ACTA ARITHMETICA
166.4 (2014)

On large values of the Riemann zeta-function on short segments of the critical line

by

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1. Introduction. In 2001–2006, A. A. Karatsuba [K2]–[K5] obtained a series of lower bound estimates for the maximum of the modulus of the Riemann zeta-function $\zeta(s)$ on circles of small radius lying in the critical strip $0 \le \Re s \le 1$, and on very short segments of the critical line $\Re s = 0.5$. These results were further developed in [G], [Fe], [C1], [C2].

In particular, it was proved in [K5] that the function

$$F(T;H) = \max_{|t-T|\le H} |\zeta(0.5+it)|$$

satisfies

$$F(T;H) \ge \frac{1}{16}\exp\left\{-\frac{5\ln T}{6(\pi/\alpha-1)(\cosh(\alpha H)-1)}\right\}, \tag{1.1}$$

where α is any fixed number, $1 \le \alpha < \pi$, $2 \le \alpha H \le \ln\ln H - c_1$, and $c_1 > 0$ is some absolute constant. Given $\varepsilon > 0$, it follows from (1.1) that for any $T \ge T_0(\varepsilon)$ and for $H \ge \pi^{-1}(1+\varepsilon)\ln\ln T - c_1$, the function $F(T;H)$ is bounded from below by some constant:

$$F(T;H) > c_2 = \tfrac{1}{16}\exp\left(-1.7\varepsilon^{-1}e^{c_1}\right) > 0.$$

In [K5], A. A. Karatsuba posed the problem of proving $F(T;H) \ge 1$ for the values of H essentially smaller than $\ln\ln T$, namely, for $H \ge \ln\ln\ln T$ (1).

In this paper, we give a conditional solution of Karatsuba's problem, based on the Riemann hypothesis. Moreover, we prove that for an arbitrarily

2010 *Mathematics Subject Classification*: Primary 11M06.
Key words and phrases: Riemann zeta function, argument of the Riemann zeta-function, Gram's law, critical line.

(1) If $\ln\ln T \ll H \le 0.1T$, then R. Balasubramanian [Ba] proved that

$$F(T;H) \gg \exp\left(\frac{3}{4}\sqrt{\frac{\ln H}{\ln\ln H}}\right).$$

This bound is supposed to be close to best possible. Thus, estimates of $F(T;H)$ for $0 < H \ll \ln\ln T$ are most interesting.

DOI: 10.4064/aa166-4-3

large fixed $A \geq 1$ there exist positive constants T_0 and c_0 that depend on A such that $F(T;H) \geq A$ for any $T \geq T_0$ and $H = (1/\pi)\ln\ln\ln T + c_0$ (see Theorem 4.1).

The method used here is applicable to the estimation of both the maxima of the function

$$|\zeta(0.5+it)| = \exp(\ln|\zeta(0.5+it)|) = \exp(\Re\ln\zeta(0.5+it)),$$

and the extremal values of the function

$$S(t) = \frac{1}{\pi}\arg\zeta(0.5+it) = \frac{1}{\pi}\Im\ln\zeta(0.5+it)$$

(for the definition and basic properties of $S(t)$, which is called the argument of the Riemann zeta-function on the critical line, see the survey [KK]).

Estimates of the maximum and minimum of $S(t)$ on very short intervals of t are of significant interest, together with the classical estimates of $\max_{T\leq t\leq 2T}(\pm S(t))$ belonging to A. Selberg [S1] and K.-M. Tsang [T1], [T2]. Estimates of the form

$$\max_{|t-T|\leq H}(\pm S(t)) \geq f(H),$$

where

$$f(H) = \frac{1}{90\pi}\sqrt{\frac{\ln H}{\ln\ln H}}, \quad (\ln T)(\ln\ln T)^{-3/2} < H < T,$$

or

$$f(H) = \frac{1}{900}\frac{\sqrt{\ln H}}{\ln\ln H}, \qquad \sqrt{\ln\ln T} \leq H \leq (\ln T)(\ln\ln T)^{-3/2},$$

have been obtained in [Ko1] and [Bo2], [Bo3], respectively.

