\mathsf{wr}}(s) < D_\triangle(s),$$

with

$$D_\triangle(s) = \frac{1}{2}\,\frac{3}{1+3^{-s}}\,\frac{1-3^{1-s}}{s-1}\,\frac{L(s,\chi_{-3})}{\zeta(2s)}\zeta(s)\zeta(2s-1),$$
$$E_\triangle(s) = \frac{3}{1+3^{-s}}L(s,\chi_{-3})\zeta(s).$$

The function $\Phi_{\triangle,\mathsf{wr}}(s)$ is meromorphic in the half-plane $\{\mathrm{Re}(s) > 1/2\}$, with a pole of order 2 at $s = 1$, and no other pole in $\{\mathrm{Re}(s) \geq 1\}$. As a consequence, the summatory function $A_\triangle(x) = \sum_{n\leq x} a_\triangle(n)$, as $x \to \infty$, has asymptotic behaviour

$$A_\triangle(x) = \frac{3\sqrt{3}\log(3)}{8\pi} x\log(x) + o(x\log(x)).$$

Sketch of proof. As before, $\Phi_{\triangle,\mathsf{wr}}(s)$ is the sum of the contributions from (4.6) and (4.7). The calculation of the upper and lower bounds can be done as in Theorem 1 via Lemma 4, this time with $\alpha = 3$ and appropriate choices for β and γ. The conclusion on the asymptotics of $A_\triangle(x)$ follows as before from Theorem 7. ∎

As for the square lattice, we can improve the error term considerably by lengthy but elementary calculations (see [32] for the details). Eventually, we obtain the following result.

THEOREM 4. *Let $a_\triangle(n)$ be the number of well-rounded sublattices of index n in the hexagonal lattice. Then the corresponding summatory function $A_\triangle(x) = \sum_{n\leq x} a_\triangle(n)$ has asymptotic behaviour*

$$\begin{aligned} A_\triangle(x) &= \frac{9\log(3)}{16}\,\frac{L(1,\chi_{-3})}{\zeta(2)} x(\log(x) - 1) + c_\triangle x + \mathcal{O}(x^{3/4}\log(x)) \\ &= \frac{3\sqrt{3}\log(3)}{8\pi} x(\log(x) - 1) + c_\triangle x + \mathcal{O}(x^{3/4}\log(x)), \end{aligned}$$

where

$$c_\triangle = L(1,\chi_{-3}) + \frac{9\log(3)L(1,\chi_{-3})}{16\zeta(2)}\Bigg(\Bigg(\gamma + \frac{L'(1,\chi_{-3})}{L(1,\chi_{-3})} - 2\frac{\zeta'(2)}{\zeta(2)}\Bigg) + 2\gamma - \frac{\log(3)}{4} - \sum_{p=1}^{\infty}\frac{1}{p}\Bigg(\log(3) - \sum_{p<q\le 3p-1}\frac{1}{q}\Bigg) - \sum_{k=0}^{\infty}\frac{4}{2k+1}\Bigg(\frac{1}{2}\log(3) - \sum_{k<\ell\le 3k}\frac{1}{2\ell+1}\Bigg)\Bigg)$$
$$\approx 0.4915036$$

is the coefficient of $(s-1)^{-1}$ *in the Laurent series of* $\sum_n a_\triangle(n)/n^s$ *around* $s=1$. ■

The number $L'(1,\chi_{-3})$ can be computed efficiently as well, via a formula involving the arithmetic-geometric mean (see [22]), and reads

$$\frac{L'(1,\chi_{-3})}{L(1,\chi_{-3})} = \log\bigg(\frac{2^{3/4}M(1,\cos(\pi/12))^2e^\gamma}{3}\bigg) = \log\bigg(\frac{2^4\pi^4e^\gamma}{3^{3/2}\Gamma(1/3)^6}\bigg) \approx 0.3682816. \tag{4.8}$$

Above and in the previous section, we have seen that the asymptotic behaviour for the hexagonal as well as for the square lattice is of the form $c_1x\log(x) + c_2x + \mathcal{O}(x^{3/4}\log(x))$. Actually, numerical calculations suggest that the error term is $\mathcal{O}(x^{1/2})$ or maybe even slightly better.

Let us now see what we can say about the other planar lattices.

5. The general case

5.1. Existence of well-rounded sublattices. Recall from Section 2 that a lattice allows a well-rounded sublattice if and only if it contains a rectangular or square sublattice. The following lemma contains several reformulations of this property.

Lemma 1. *Let Γ be any planar lattice. There are natural bijections between the following objects:*

(1) Rational orthogonal frames *for Γ, that is, unordered pairs $\mathbb{Q}w, \mathbb{Q}z$ of perpendicular ($w \perp z$), one-dimensional subspaces of the rational space $\mathbb{Q}\Gamma$ generated by Γ (so we may assume $w, z \in \Gamma$).*
(2) *Unordered pairs $\{\pm R\}$ of coincidence reflections of Γ; from now on, we shall simply write $\pm R$ for such a pair.*
(3) Basic *rectangular or square sublattices $\Lambda \subseteq \Gamma$, where 'basic' means that $\Lambda = \langle w, z\rangle_{\mathbb{Z}}$ with w, z primitive in Γ (so $\mathbb{Q}w \cap \Gamma = \mathbb{Z}w$ and $\mathbb{Q}z \cap \Gamma = \mathbb{Z}z$). We shall call them* BRS sublattices *for short.*

(4) *Four-element subsets* $\{\pm w, \pm z\} \subset \Gamma$ *of non-zero primitive lattice vectors with* $w \perp z$.

Given Γ, we use the notation $\mathcal{R} = \mathcal{R}_\Gamma$ for the set of all pairs $\pm R$ of coincidence reflections of Γ. So $\mathcal{R}_\Gamma$ is in natural bijection with any of the four sets described in Lemma 1. For the rest of the paper, we introduce the following notation, based on Lemma 1. For $\pm R \in \mathcal{R}_\Gamma$, we denote by Γ_R (rather than $\Gamma_{\pm R}$) the corresponding BRS sublattice. Explicitly, this is

$$\begin{aligned}\Gamma_R &= \Gamma \cap \mathrm{Fix}(R) \oplus \Gamma \cap \mathrm{Fix}(-R)\\ &= \mathbb{Z}w \oplus \mathbb{Z}z, \qquad \text{where } Rw = w,\ Rz = -z\end{aligned}$$

(thus w, z are primitive in Γ). In accordance with part (2) of Lemma 1, we have $\Gamma_R = \Gamma_{-R}$, with the roles of w and z interchanged. If we start with an arbitrary primitive vector $w \in \Gamma$, we similarly write

$$\Gamma_w := \mathbb{Z}w \oplus \mathbb{Z}z, \qquad \text{where } z \perp w \text{ and } z \text{ is primitive in } \Gamma.$$

The four-element set $\{\pm w, \pm z\}$ is uniquely determined by any of its members, and Γ_w is the unique BRS sublattice belonging to this set, according to part (4) of the lemma.

In addition to Γ_R, there is a second sublattice of Γ which is invariant under R and contains w, z as primitive vectors. This is

$$\widetilde{\Gamma}_R := \left\langle \frac{w+z}{2}, \frac{w-z}{2} \right\rangle_{\mathbb{Z}}, \tag{5.1}$$

the unique superlattice of Γ_R containing Γ_R with index 2 in such a way that w, z are still primitive in $\widetilde{\Gamma}_R$. By the way, it is a purely algebraic fact that, if R is a non-trivial automorphism of order 2 of an abstract lattice Λ (free $\mathbb{Z}$-module) of rank 2, i.e. $R^2 = \mathrm{id} \neq \pm R$, then either Λ has a $\mathbb{Z}$-basis w, z of eigenvectors of R (so $Rz = z$, $Rw = -w$), or Λ has a $\mathbb{Z}$-basis u, v with $Ru = v$. Thus, already on the level of abstract reflections, one can distinguish between 'rectangular type' and 'rhombic type' of a reflection acting on a lattice. In the situation considered above, the reflection R on Γ_R is of rectangular type, and so the lattice Γ_R itself is of rectangular or square Bravais type, whereas the reflection R on $\widetilde{\Gamma}_R$ is of rhombic type, which implies that $\widetilde{\Gamma}_R$ is of rhombic-cr, square or hexagonal Bravais type. The significance of $\widetilde{\Gamma}_R$ is explained by the following lemma.