In this paper, we prove the existence of positive and negative values of $S(t)$ with modulus exceeding 3, on each interval of length $H = 0.8\ln\ln\ln t + c_0$ (see Theorems 4.3–4.6). For comparison, we note that from the calculation of the first 200 billions zeros of $\zeta(s)$ on the critical line (S. Wedeniwski [W], 2003) it turns out that

$$\begin{aligned}
|S(t)| &< 1 \quad \text{if } 7 < t < 280,\\
|S(t)| &< 2 \quad \text{if } 7 < t < 6\,820\,050,\\
|S(t)| &< 3 \quad \text{if } 7 < t < 16\,220\,609\,807.
\end{aligned}$$

The first values of $S(t)$ which exceed 3 in modulus are located near the Gram points t_n (see §4) with $n = 53365784979$ and $n = 67976501145$ and are equal to 3.0214 and -3.2281, respectively. At present, no values of t such that $|S(t)| \geq 4$ are known.

Since the function $S(t)$ is "responsible" for the irregularity in the distribution of zeros of $\zeta(s)$, Theorems 4.4 and 4.6 imply some conditional results related to the distribution of Gram intervals $G_n = (t_{n-1}, t_n]$ which contain an "abnormal" (that is, $\neq 1$) number of ordinates of zeros of $\zeta(s)$ (see Theorems 5.1, 5.2).

The paper ends with a proof of Theorem 5.3 on the distribution of non-zero values of an integer-valued function Δ_n introduced by A. Selberg [S2] in connection with the so-called Gram law.

In this paper, we use the following notation: $\Lambda(n)$ is the von Mangoldt function, equal to $\ln p$ for prime p and for $n = p^k$, $k = 1, 2, \ldots$, and equal to zero otherwise; $\Lambda_1(n) = \Lambda(n)/\ln n$ $(n \geq 2)$; $\cosh z = (e^z + e^{-z})/2$; $K_a(z) = \exp(-a\cosh z)$ $(a > 0)$; $\hat{f}$ denotes the Fourier transform of the function f, that is,

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}\,dx;$$

$\|\alpha\| = \min(\{\alpha\}, 1 - \{\alpha\})$ is the distance between α and the closest integer; $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$ are the primes indexed in ascending order; $\Omega(n)$ is the number of prime factors of n counted with multiplicity; $\theta, \theta_1, \theta_2, \ldots$ are complex numbers, different in different formulae, whose absolute value does not exceed one. All other notation is explained in the text.

2. Auxiliary assertions. In this section, we give some auxiliary lemmas.

LEMMA 2.1. *For any $m \geq 1$, the numbers*

$$1, \quad \frac{1}{2\pi}\ln 2, \quad \frac{1}{2\pi}\ln 3, \quad \frac{1}{2\pi}\ln 5, \quad \ldots, \quad \frac{1}{2\pi}\ln p_m$$

are linearly independent over the field of rationals.

Proof. Assume to the contrary that there exist integers $k \geq 0, k_1, \ldots, k_m$ not all zero and such that

$$k + \frac{k_1}{2\pi}\ln 2 + \frac{k_2}{2\pi}\ln 3 + \cdots + \frac{k_m}{2\pi}\ln p_m = 0,$$

or, what is the same,

$$k - \frac{1}{2\pi}\ln\frac{a}{b} = 0, \tag{2.1}$$

where a and b are coprime integers not both 1, whose prime factors do not exceed p_m. Exponentiating (2.1), we get

$$e^{2\pi k} = a/b. \tag{2.2}$$

If $k = 0$ then (2.2) contradicts the fundamental theorem of arithmetic. If $k \geq 1$ then e^{π} appears to be the root of the polynomial $bz^{2k} - a$. This is impossible in view of the transcendence of e^{π} (see for example [F, §2.4]). ∎

LEMMA 2.2. $|\widehat{K}_a(\lambda)| \le \kappa e^{-b|\lambda|}$ *for any real* λ *with*

$$\kappa = \kappa(a,b) = 2\int_0^\infty \exp\bigl(-a(\cos b)\cosh u\bigr)\,du,$$

where b *is any number with* $0 < b < \pi/2$.

The proof repeats almost verbatim that of [Ko2, Lemma 4].

LEMMA 2.3. *Suppose that* λ *is real and* $|\lambda| \ge a\sqrt{2}$. *Then*

$$\widehat{K}_a(\lambda) = \frac{2\sqrt{2\pi}}{\sqrt[4]{\lambda^2 - a^2}}\exp\biggl(-\frac{\pi|\lambda|}{2}\biggr)\bigl(\cos g_a(\lambda) + r_a(\lambda)\bigr),$$

where

$$g_a(\lambda) = \sqrt{\lambda^2 - a^2} - |\lambda|\ln\biggl(\frac{|\lambda|}{a} + \frac{\sqrt{\lambda^2 - a^2}}{a}\biggr) + \frac{\pi}{4}, \quad |r_a(\lambda)| \le c_a|\lambda|^{-0.1},$$

and

$$c_a = \begin{cases} 9.3 & \text{if } a \ge 1/\sqrt{2}, \\ 8.2a^{-0.4} & \text{if } 0 < a < 1/\sqrt{2}. \end{cases}$$

Proof. Without loss of generality, we assume that $\lambda > 0$. Take $R > 1$ and denote by Γ_R the boundary of the rectangle with vertices at $\pm R$, $\pm R - \pi i/2$, traversed counterclockwise. Cauchy's residue theorem yields

$$\int_{\Gamma_R} K_a(z)e^{-i\lambda z}\,dz = \sum_{k=1}^{4} I_k = 0,$$

where I_1, I_3 are the integrals along the upper and lower sides of the contour and I_2, I_4 are the integrals over the lateral sides.

Further, it is easy to note that

$$-I_1 = \int_{-R}^{R} K_a(u)e^{-i\lambda u}\,du,$$

$$I_3 = \int_{-R}^{R} K_a\biggl(u - \frac{\pi i}{2}\biggr)e^{-i\lambda(u-\pi i/2)}\,du = e^{-\pi\lambda/2}\int_{-R}^{R} e^{i\varphi_a(u)}\,du,$$

where $\varphi_a(u) = a\sinh u - \lambda u$. Set $z = R - \pi i t/2$, where $0 \le t \le 1$. Since $|K_a(z)| = e^{-a\cosh(R)\cos(\pi t/2)}$, we get

$$|I_4| \le \frac{\pi}{2}\int_0^1 e^{-a\cosh(R)\cos(\pi t/2)}\,dt = \frac{\pi}{2}\int_0^1 e^{-a\cosh(R)\sin(\pi t/2)}\,dt$$

$$\le \frac{\pi}{2}\int_0^1 e^{-at\cosh(R)}\,dt \le \frac{\pi}{2a\cosh R}.$$

The same bound is valid for the integral I_2. Hence,

$$\int_{-R}^{R} K_a(u)e^{-i\lambda u}\,du = e^{-\pi\lambda/2}\int_{-R}^{R} e^{i\varphi_a(u)}\,du + \frac{\pi\theta}{a\cosh R}.$$

Letting R tend to infinity, we obtain

$$\widehat{K}_a(\lambda) = e^{-\pi\lambda/2}\int_{-\infty}^{\infty} e^{i\varphi_a(u)}\,du = 2e^{-\pi\lambda/2}\Re j_a(\lambda), \quad j_a(\lambda) = \int_{0}^{\infty} e^{i\varphi_a(u)}\,du.$$

The derivative $\varphi_a'(u)$ has a unique zero on the ray of integration at the point

$$u_a = \operatorname{arccosh}(\lambda/a) = \ln(\lambda/a + \sqrt{\lambda^2/a^2 - 1}).$$

Setting $u = u_a + v$, where $-u_a \le v < \infty$, and noting that

$$\varphi_a(u) = a(\sinh u_a \cosh v + \cosh u_a \sinh v) - \lambda(u_a + v) = -\lambda u_a + \lambda\psi_a(v),$$

where $\psi_a(v) = \alpha\cosh v + \sinh v - v$, $\alpha = \sqrt{1-(a/\lambda)^2}$, we find that

$$j_a(\lambda) = e^{-i\lambda u_a}\int_{-u_a}^{\infty} e^{i\lambda\psi_a(v)}\,dv.$$

Suppose that $0 < \delta < \min(1, u_a, \lambda^{-1/3})$. Then we represent $j_a(\lambda)$ as the sum

$$e^{-i\lambda u_a}\Big(\int_{-\delta}^{\delta} + \int_{-u_a}^{-\delta} + \int_{\delta}^{\infty}\Big)e^{i\lambda\psi_a(v)}\,dv = e^{-i\lambda u_a}(j_1 + j_2 + j_3).$$