Lemma 2. *Given* Γ *and* $\pm R \in \mathcal{R}_\Gamma$ *as above, let* $\Lambda \supseteq \Gamma_R = \langle w, z \rangle$ *be an* R*-invariant superlattice containing* w, z *as primitive vectors. Then either* $\Lambda = \Gamma_R$ *or* $\Lambda = \widetilde{\Gamma}_R$.

Proof. Since z is primitive, Λ has a $\mathbb{Z}$-basis u, z, where u is of the form $u = \frac{1}{m}w + \frac{k}{m}z$ with $m = [\Lambda : \Gamma_R]$ and $0 \le k < m$. The condition $Ru \in \Lambda$ immediately leads to $m \in \{1, 2\}$ and $k \in \{0, 1\}$, where $k = 1$ for $m = 2$. ∎

LEMMA 3. *Given Γ and $\pm R \in \mathcal{R}_\Gamma$ as above, $\widetilde{\Gamma}_R$ is contained in Γ if and only if the index $[\Gamma : \Gamma_R]$ is even.*

Proof. If $[\Gamma : \Gamma_R] = [\Gamma : \langle w, z\rangle]$ is even and $\frac{1}{2}(aw + bz)$ with $a, b \in \{0,1\}$ represents an element of order 2 in the factor group Γ/Γ_R, then, since $w/2, z/2 \notin \Gamma$, we must have $a = b = 1$, leading to the sublattice $\widetilde{\Gamma}_R$. The converse is clear. ∎

COROLLARY 1. *For any pair of coincidence reflections $\pm R \in \mathcal{R}_\Gamma$, the coincidence site lattice $\Gamma(R) = \Gamma \cap R\Gamma$ is equal to Γ_R or to $\widetilde{\Gamma}_R$. The latter occurs if and only if the index $[\Gamma : \Gamma_R]$ is even.* ∎

The following basic result partitions the set of all planar lattices admitting a well-rounded (or rectangular) sublattice into two disjoint classes, as announced at the end of the introduction. Clearly, a rational lattice has infinitely many BRS sublattices, since for any non-zero lattice vector v, the orthogonal subspace of v also contains a non-zero lattice vector (simply by solving a linear equation with rational coefficients). In contrast, the non-rational case can be analysed as follows.

PROPOSITION 1. *Let Γ be non-rational planar lattice which has a rectangular sublattice, so that $\mathcal{R}_\Gamma \neq \emptyset$ by Lemma 1. Then $|\mathcal{R}_\Gamma| = 1$, whence Γ has exactly one BRS sublattice, and one pair of coincidence reflections.*

Proof. Observe that Γ has a sublattice Λ with an orthogonal basis v, w, where we may assume $|v| = 1$ and $|w|^2 = c > 0$. Now assume that there is a further vector $u = rv + sw$ with $rs \neq 0$ admitting an orthogonal, non-zero vector $u' = r'v + s'w$. Then $rr' + css' = 0$ and necessarily $s' \neq 0$, thus $c = -rr'/ss' \in \mathbb{Q}$. Therefore Λ, and thus also Γ, is rational. ∎

The previous result (with a slightly more complicated proof) is also found in [17, Lemma 2.5 and Remark 2.6]. Our approach suggests the following distinction of cases.

PROPOSITION 2. *Let $\Gamma = \langle 1, \tau\rangle_\mathbb{Z}$ be a lattice in $\mathbb{R}^2 \simeq \mathbb{C}$, and write $n = |\tau|^2$ and $t = \tau + \bar{\tau}$. Then Γ has a well-rounded sublattice if and only if one of the following conditions is satisfied:*

(1) *Γ is rational, i.e. both t and n are rational;*
(2) *t is rational, but n is not;*
(3) *t is irrational, and there exist $q, r \in \mathbb{Q}$ with $\sqrt{q + r^2} \in \mathbb{Q}$ and with $n = q + rt$.*

Note that case (3) includes both rational and irrational n. If n is rational, it has then to be a rational square.

Proof of Proposition 2. Recall that Γ has a well-rounded sublattice if and only if it has a rectangular or a square sublattice. This happens if and only if there exist integers a, b, c, d such that the non-zero vectors $a + b\tau$ and $c + d\tau$ are orthogonal. The latter condition holds if and only if

$$ac + bdn + (ad + bc)t/2 = 0 \tag{5.2}$$

has a non-trivial integral solution, where $n = |\tau|^2$ and $t = \tau + \bar{\tau}$ are the norm and the trace of τ, respectively. In fact, there exists an integral solution if and only if there exists a rational one. This leads to the following three cases:

(1) Clearly, (5.2) has a solution if both t and n are rational.
(2) Let $t \in \mathbb{Q}, n \notin \mathbb{Q}$. Then condition (5.2) is equivalent to the relations $bd = 0 = ac + (ad + bc)t/2$. With $t/2 = p/q$, $p, q \in \mathbb{Z}$, an integer solution is given by $a = 1$, $b = 0$, $c = p$, $d = -q$.
(3) Let $t \notin \mathbb{Q}$ with $n = q + rt$. As $n > 0$, at least one of q and r is non-zero. Here, condition (5.2) is equivalent to $ac + bdq = 0$ and $2bdr + (ad + bc) = 0$. As $a = c = 0$ would imply $a + b\tau = 0$ or $c + d\tau = 0$, we may assume without loss of generality that $a \neq 0$. This gives $c = -bdq/a$ and $1 + 2br/a - (b/a)^2 q = 0$, where we have assumed $d \neq 0$ in the latter equation (since otherwise $c + d\tau = 0$). The equation has a rational solution if and only if $r^2 + q$ is a square.

Finally, we have to check that the remaining case does not allow for integral solutions. Let t and n be irrational and assume that they are independent over $\mathbb{Q}$. This clearly requires $ac = bd = ad + bc = 0$, which implies $a + b\tau = 0$ or $c + d\tau = 0$. ∎

Remark 1. After we had arrived at Proposition 2, we became aware of an essentially equivalent result by Kühnlein [17, Lemma 2.5], where the invariant $\delta(\Gamma) = \dim\langle 1, t, n\rangle_{\mathbb{Q}}$ is introduced. Clearly, condition (1) of Proposition 2 is equivalent to $\delta(\Gamma) = 1$, and our conditions (2) and (3) are equivalent to $\delta(\Gamma) = 2$ together with the condition that Kühnlein's 'strange invariant' $\sigma(\Gamma)$ is the class of all squares in $\mathbb{Q}^{\times}$. Here, $\sigma(\Gamma)$ is the square class of $-\det(X)$, where $X = \left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$ is a non-trivial integral matrix satisfying $\operatorname{tr}(XG) = 0$, with $G = \left(\begin{smallmatrix} 1 & t/2 \\ t/2 & n \end{smallmatrix}\right)$ being the Gram matrix of Γ. Altogether, this shows that our criterion is equivalent to Kühnlein's.

In the situation of Proposition 1, let R be the unique (up to sign) coincidence reflection and $\Gamma_R = \langle w, z\rangle$ the unique BRS sublattice. We get all well-rounded sublattices by considering the rectangular sublattices generated by kw, ℓz with the constraint

$$k \frac{1}{\sqrt{3}} \frac{|w|}{|z|} \leq \ell \leq k\sqrt{3}\, \frac{|w|}{|z|}, \tag{5.3}$$

whose superlattice $\left\langle \frac{1}{2}kw \pm \frac{1}{2}\ell z\right\rangle_{\mathbb{Z}}$ is a sublattice of Γ. The latter requires that k and ℓ have the same parity. By Lemma 3, odd values k, ℓ occur if and only if the index $\sigma = \sigma_\Gamma := [\Gamma : \Gamma_R]$ is even. This gives the following result.

PROPOSITION 3. *Let $\varGamma$ be a lattice that has a well-rounded sublattice and assume that $\varGamma$ is not rational* (*cf. Proposition* 1). *Let σ be the index of its unique BRS sublattice $\varGamma_R$, and κ be the ratio of the lengths of its orthogonal basis vectors. The generating function for the number of well-rounded sublattices then reads as follows:*

(1) *If σ is odd, one has*

$$\varPhi_{\varGamma,\mathsf{wr}}(s) = \frac{1}{\sigma^s}\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s)$$

with

$$\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s) = \frac{1}{2^s}\sum_{k\in\mathbb{N}}\ \sum_{\frac{\kappa}{\sqrt{3}}k\le\ell\le\sqrt{3}\,\kappa k}\frac{1}{k^s\ell^s}.$$

(2) *If σ is even, one has*

$$\varPhi_{\varGamma,\mathsf{wr}}(s) = \frac{1}{\sigma^s}\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s) + \frac{2^s}{\sigma^s}\phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s)$$

with $\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s)$ as above and

$$\phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s) = \sum_{k\in\mathbb{N}}\ \sum_{\frac{\kappa}{\sqrt{3}}(k+\frac{1}{2})-\frac{1}{2}\le\ell\le\sqrt{3}\,\kappa(k+\frac{1}{2})-\frac{1}{2}}\frac{1}{(2k+1)^s(2\ell+1)^s}.$$

REMARK 2. The quantity $\kappa = |w|/|z|$ is unique up to taking its inverse. Note that $\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s) = \phi_{\mathsf{wr},\mathrm{even}}(1/\kappa;s)$ and $\phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s) = \phi_{\mathsf{wr},\mathrm{odd}}(1/\kappa;s)$. Hence, there is no ambiguity in the definition of the generating functions.

In the cases of the square and hexagonal lattices we have been able to give lower and upper bounds for the generating functions $\varPhi_{\mathsf{wr}}$. In a similar way we obtain the following result.

REMARK 3. We have the following inequalities for real $s>1$:

$$D_{\mathrm{even}}(\kappa;s) - E_{\mathrm{even}}(\kappa;s) < \phi_{\mathsf{wr},\mathrm{even}}(\kappa;s) < D_{\mathrm{even}}(\kappa;s) + E_{\mathrm{even}}(\kappa;s),$$
$$D_{\mathrm{odd}}(\kappa;s) - E_{\mathrm{odd}}(\kappa;s) < \phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s) < D_{\mathrm{odd}}(\kappa;s) + E_{\mathrm{odd}}(\kappa;s),$$

with the generating functions

$$D_{\mathrm{even}}(\kappa;s) = \frac{1}{2^s}\left(\frac{\sqrt{3}}{\kappa}\right)^{s-1}\frac{1-3^{1-s}}{s-1}\zeta(2s-1),$$
$$E_{\mathrm{even}}(\kappa;s) = \frac{1}{2^s}\left(\frac{\sqrt{3}}{\kappa}\right)^{s}\zeta(2s),$$
$$D_{\mathrm{odd}}(\kappa;s) = \frac{1}{2}\left(\frac{\sqrt{3}}{\kappa}\right)^{s-1}\frac{1-3^{1-s}}{s-1}\left(1-\frac{1}{2^{2s-1}}\right)\zeta(2s-1),$$
$$E_{\mathrm{odd}}(\kappa;s) = \left(\frac{\sqrt{3}}{\kappa}\right)^{s}\left(1-\frac{1}{2^{2s}}\right)\zeta(2s).$$

Let us now take a closer look at the analytic properties of $\Phi_{\Gamma,\mathsf{wr}}$. Before formulating the theorem, we observe that the two cases of Proposition 3 can be unified by considering the index $\Sigma := [\Gamma : \Gamma(R)]$ of the unique non-trivial CSL in Γ. By Corollary 1, $\sigma = \Sigma$ if σ is odd and $\sigma = 2\Sigma$ if σ is even. We can now formulate a refinement of [17, Lemma 3.3 and Corollary 3.4] as follows.

PROPOSITION 4. *Let Γ be a lattice with a well-rounded sublattice and assume that Γ is not rational, so that Γ has exactly one non-trivial CSL. Let Σ be its index in Γ. Then the generating function $\Phi_{\Gamma,\mathsf{wr}}$ for the number of well-rounded sublattices has an analytic continuation to the open half-plane* $\{\mathrm{Re}(s) > 1/2\}$ *except for a simple pole at $s = 1$, with residue* $\log(3)/(4\Sigma)$.

Proof. We proceed as in the proof of Theorem 1 by applying Euler's summation formula to the inner sum. This shows that $\phi_{\mathsf{wr,even}}(\kappa; s) - D_{\mathrm{even}}(\kappa; s)$ and $\phi_{\mathsf{wr,odd}}(\kappa; s) - D_{\mathrm{odd}}(\kappa; s)$ are both analytic in $\{\mathrm{Re}(s) > 1/2\}$. Moreover, the explicit formulas above show that both $D_{\mathrm{even}}(\kappa; s)$ and $D_{\mathrm{odd}}(\kappa; s)$ are analytic in the whole complex plane except at $s = 1$, where they have a simple pole with residue $\log(3)/4$ and $\log(3)/8$, respectively. Inserting this result into the expressions for $\Phi_{\Gamma,\mathsf{wr}}(s)$, we compute the residue at $s = 1$ to be $\log(3)/(4\Sigma)$, where we have used the fact that $\sigma = \Sigma$ if σ is odd and $\sigma = 2\Sigma$ if σ is even. ∎

Using similar arguments to those in the proofs of Theorems 1 and 2, one can derive from Proposition 4 the asymptotic behaviour of the number of well-rounded sublattices as follows.

THEOREM 5. *Under the assumptions of Proposition 4, the summatory function $A_\Gamma(x) = \sum_{n\le x} a_\Gamma(n)$ has asymptotic behaviour*

$$A_\Gamma(x) = \frac{\log(3)}{4\Sigma} x + \mathcal{O}(\sqrt{x})$$

as $x \to \infty$. ∎

5.2. The rational case. A rational lattice Γ contains infinitely many BRS sublattices Γ_R. Using the same considerations as in the previous subsection, for any given pair $\pm R$ we can count the number of well-rounded sublattices invariant under $\pm R$ (that is, contained in $\widetilde{\Gamma}_R$). Counting all possible well-rounded sublattices then amounts to summing over all possible pairs $\pm R$. However, some care is needed in the case of square and hexagonal lattices.

For convenience, we will use the notation $\mathcal{R}_1 := \{\pm R \mid \widetilde{\Gamma}_R \nsubseteq \Gamma\}$ and $\mathcal{R}_2 := \{\pm R \mid \widetilde{\Gamma}_R \subseteq \Gamma\}$, which, by Lemma 3, is a partition of $\mathcal{R}$ into sets of odd and even index of Γ_R, which is reflected by the subscripts 1 and 2.