We have

$$\psi_a(v) = \psi_a(0) + \psi_a'(0)v + \psi_a''(0)\frac{v^2}{2} + \psi_a^{(3)}(\xi)\frac{v^3}{6}$$

for $|v| \le \delta$, where ξ lies between 0 and v. Since $\psi_a'(v) = \alpha\sinh v + \cosh v - 1$, $\psi_a''(v) = \alpha\cosh v + \sinh v$ and $\psi_a^{(3)}(v) = \alpha\sinh v + \cosh v$, we have $\psi_a(0) = \psi_a''(0) = \alpha$, $\psi_a'(0) = 0$, and

$$|\psi_a^{(3)}(\xi)| = |\alpha\sinh\xi + \cosh\xi| \le \sinh|\xi| + \cosh\xi = e^{|\xi|} \le e^{\delta} < e.$$

Hence,

$$\lambda\psi_a(v) = \mu + \mu\frac{v^2}{2} + e\lambda\frac{\theta v^3}{6}, \quad \mu = \alpha\lambda = \sqrt{\lambda^2 - a^2}.$$

Let us define $\varrho(v)$ by the relation $\exp(ie\theta\lambda v^3/6) = 1 + \varrho(v)$. Thus we get

$$\begin{aligned}|\varrho(v)| &= \left|\frac{ie\lambda}{6}\theta v^3 + \frac{1}{2!}\left(\frac{ie\lambda}{6}\theta v^3\right)^2 + \frac{1}{3!}\left(\frac{ie\lambda}{6}\theta v^3\right)^3 + \cdots\right| \\ &\le \frac{e\lambda}{6}|v|^3\left(1 + \frac{1}{2!}\,\frac{e}{6} + \frac{1}{3!}\left(\frac{e}{6}\right)^2 + \cdots\right) = (e^{e/6} - 1)\lambda|v|^3 < \frac{3\lambda}{5}|v|^3.\end{aligned}$$

Therefore,

$$j_1 = \int_{-\delta}^{\delta} \exp\biggl(i\mu + \frac{i\mu v^2}{2}\biggr)(1+\varrho(v))\,dv$$

$$= e^{i\mu}\int_{-\delta}^{\delta} \exp\biggl(\frac{i\mu v^2}{2}\biggr)\,dv + 2\theta_1\int_0^{\delta}\frac{3\lambda}{5}v^3\,dv = e^{i\mu}\sqrt{\frac{2}{\mu}}\int_0^{\frac{\mu}{2}\delta^2}\frac{e^{iw}\,dw}{\sqrt{w}} + \frac{3\theta_1}{10}\lambda\delta^4.$$

Replacing the last integral by an improper one and noting that

$$\int_0^{\infty}\frac{e^{iw}\,dw}{\sqrt{w}} = e^{\pi i/4}\sqrt{\pi}, \qquad \biggl|\int_u^{\infty}\frac{e^{iw}\,dw}{\sqrt{w}}\biggr| \le \frac{2}{\sqrt{u}},$$

we find that

$$j_1 = e^{i\mu}\sqrt{\frac{2}{\mu}}\biggl(\sqrt{\pi}\,e^{\pi i/4} + \frac{2\theta_2\sqrt{2}}{\sqrt{\mu\delta^2}}\biggr) + \frac{3\theta_1}{10}\lambda\delta^4$$

$$= \sqrt{\frac{2\pi}{\mu}}\,e^{i(\mu+\pi i/4)} + \theta_3\biggl(\frac{4}{\mu\delta} + \frac{3\lambda\delta^4}{10}\biggr)$$

for any $u > 0$. Further, integration by parts in j_2 yields

$$j_2 = \frac{1}{i\lambda}\biggl(\frac{e^{i\lambda\psi_a(-\delta)}}{\psi_a'(-\delta)} - \frac{e^{i\lambda\psi_a(-u_a)}}{\psi_a'(-u_a)} - \int_{-u_a}^{-\delta} e^{i\lambda\psi_a(v)}\,d\frac{1}{\psi_a'(v)}\biggr)$$

and hence

$$|j_2| \le \frac{1}{\lambda}\biggl(\frac{1}{|\psi_a'(-\delta)|} + \frac{1}{|\psi_a'(-u_a)|} + \int_{-u_a}^{-\delta}\biggl|d\frac{1}{\psi_a'(v)}\biggr|\biggr).$$

Since

$$\alpha = \frac{\sqrt{\lambda^2 - a^2}}{\lambda} = \frac{\sinh u_a}{\cosh u_a} = \tanh u_a,$$

the derivative $\psi_a''(v) = \cosh v\,(\alpha + \tanh v)$ is positive for $v > -u_a$. Thus, the function $1/\psi_a'(v)$ decreases for $v > -u_a$. Hence,

$$|j_2| \le \frac{1}{\lambda}\biggl(\frac{1}{|\psi_a'(-\delta)|} + \frac{1}{|\psi_a'(-u_a)|} - \int_{-u_a}^{-\delta} d\frac{1}{\psi_a'(v)}\biggr)$$

$$= \frac{1}{\lambda}\biggl(\frac{1}{|\psi_a'(-\delta)|} + \frac{1}{|\psi_a'(-u_a)|} - \frac{1}{\psi_a'(-\delta)} + \frac{1}{\psi_a'(-u_a)}\biggr).$$

Since $\psi_a'(0) = 0$, we have $\psi_a'(v) < 0$ for negative v and therefore

$$|j_2| \le \frac{2}{\lambda|\psi_a'(-\delta)|}.$$

Further, we have

$$|\psi_a'(-\delta)| = |\alpha\sinh\delta - \cosh\delta + 1| = 2\sinh\frac{\delta}{2}\left|\alpha\cosh\frac{\delta}{2} - \sinh\frac{\delta}{2}\right|$$
$$> \delta\left|\alpha\cosh\frac{\delta}{2} - \sinh\frac{\delta}{2}\right|.$$

Since $\lambda \ge a\sqrt{2}$, it follows that $\alpha \ge 1/\sqrt{2}$ and hence

$$\alpha\cosh\frac{\delta}{2} - \sinh\frac{\delta}{2} \ge \frac{1}{\sqrt{2}}\cosh\frac{\delta}{2} - \sinh\frac{\delta}{2}$$
$$\ge \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2!}\left(\frac{\delta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\delta}{2}\right)^4 + \cdots\right)$$
$$- \left(\frac{\delta}{2} + \frac{1}{3!}\left(\frac{\delta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\delta}{2}\right)^5 + \cdots\right) > \frac{1}{\sqrt{2}} - \frac{\delta}{2} > \frac{1}{5}.$$

Finally we get

$$|\psi_a'(-\delta)| > \frac{\delta}{5}, \quad |j_2| < \frac{10}{\lambda\delta} < \frac{10}{\mu\delta}.$$

The proof of the inequality $|j_3| \le 2(\lambda\psi_a'(\delta))^{-1}$ is just the same. By the relations $\psi_a'(\delta) = \alpha\sinh\delta + \cosh\delta - 1 > \alpha\delta \ge \delta/\sqrt{2}$, this implies that

$$|j_3| \le \frac{2\sqrt{2}}{\lambda\delta} < \frac{3}{\mu\delta}.$$

Therefore,

$$j_1 + j_2 + j_3 = \sqrt{2\pi/\mu}\, e^{i(\mu+\pi/4)} + r_1,$$

where

$$|r_1| \le \frac{4}{\mu\delta} + \frac{3\lambda\delta^4}{10} + \frac{10}{\mu\delta} + \frac{3}{\mu\delta} = \frac{17}{\mu\delta} + \frac{3\lambda\delta^4}{10}.$$

Thus we conclude that

$$j_a(\lambda) = \sqrt{2\pi/\mu}\, e^{i(\mu+\pi/4-\lambda u_a)}(1 + r_2),$$

where

$$|r_2| \le \sqrt{\frac{\mu}{2\pi}}\left(\frac{17}{\mu\delta} + \frac{3\lambda\delta^4}{10}\right) \le \frac{1}{\sqrt{\pi}}\left(\frac{17}{\sqrt[4]{2}\,\delta\sqrt{\lambda}} + \frac{3\lambda^{3/2}\delta^4}{10\sqrt{2}}\right).$$

If $a\sqrt{2} \ge 1$, we put $\delta = (7/8)\lambda^{-2/5}$. Since $\lambda \ge a\sqrt{2} \ge 1$, the inequalities $\delta < 1$, $\delta < \lambda^{-1/3}$ are obvious. Moreover,

$$u_a = \ln(\lambda/a + \sqrt{(\lambda/a)^2 - 1}) \ge \ln(\sqrt{2} + 1) > 7/8 \ge \delta,$$

and hence $\delta < \min(1, \lambda^{-1/3}, u_a)$. Thus, in this case we have

$$|r_2| \le \frac{1}{\sqrt{\pi}}\left(\frac{8\cdot 17}{7\sqrt[4]{2}} + \frac{3}{10\sqrt{2}}\left(\frac{7}{8}\right)^4\right)\lambda^{-1/10} < 9.3\lambda^{-0.1}.$$