PROPOSITION 5. *Let Γ be a rational lattice and let $\Phi^{\triangle}_{\Gamma}(s)$ be the generating function of all hexagonal sublattices of Γ. Now, for any pair of* *coincidence reflections* $\pm R \in \mathcal{R}_\Gamma$*, let* $\sigma(R) = [\Gamma : \Gamma_R]$ *and let* $\kappa(R)$ *be the length ratio of orthogonal basis vectors of* Γ_R*. Then the generating function for the number of well-rounded sublattices is*

$$\begin{aligned}(5.4)\qquad \Phi_{\Gamma,\mathsf{wr}}(s) = &\sum_{\pm R\in\mathcal{R}_1} \frac{1}{\sigma(R)^s}\phi_{\mathsf{wr},\mathrm{even}}(\kappa(R);s)\\ &+ \sum_{\pm R\in\mathcal{R}_2} \frac{1}{\sigma(R)^s}\big(\phi_{\mathsf{wr},\mathrm{even}}(\kappa(R);s) + 2^s\phi_{\mathsf{wr},\mathrm{odd}}(\kappa(R);s)\big)\\ &- 2\Phi_\Gamma^{\triangle}(s),\end{aligned}$$

where $\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s)$ *and* $\phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s)$ *are as in Proposition* 3.

Keep in mind that we sum over pairs of coincidence reflections $\pm R$ here. According to Lemma 1, we could alternatively sum over BRS sublattices or rational orthogonal frames. Furthermore, note that $\Phi_\Gamma^{\triangle}(s) = 0$ unless Γ is commensurate to a hexagonal lattice.

Before proving Proposition 5, let us have a closer look at some special cases.

Remark 4. If Γ is not commensurate to a square or a hexagonal lattice, all well-rounded sublattices are rhombic. Likewise, all CSLs $\Gamma(R)$ generated by a reflection are either rectangular or rhombic-cr. In fact, there exists a bijection between BRS sublattices Γ_R and the corresponding CSLs $\Gamma(R)$, which implies that the summation in (5.4) could be carried out over CSLs as well. In particular, we have $\mathcal{R}_1 = \mathcal{R}_{\mathrm{rec}} := \{\pm R \mid \Gamma(R) \text{ rectangular}\}$ and $\mathcal{R}_2 = \mathcal{R}_{\text{rh-cr}} := \{\pm R \mid \Gamma(R) \text{ rhombic-cr}\}$ by Lemma 3.

The case that Γ is commensurate to a hexagonal lattice is the only one where the additional term $-2\Phi_\Gamma^{\triangle}(s)$ is non-trivial, which compensates for the fact that the sum over $\pm R \in \mathcal{R}_2$ counts every hexagonal sublattice thrice. Here, we do not have the bijection between the BRS sublattices Γ_R and the CSLs $\Gamma(R)$ any more, and the sums cannot be replaced by sums over CSLs. Still, we have a characterisation of the sets $\mathcal{R}_1$ and $\mathcal{R}_2$ via CSLs, namely $\mathcal{R}_1 = \mathcal{R}_{\mathrm{rec}} := \{\pm R \mid \Gamma(R) \text{ rectangular}\}$ and $\mathcal{R}_2 = \mathcal{R}_{\text{rh-cr-hex}} := \{\pm R \mid \Gamma(R) \text{ rhombic-cr or hexagonal}\}$.

If Γ is commensurate to a square lattice, no simple characterisation of $\mathcal{R}_1$ and $\mathcal{R}_2$ via CSLs is possible. This is due to the fact that square CSLs may appear both in $\mathcal{R}_1$ and in $\mathcal{R}_2$.

Proof of Proposition 5. As indicated above, counting all well-rounded sublattices that are invariant under a given pair $\pm R$ (that is, contained in $\widetilde{\Gamma}_R$) gives a contribution

$$\frac{1}{\sigma(R)^s}\phi_{\mathsf{wr},\mathrm{even}}(\kappa(R);s)$$

if $\widetilde{\Gamma}_R \not\subseteq \Gamma$, and

$$\frac{1}{\sigma(R)^s}\left(\phi_{\mathsf{wr},\mathrm{even}}(\kappa(R);s) + 2^s\phi_{\mathsf{wr},\mathrm{odd}}(\kappa(R);s)\right)$$

if $\widetilde{\Gamma}_R \subseteq \Gamma$. If Γ is not commensurate to a hexagonal or a square lattice, every well-rounded sublattice is of rhombic type and belongs to a unique pair $\pm R$ of coincidence reflections. Thus, summing over all pairs $\pm R$ immediately gives the result in this case.

The situation is more complex for lattices that are commensurate to a hexagonal or a square lattice, since some well-rounded sublattices may be of hexagonal or square type, respectively, and hence there may be more than one pair $\pm R$ of coincidence reflections associated with it. The rhombic well-rounded sublattices may still be treated in the same way as above, but the hexagonal and square sublattices need extra care.

A hexagonal sublattice corresponds to exactly three pairs of coincidence reflections. Thus we count the hexagonal lattices thrice if we sum over all pairs of coincidence reflections, which we compensate for by subtracting the term $2\Phi_\Gamma^{\triangle}(s)$.

Similarly, a square sublattice Λ is invariant under two pairs $\pm R, \pm S$ of coincidence reflections. However, these two pairs play different roles, as exactly one of these pairs, say $\pm S$, has eigenvectors which form a reduced basis of Λ. This implies that Λ is only counted in the set of rhombic and square lattices which emerge from Γ_R. Hence, we have a unique pair $\pm R$ in this case as well, and no correction term is needed here. ∎

THEOREM 6. *For any rational lattice Γ, the generating function $\Phi_{\Gamma,\mathsf{wr}}(s)$ has an analytic continuation to the half-plane $\{\mathrm{Re}(s) > 1/2\}$ except for a pole of order 2 at $s = 1$. Hence there exists a constant $c > 0$ such that the asymptotic behaviour, as $x \to \infty$, is*

$$A_\Gamma(x) = \sum_{n \le x} a_\Gamma(n) \sim cx\log(x).$$

Proof. We have already shown that $\phi_{\mathsf{wr},\mathrm{even}}(\kappa;s)$ and $\phi_{\mathsf{wr},\mathrm{odd}}(\kappa;s)$ are analytic in $\{\mathrm{Re}(s) > 1/2\}$ except at $s = 1$, where both functions have a simple pole. The same holds true for $\Phi_\Gamma^{\triangle}(s)$. It thus remains to analyse the sums over the pairs of coincidence reflections in Proposition 5. By Lemma 1, summing over all pairs of coincidence reflections is equivalent to summing over all four-element subsets $\{\pm w, \pm z\}$ of primitive orthogonal lattice vectors. Since these sets are disjoint, we can as well sum over all primitive vectors in Γ, obtaining each summand exactly four times. As earlier, we denote by Γ_w the BRS-sublattice corresponding to $\{\pm w, \pm z\}$, and we define $\sigma(w) := [\Gamma : \Gamma_w]$, the index of Γ_w in Γ. Finally, we use the notation

$\kappa(w) = |w|/|z|$ for the quantity κ introduced in Remark 2. We thus obtain

$$\begin{aligned}\varPhi_{\varGamma,\mathsf{wr}}(s) - 2\varPhi_{\varGamma}^{\triangle}(s) &= \frac{1}{4} \sum_{\substack{w \text{ primitive} \\ \sigma(w) \text{ odd}}} \frac{1}{\sigma(w)^s} \phi_{\mathsf{wr},\text{even}}(\kappa(w); s) \\ &\quad + \frac{1}{4} \sum_{\substack{w \text{ primitive} \\ \sigma(w) \text{ even}}} \frac{1}{\sigma(w)^s} \big(\phi_{\mathsf{wr},\text{even}}(\kappa(w); s) + 2^s \phi_{\mathsf{wr},\text{odd}}(\kappa(w); s)\big),\end{aligned}$$

where the factor $1/4$ reflects the four elements of $\{\pm w, \pm z\}$, as observed above.

From now on, we assume without loss of generality that $\varGamma$ is integral and primitive. Then, by Proposition 6 of Appendix C, we have $\sigma(w) = (w,w)/g^*(w)$, and $\kappa(w) = g^*(w)/\sqrt{d}$, where d is the discriminant of $\varGamma$ and $g^*(w)$ is the coefficient of w in $\varGamma^*$. By Proposition 6, $g^*(w)$ is a divisor of d, and can therefore take only a finite number of distinct values. As a consequence, also $\kappa(w)$ takes only finitely many values. Moreover, $g^*(w)$ and $\kappa(w)$ are constant on the cosets of an appropriate sublattice of $\varGamma$. Accordingly, we can subdivide the above summation into finitely many sums of simpler type.

To work this out explicitly, we choose a basis $\{v_1, v_2\}$ of $\varGamma^*$ such that $\{v_1, dv_2\}$ is a basis of $\varGamma$, as in Appendix C. Using the quadratic form $Q(m,n) := |mv_1 + ndv_2|^2$, and similarly setting $g^*(m,n) := g^*(mv_1 + ndv_2)$, $\sigma(m,n) := \sigma(mv_1 + ndv_2)$ and $\kappa(m,n) := \kappa(mv_1 + ndv_2)$, for $(m,n) \in \mathbb{Z}^2$, we have $g^*(m,n) = \gcd(m,d)$ and $\sigma(m,n) = Q(m,n)/g^*(m,n)$, by formula (C.1), assuming $\gcd(m,n) = 1$. It follows from Proposition 7 that the parity of $\sigma(m,n)$ only depends on $\gcd(m,D)$ and $\gcd(n,2)$, where $D = \operatorname{lcm}(2,d)$, and if the residues $m \bmod D$ and $n \bmod 2$ are fixed, the index $\sigma(m,n)$ only depends on $Q(m,n)$. Hence,

$$\begin{aligned}\varPhi_{\varGamma,\mathsf{wr}}(s) - 2\varPhi_{\varGamma}^{\triangle}(s) &= \frac{1}{4} \sum_{\gcd(m,n)=1} \frac{\gcd(m,d)^s}{Q(m,n)^s} \\ &\qquad \times \big(\phi_{\mathsf{wr},\text{even}}(\kappa(m,n); s) + \delta_\sigma(m,n)\, 2^s \phi_{\mathsf{wr},\text{odd}}(\kappa(m,n); s)\big) \\ &= \frac{1}{4} \sum_{k|D} \sum_{\ell|2} \big(\phi_{\mathsf{wr},\text{even}}(\kappa(k,\ell); s) + \delta_\sigma(k,\ell) 2^s \phi_{\mathsf{wr},\text{odd}}(\kappa(k,\ell); s)\big) \\ &\qquad \times \sum_{\substack{\gcd(m,n)=1 \\ \gcd(m,D)=k \\ \gcd(n,2)=\ell}} \frac{\gcd(k,d)^s}{Q(m,n)^s},\end{aligned}$$

where δ_σ is defined by

$$\delta_\sigma(m,n) := \begin{cases} 1 & \text{if } \sigma(m,n) \text{ is even}, \\ 0 & \text{if } \sigma(m,n) \text{ is odd} \end{cases}$$

and depends on $\gcd(m, D)$ and $\gcd(n, 2)$ only. By Remark 3, $\phi_{\mathsf{wr,odd}}(\kappa(k,\ell); s)$ and $\phi_{\mathsf{wr,even}}(\kappa(k,\ell); s)$ are both analytic in $\{\mathrm{Re}(s) > 1/2\}$ except at $s = 1$, where both have a simple pole. Invoking Appendix D, this is true of

$$\sum_{\substack{\gcd(m,n)=1\\ \gcd(m,D)=k\\ \gcd(n,2)=\ell}} \frac{1}{Q(m,n)^s}$$

as well, which shows that $\Phi_{\Gamma,\mathsf{wr}}(s) - 2\Phi_{\Gamma}^{\triangle}(s)$, and thus $\Phi_{\Gamma,\mathsf{wr}}(s)$, has a pole of order 2 at $s = 1$ and is analytic elsewhere in $\{\mathrm{Re}(s) > 1/2\}$, as claimed. The asymptotic behaviour now follows from an application of Delange's theorem (compare Theorem 7). ∎

At this stage, it remains an open question whether, in the general rational case, the asymptotics is $c_1 x \log(x) + c_2 x + \mathcal{O}(x)$, like for the square and hexagonal lattices.

A. Some results from analytic number theory. In what follows, we summarise some results from analytic number theory that we need to determine certain asymptotic properties of the coefficients of Dirichlet series generating functions. For the general background, we refer to [1] and [30].

Consider a Dirichlet series of the form $F(s) = \sum_{m=1}^{\infty} a(m) m^{-s}$. We are interested in the summatory function $A(x) = \sum_{m \le x} a(m)$ and its behaviour for large x. Let us give one classical result for the case that $a(m)$ is real and non-negative.

THEOREM 7. *Let $F(s)$ be a Dirichlet series with non-negative coefficients which converges for $\mathrm{Re}(s) > \alpha > 0$. Suppose that $F(s)$ is holomorphic at all points of the line $\{\mathrm{Re}(s) = \alpha\}$ except at $s = \alpha$. Here, when approaching α from the half-plane to the right of it, we assume $F(s)$ to have a singularity of the form $F(s) = g(s) + h(s)/(s-\alpha)^{n+1}$ where n is a non-negative integer, and both $g(s)$ and $h(s)$ are holomorphic at $s = \alpha$. Then, as $x \to \infty$, we have*

$$A(x) := \sum_{m \le x} a(m) \sim \frac{h(\alpha)}{\alpha \cdot n!} x^{\alpha} (\log(x))^n. \tag{A.1}$$

The proof follows easily from Delange's theorem, for instance by taking $q = 0$ and $\omega = n$ in Tenenbaum's formulation of it (see [28, §II.7, Thm. 15] and references given there).

The critical assumption in Theorem 7 is the behaviour of $F(s)$ along the line $\{\mathrm{Re}(s) = \alpha\}$. In all our applications, this can be checked explicitly. To do so, we have to recall a few properties of the Riemann zeta function $\zeta(s)$ and of the Dedekind zeta functions of imaginary quadratic fields.

It is well-known that $\zeta(s)$ is a meromorphic function in the complex plane, and has a sole simple pole at $s = 1$ with residue 1 (see [1, Thm.

12.5(a)]). It has no zeros in the half-plane $\{\mathrm{Re}(s) \geq 1\}$ (cf. [28, §II.3, Thm. 9]). The values of $\zeta(s)$ at positive even integers are known [1, Thm. 12.17] and we have

$$\zeta(2) = \pi^2/6. \tag{A.2}$$

This is all we need to know for this case.

Let us now consider an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d}\,)$ with $d < 0$ squarefree. The corresponding discriminant is

$$D = \begin{cases} 4d & \text{if } d \equiv 2, 3 \bmod 4, \\ d & \text{if } d \equiv 1 \bmod 4 \end{cases}$$

(see [30, §10]). We need the Dedekind zeta function of K (with fundamental discriminant $D < 0$). By [30, §11, eq. (10)] it can be written as

$$\zeta_K(s) = \zeta(s) \cdot L(s, \chi_D) \tag{A.3}$$

where $L(s, \chi_D) = \sum_{m=1}^{\infty} \chi_D(m) m^{-s}$ is the L-series [1, §6.8] of the primitive Dirichlet character χ_D. The latter is a totally multiplicative arithmetic function, and thus completely specified by

$$\chi_D(p) = \left(\frac{D}{p}\right) \tag{A.4}$$

for odd primes p, where $\left(\frac{D}{p}\right)$ is the usual Legendre symbol, together with

$$\left(\frac{D}{2}\right) = \begin{cases} 0 & \text{if } D \equiv 0 \bmod 4, \\ 1 & \text{if } D \equiv 1 \bmod 8, \\ -1 & \text{if } D \equiv 5 \bmod 8. \end{cases}$$

Since $L(s, \chi_D)$ is an entire function [1, Thm. 12.5], $\zeta_K(s)$ is meromorphic, and its only pole is simple and located at $s = 1$. The residue is $L(1, \chi_D)$, and from [30, §9, Thm. 2] we get the simple formula

$$L(1, \chi_D) = -\frac{\pi}{|D|^{3/2}} \sum_{n=1}^{|D|-1} n \chi_D(n). \tag{A.5}$$

In particular, for the two fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, one has the values $\pi/4$ and $\pi/(3\sqrt{3})$, respectively.

Our next goal is an estimate on sums of the form $\sum_{\ell < n < \alpha\ell} n^{-s}$ for $\ell \in \mathbb{N}$, $\alpha > 1$ and $s > 0$. Invoking Euler's summation formula from [1, Thm. 3.1], one has

$$\sum_{\ell < n \leq \alpha\ell} \frac{1}{n^s} = \int_{\ell}^{\alpha\ell} \frac{dx}{x^s} - \int_{\ell}^{\alpha\ell} (x - [x]) \frac{s\,dx}{x^{s+1}} + \frac{[\alpha\ell] - \alpha\ell}{(\alpha\ell)^s} - \frac{[\ell] - \ell}{\ell^s}. \tag{A.6}$$

The last term vanishes (since $\ell \in \mathbb{N}$), while the second last does whenever $\alpha\ell \in \mathbb{N}$ (otherwise it is negative). Since the second integral on the right hand side is strictly positive (due to $\alpha > 1$), we see that

$$\sum_{\ell<n<\alpha\ell} \frac{1}{n^s} \le \sum_{\ell<n\le\alpha\ell} \frac{1}{n^s} < I_s := \int_{\ell}^{\alpha\ell} \frac{dx}{x^s} = \frac{1-\alpha^{1-s}}{s-1}\ell^{1-s}. \tag{A.7}$$

Observing next (once again due to $\alpha > 1$) that

$$\int_{\ell}^{\alpha\ell} (x-[x])\frac{s\,dx}{x^{s+1}} < \frac{1}{\ell^s} - \frac{1}{(\alpha\ell)^s},$$

one can separately consider the two cases $\alpha\ell \notin \mathbb{N}$ and $\alpha\ell \in \mathbb{N}$ to verify that we always get

$$\sum_{\ell<n<\alpha\ell} \frac{1}{n^s} > I_s - \frac{1}{\ell^s}.$$

This can directly be generalised to sums of the form $\sum_{\ell<n<\alpha\ell+\beta}(n+\gamma)^{-s}$ with $\beta,\gamma \ge 0$, which we summarise as follows.

LEMMA 4. *Let $\ell \in \mathbb{N}$, $\alpha > 1$, $\beta \ge 0$ and $0 \le \gamma < 1$. If $s \ge 0$, then*

$$I_s - \frac{1}{(\ell+\gamma)^s} < \sum_{\ell<n<\alpha\ell+\beta} \frac{1}{(n+\gamma)^s} < I_s,$$

with the integral $I_s = \int_{\ell}^{\alpha\ell+\beta} \frac{dx}{(x+\gamma)^s}$ generalising that in (A.7). ■

Let us finally mention that

$$\frac{1-\alpha^{1-s}}{s-1} = \log(\alpha)\sum_{m\ge0} \frac{(\log(\alpha)(1-s))^m}{(m+1)!},$$

so that this function is analytic in the entire complex plane. In particular, one has the asymptotic expression $\frac{1-\alpha^{1-s}}{s-1} = \log(\alpha) + \mathcal{O}(|1-s|)$ for $s \to 1$.

B. Asymptotics of similar sublattices. We have sketched how to determine the asymptotics of the number of well-rounded sublattices of the square and hexagonal lattices. As a by-product, and as a refinement of the results from [3], we obtain the asymptotics of the number of similar and primitive similar sublattices as follows.

THEOREM 8. *The asymptotic behaviour of the number of similar and of primitive similar sublattices of the square lattice is given by*

$$\sum_{n\le x} b_{\square}(n) = L(1,\chi_{-4})x + \mathcal{O}(\sqrt{x}) = \frac{\pi}{4}x + \mathcal{O}(\sqrt{x}) \tag{B.1}$$

and

$$\sum_{n\le x} b_{\square}^{\mathsf{pr}}(n) = \frac{L(1,\chi_{-4})}{\zeta(2)}x + \mathcal{O}(\sqrt{x}\log(x)) = \frac{3}{2\pi}x + \mathcal{O}(\sqrt{x}\log(x)). \tag{B.2}$$

Sketch of proof. Note that $b_{\square}(n) = (\chi_{-4} * 1)(n)$. We now get the asymptotics of its summatory function by an application of (3.17). Observe that $b_{\square}^{\mathsf{pr}} = \nu * b_{\square}$, where $\nu(n) := \mu(\sqrt{n})$ is defined to be 0 if n is not a square and μ is the Möbius function. An application of (3.16) then yields the result. ∎

Similarly, one proves the following result.

THEOREM 9. *The asymptotic behaviour of the number of similar and of primitive similar sublattices of the hexagonal lattice is given by*

$$\sum_{n\le x} b_{\triangle}(n) = L(1,\chi_{-3})\,x + \mathcal{O}(\sqrt{x}\,) = \frac{\pi}{3\sqrt{3}}\,x + \mathcal{O}(\sqrt{x}\,) \tag{B.3}$$

and

$$\begin{aligned}\sum_{n\le x} b_{\triangle}^{\mathsf{pr}}(n) &= \frac{L(1,\chi_{-3})}{\zeta(2)}x + \mathcal{O}(\sqrt{x}\log(x)) \\ &= \frac{2}{\pi\sqrt{3}}x + \mathcal{O}(\sqrt{x}\log(x)),\end{aligned} \tag{B.4}$$

as $x \to \infty$. ∎

C. The index of BRS sublattices. Let us complement the discussion of rational orthogonal frames and BRS sublattices as introduced in Lemma 1. We start with an arbitrary rational, primitive, planar lattice Γ and denote by $(v,w) \in \mathbb{Z}$ with $v,w \in \Gamma$ the given positive definite integer-valued primitive symmetric bilinear form on Γ, extended to the rational space $\mathbb{Q}\Gamma$. Primitivity means that the form is not a proper integral multiple of another form; this is equivalent to the condition that $\gcd(a,b,c) = 1$, where $G = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is the Gram matrix with respect to an arbitrary basis v_1, v_2 of Γ.

In the following, we need the notion of the *coefficient* $g_{\Gamma}(v)$ of an arbitrary vector $v \in \mathbb{Q}\Gamma$ with respect to Γ. This is the unique positive rational number g such that $v = gv_0$, where $v_0 \in \Gamma$ is primitive in Γ. Equivalently, $g_{\Gamma}(v)$ is the unique positive generator of the rank one $\mathbb{Z}$-submodule of $\mathbb{Q}$ consisting of all $q \in \mathbb{Q}$ such that $q^{-1}v \in \Gamma$. So, a vector v is primitive in Γ if and only if $g_{\Gamma}(v) = 1$, in accordance with the first definition. Another description of $g_{\Gamma}(v)$ is the gcd (taken in $\mathbb{Q}$) of the coefficients of v with respect to an arbitrary $\mathbb{Z}$-basis of Γ. Below, we shall use the coefficient $g^* := g_{\Gamma^*}$ in particular with respect to the dual lattice $\Gamma^* := \{w \in \mathbb{Q}\Gamma \mid \forall v \in \Gamma : (v,w) \in \mathbb{Z}\}$.

For an arbitrary primitive vector $w \in \Gamma$, we recall the notation Γ_w for the BRS sublattice spanned by w and its orthogonal sublattice $w^{\perp} \cap \Gamma$, i.e. by w and z, where z is the primitive lattice vector orthogonal to w (unique up to sign). The main result of this appendix is to compute the index of $\Gamma_w \in \Gamma$ as follows.

PROPOSITION 6. *Let w be a primitive vector in a planar lattice Γ with primitive symmetric bilinear form, and let $g^*(w)$ denote its coefficient in the dual lattice $\Gamma^* \supseteq \Gamma$. Then $g^*(w)$ is a divisor of the discriminant d of the lattice, and*

$$[\Gamma : \Gamma_w] = \frac{(w,w)}{g^*(w)}.$$

Proof. The first claim follows easily from the fact that d is equal to the order of the factor group Γ^*/Γ, but it is also a consequence of the following computation leading to a proof of the second claim. Since w is primitive, we can complement it to a basis $v_1 = w, v_2$ of Γ. Consider the dual basis v_1^*, v_2^* with respect to the given scalar product; it is a $\mathbb{Z}$-basis of Γ^*. Writing the above vector z as $z = sv_1^* + tv_2^*$ with $s, t \in \mathbb{Z}$ clearly leads to $s = 0$, and t is the smallest integer such that $tv_2^* \in \Gamma$. If G is the Gram matrix with respect to v_1, v_2 as above, then G is also the transformation matrix which expresses the original basis vectors v_1, v_2 in terms of their dual vectors, in particular $v_1 = av_1^* + bv_2^*$, which shows that the coefficient of $w = v_1$ in Γ^* is

$$g^*(w) = \gcd(a,b).$$

On the other hand, with $d := ac - b^2$,

$$G^{-1} = \frac{1}{d}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

is the transformation matrix expressing the dual basis in terms of the original basis. In particular

$$v_2^* = \frac{1}{d}(-bv_1 + av_2),$$

which implies that

$$t = \frac{d}{\gcd(a,b)}.$$

To compute the index of Γ_w in Γ, we use the bases v_1, v_2 of Γ and v_1, tv_2^* of Γ_w. The corresponding transformation matrix is $\left(\begin{smallmatrix} 1 & -(b/d)t \\ 0 & (a/d)t \end{smallmatrix}\right)$, which has determinant

$$\frac{a}{d}t = \frac{a}{d}\,\frac{d}{\gcd(a,b)} = \frac{a}{g^*(w)},$$

as claimed. ∎

Since the vector w was assumed primitive in Γ, it is even true that $g^*(w)$ is a divisor of the exponent of the factor group Γ^*/Γ. But from the primitivity of the bilinear form it follows that this factor group is actually cyclic of order d, so its exponent is equal to d, and we do not get an improvement: all divisors of the discriminant d can occur as a value $g^*(w)$.

It is easy to see that the quantity $g^*(w)$ only depends on an appropriate coset of w; in fact, under the assumptions of the last proposition, the coset modulo $d\Gamma^*$ suffices. For purposes of reference, we state this as an explicit remark.

Remark 5. Under the assumptions of Proposition 6, let w, w' be primitive such that $w \equiv w' \bmod d\Gamma^*$. Then $g^*(w) = g^*(w')$.

For explicit computations involving g^*, it is convenient to use a basis corresponding to the elementary divisors of Γ in Γ^*, that is, a basis $\{v_1, v_2\}$ of Γ^* such that $\{v_1, dv_2\}$ is a basis of Γ. The primitive vectors in Γ read $w = mv_1 + ndv_2$ with $\gcd(m, n) = 1$. Using $g := \gcd(m, d)$, we can rewrite this as $w = g((m/g)v_1 + n(d/g)v_2)$, where the coefficients m/g and $n(d/g)$ are coprime, in other words, $(m/g)v_1 + n(d/g)v_2$ is primitive in Γ^*. This proves

(C.1) $$g^*(mv_1 + ndv_2) = \gcd(m, d) \quad \text{if } \gcd(m, n) = 1.$$

Notice that this formula again proves Remark 5.

For our application to well-rounded sublattices, we also have to consider the parity of the index $[\Gamma : \Gamma_w]$. For this, we need the following refinement of Remark 5.

Proposition 7. *Under the assumptions of Proposition* 6, *let* w, w' *be primitive such that* $w \equiv w' \bmod d\Gamma^*$ *and* $w \equiv w' \bmod 2\Gamma$. *Then*

$$[\Gamma : \Gamma_w] \equiv [\Gamma : \Gamma_{w'}] \bmod 2.$$

Proof. The proof is of course based on Proposition 6, taking into account that, under our assumptions, $g := g^*(w) = g^*(w')$, by Remark 5. First of all, recall that g divides d. Now, we write $w' = w + u = w + du'$ with $u' \in \Gamma^*$ and $u \in 2\Gamma$, and we compute explicitly

$$\frac{(w', w')}{g} = \frac{(w, w)}{g} + 2\frac{d}{g}(w, u') + \frac{d}{g}(u, u') \equiv \frac{(w, w)}{g} \bmod 2.$$

Notice that the last inner product (u, u') is indeed in $2\mathbb{Z}$, since $u \in 2\Gamma$ and $u' \in \Gamma^*$. ∎

D. Epstein's ζ-function. For a quadratic form $Q(m, n) = am^2 + 2bmn + cn^2$, the Epstein ζ-function is defined as

(D.1) $$\zeta_Q(s) := \sum_{(m,n) \neq (0,0)} \frac{1}{Q(m, n)^s},$$

where the sum runs over all non-zero vectors $(m, n) \in \mathbb{Z}^2$. The series converges in the half-plane $\{\mathrm{Re}(s) > 1\}$. It has an analytic continuation which is a meromorphic function in the whole complex plane with a single simple pole at $s = 1$ with residue $\pi/\sqrt{d}$, where $d = ac - b^2$ as before (see [16, 26]). It is closely connected to

$$\zeta_Q^{\mathsf{pr}}(s) := \sum_{(m,n)=1} \frac{1}{Q(m,n)^s} = \frac{1}{\zeta(2s)}\zeta_Q(s), \tag{D.2}$$

where the sum runs over all pairs of integers that are relatively prime. In the explicit summations, we now use (m, n) instead of $\gcd(m, n)$.

In Section 5.2, we need the sum

$$\sum_{\substack{(m,n)=1 \\ (m,D)=k \\ (n,C)=\ell}} \frac{1}{Q(m,n)^s}, \tag{D.3}$$

where C, D, k, ℓ are some fixed positive integers with k, ℓ relatively prime. Using the Möbius μ-function, we can express

$$\sum_{\substack{(m,n)=1 \\ (m,D)=k \\ (n,C)=\ell}} \frac{1}{Q(m,n)^s} = \sum_{\substack{(m,n)=1 \\ (m,\ell D/k)=1 \\ (n,kC/\ell)=1}} \frac{1}{Q(km,\ell n)^s} = \sum_{c|(\ell D/k)} \mu(c)\varphi_Q\left(c\frac{kC}{\ell}; ck, \ell; s\right)$$

in terms of

$$\varphi_Q(a; k, \ell; s) := \sum_{\substack{(m,n)=1 \\ (n,a)=1}} \frac{1}{Q(km,\ell n)^s}. \tag{D.4}$$

As $Q(m, n)$ is homogeneous of degree 2, we have

$$\varphi_Q(a; kb, \ell b; s) = \frac{1}{b^{2s}}\varphi_Q(a; k, \ell; s). \tag{D.5}$$

Furthermore, observe that $\varphi_Q(a; k, \ell; s) = \varphi_Q(b; k, \ell; s)$ whenever a and b have the same prime factors. In particular, we may assume that a is squarefree in the following. Using the same methods as above, we can derive the following recursion:

$$\varphi_Q(a; k, \ell; s) = \sum_{b|a} \sum_{c|(a/b)} \mu(c)\frac{1}{b^{2s}}\varphi_Q(b; k, c\ell; s), \tag{D.6}$$

where we have made use of the assumption that a is squarefree and employed the multiplicativity of μ. This recursion has the solution

$$\varphi_Q(a; k, \ell; s) = \left(\prod_{p|a} \frac{1}{1 - p^{-2s}}\right)\left(\sum_{b|a} \mu(b)\varphi_Q(1; k, b\ell; s)\right), \tag{D.7}$$

where the product is taken over all primes p dividing a. As $\varphi_Q(1; k, b\ell; s)$ is the primitive Epstein ζ-function $\zeta_{\tilde{Q}}^{\mathsf{pr}}(s)$ corresponding to the quadratic form

$\tilde{Q}(m,n) = Q(km, b\ell n)$, this shows that $\varphi_Q(a;k,\ell;s)$ and thus

$$\sum_{\substack{(m,n)=1\\(m,D)=k\\(n,C)=\ell}} \frac{1}{Q(m,n)^s}$$

are sums of Epstein zeta functions, and thus are meromorphic functions with a simple pole at $s=1$ and analytic elsewhere in $\{\mathrm{Re}(s) > 1/2\}$.

Alternatively, we can obtain this result by an application of [26, Theorem 3 (p. 45)] (see also [18]). Applied to our situation, it states that

$$\psi_Q(D,C,i,j;s) := \sum_{\substack{m\equiv i \bmod D\\ n\equiv j \bmod C}} \frac{1}{Q(m,n)^s} \tag{D.8}$$

has an analytic continuation onto the entire complex plane except for a simple pole at $s=1$ with residue $\pi/\sqrt{\det(Q')}$, where $Q'(m,n) := Q(Dm, Cn)$. Using methods similar to those in [5, 23], we first observe that for k, ℓ coprime,

$$\begin{aligned}
\sum_{\substack{(m,n)=1\\(m,D)=k\\(n,C)=\ell}} \frac{1}{Q(m,n)^s} &= \sum_{\substack{(m,D)=k\\(n,C)=\ell}} \frac{1}{Q(m,n)^s} \sum_{r|(m,n)} \mu(r)\\
&= \sum_{r\in\mathbb{N}} \mu(r)\frac{1}{r^{2s}} \sum_{\substack{(rm,D)=k\\(rn,C)=\ell}} \frac{1}{Q(m,n)^s}\\
&= \sum_{u|k}\sum_{v|\ell}\sum_{\substack{r\in\mathbb{N}\\(r,CD)=1}} \frac{\mu(uvr)}{(uvr)^{2s}} \sum_{\substack{(uvrm,D)=k\\(uvrn,C)=\ell}} \frac{1}{Q(m,n)^s}.
\end{aligned}$$

As r is coprime with C and D we see that

$$\sum_{\substack{(uvrm,D)=k\\(uvrn,C)=\ell}} \frac{1}{Q(m,n)^s} = \sum_{\substack{(vm,D/u)=k/u\\(un,C/v)=\ell/v}} \frac{1}{Q(m,n)^s} \tag{D.9}$$

is independent of r. Moreover, the latter sum can be written as a (finite) sum of suitable functions of the form $\psi_Q(D,C,i,j;s)$ and therefore it is analytic in the entire complex plane except for a simple pole at $s=1$. As u,v,r are coprime, $\mu(uvr) = \mu(u)\mu(v)\mu(r)$, and hence the only remaining infinite sum

$$\sum_{\substack{r\in\mathbb{N}\\(r,CD)=1}} \frac{\mu(r)}{r^{2s}} = \frac{1}{\zeta(2s)} \prod_{p|CD} \frac{1}{1-p^{2s}} \tag{D.10}$$

is analytic in $\{\mathrm{Re}(s) > 1/2\}$, which again shows that

$$\sum_{\substack{(m,n)=1 \\ (m,D)=k \\ (n,C)=\ell}} \frac{1}{Q(m,n)^s}$$

is a meromorphic function with a simple pole at $s = 1$ and analytic elsewhere in $\{\mathrm{Re}(s) > 1/2\}$.

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] M. Baake, *Solution of the coincidence problem in dimensions* $d \le 4$, in: The Mathematics of Long-Range Aperiodic Order, R. V. Moody (ed.), Kluwer, Dordrecht, 1997, 9–44; rev. version: arXiv:math/0605222.

[3] M. Baake and U. Grimm, *Bravais colourings of planar modules with* N*-fold symmetry*, Z. Kristallogr. 219 (2004), 72–80.

[4] M. Baake, U. Grimm, M. Heuer and P. Zeiner, *Coincidence rotations of the root lattice* A_4, Eur. J. Combin. 29 (2008), 1808–1819.

[5] M. Baake, R. V. Moody and P. A. B. Pleasants, *Diffraction from visible lattice points and* k*th power free integers*, Discrete Math. 221 (2000), 3–42.

[6] M. Baake, R. Scharlau and P. Zeiner, *Similar sublattices of planar lattices*, Canad. J. Math. 63 (2011), 1220–1237.

[7] Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, New York, 1966.

[8] J. Brüdern, *Einführung in die analytische Zahlentheorie*, Springer, Berlin, 1995.

[9] C. J. Bushnell and I. Reiner, *Zeta functions of arithmetic orders and Solomon's conjectures*, Math. Z. 173 (1980), 135–161.

[10] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, New York, 2010.

[11] L. Fukshansky, *On distribution of well-rounded sublattices of* $\mathbb{Z}^2$, J. Number Theory 128 (2008), 2359–2393.

[12] L. Fukshansky, *Well-rounded zeta-function of planar arithmetic lattices*, Proc. Amer. Math. Soc. 142 (2014), 369–380.

[13] L. Fukshansky, D. Moore, R. A. Ohana and W. Zeldow, *On well-rounded sublattices of the hexagonal lattice*, Discrete Math. 310 (2010), 3287–3302.

[14] F. J. Grunewald, D. Segal and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. 93 (1988), 185–223.

[15] M. Klemm, *Symmetrien von Ornamenten und Kristallen*, Springer, Berlin, 1982.

[16] M. Koecher und A. Krieg, *Elliptische Funktionen und Modulformen*, 2nd ed., Springer, Berlin, 2007.

[17] S. Kühnlein, *Well-rounded sublattices*, Int. J. Number Theory 8 (2012), 1133–1144.
[18] S. Kühnlein and R. Schwerdt, *Well-rounded sublattices and twisted Epstein zeta functions*, preprint, 2014.
[19] A. Lubotzky and D. Segal, *Subgroup Growth*, Birkhäuser, Basel, 2003.
[20] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Springer, Berlin, 2010.
[21] C. McMullen, *Minkowski's conjecture, well-rounded lattices and topological dimension*, J. Amer. Math. Soc. 18 (2005), 711–734.
[22] P. Moree, *Chebyshev's bias for composite numbers with restricted prime divisors*, Math. Comp. 73 (2004), 425–449.
[23] P. A. B. Pleasants and C. Huck, *Entropy and diffraction of the k-free points in n-dimensional lattices*, Discrete Comput. Geom. 50 (2013), 39–68.
[24] M. du Sautoy and F. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. 152 (2000), 793–833.
[25] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, NJ, 1971.
[26] C. L. Siegel, *Advanced Analytic Number Theory*, 2nd ed., Tata Inst. Fund. Res., Bombay, 1980.
[27] L. Solomon, *Zeta functions and integral representation theory*, Adv. Math. 26 (1977), 306–326.
[28] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, Cambridge, 1995.
[29] B. L. van der Waerden, *Die Reduktionstheorie der positiven quadratischen Formen*, Acta Math. 96 (1956), 265–309.
[30] D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer, Berlin, 1981.
[31] P. Zeiner, *Coincidences of hypercubic lattices in 4 dimensions*, Z. Kristallogr. 221 (2006), 105–114.
[32] P. Zeiner, *Supplement to "Well-rounded sublattices of planar lattices"*, available from the author, or as supplement from arXiv:1311.6306.
[33] Y. M. Zou, *Structures of coincidence symmetry groups*, Acta Cryst. A 62 (2006), 109–114.

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