If $a\sqrt{2}<1$ then we set $\delta=(a/\lambda)^{2/5}$. Then $\lambda\ge a\sqrt{2}$ implies that

$$\delta\le(1/\sqrt{2})^{2/5}<1,\quad a^6<a(1/\sqrt{2})^5=a\sqrt{2}/8\le\lambda/8<\lambda,$$

and $a^{2/5}<\lambda^{1/15}=\lambda^{2/5-1/3}$. Thus, $\delta<\lambda^{-1/3}$. Finally, since $x^{-2/5}<\ln(x+\sqrt{x^2-1})$ for any $x\ge\sqrt{2}$, we find $\delta<u_a$. Therefore, in this case, the inequality $\delta<\min(1,\lambda^{-1/3},u_a)$ is also valid. Thus

$$|r_2|\le\frac{1}{\sqrt{\pi}}\biggl(\frac{17}{\sqrt[4]{2}}+\frac{3a^2}{10\sqrt{2}}\biggr)a^{-2/5}\lambda^{-1/10}<8.2a^{-0.4}\lambda^{-0.1}.$$

Finally we get

$$\begin{aligned}\widehat{K}_a(\lambda)&=2e^{-\pi\lambda/2}\sqrt{\frac{2\pi}{\mu}}\Re(e^{i(\mu-\lambda u_a+\pi/4)}(1+r_2))\\&=2\sqrt{\frac{2\pi}{\mu}}e^{-\pi\lambda/2}(\cos(\mu-\lambda u_a+\pi/4)+r),\end{aligned}$$

where $|r|\le c_a\lambda^{-0.1}$ is such that $c_a=9.3$ for $a\sqrt{2}\ge1$ and $c_a=8.2a^{-0.4}$ for $0<a\sqrt{2}<1$. Lemma 2.3 is proved. ∎

Corollary 2.4. *Under the conditions of Lemma* 2.3,

$$|\widehat{K}_a(\lambda)|<\kappa_a\frac{e^{-\pi|\lambda|/2}}{\sqrt{|\lambda|}},$$

where $\kappa_a=61.5$ *for* $a\sqrt{2}\ge1$ *and* $\kappa_a=54.1a^{-0.4}$ *for* $0<a\sqrt{2}<1$.

Proof. Lemma 2.3 together with the condition $|\lambda|\ge a\sqrt{2}$ implies that

$$|\widehat{K}_a(\lambda)|<\frac{2\sqrt{2\pi}}{\sqrt{|\lambda|}}\,\frac{e^{-\pi|\lambda|/2}}{\sqrt[4]{1-(a/\lambda)^2}}(1+r)\le\frac{2^{7/4}\sqrt{\pi}}{\sqrt{|\lambda|}}e^{-\pi|\lambda|/2}(1+r),$$

where $r=c_a|\lambda|^{-1/10}$. Using the above expressions for c_a, we get the desired bound. ∎

Lemma 2.5. *Suppose that the function* $f(z)$ *is analytical in the strip* $|\Im z|\le0.5+\alpha$, *where it satisfies* $|f(z)|\le c(|z|+1)^{-(1+\beta)}$ *with some positive* β *and* c. *Then*

$$\int_{-\infty}^{\infty}f(u)\ln\zeta(0.5+i(t+u))\,du=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\hat{f}(\ln n)+2\pi\biggl(\sum_{\beta>0.5}\int_0^{\beta-0.5}f(\gamma-t-iv)\,dv-\int_0^{0.5}f(-t-iv)\,dv\biggr)\tag{2.3}$$

for any t, *where* $\varrho=\beta+i\gamma$ *in the last sum runs through all complex zeros of* $\zeta(s)$ *to the right of the critical line.*

This assertion goes back to A. Selberg (see for example [S1, Lemma 16]). In [KK, Ch. II, §2], [T1], there are some variants of this lemma, where $f(z)$

satisfies slightly different conditions. These proofs can be easily adapted to the case under consideration.

LEMMA 2.6. *If the Riemann hypothesis is true then*

$$\int_{-\infty}^{\infty} K_a(\pi u)\ln\zeta(0.5+i(t+u))\,du = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\left(\frac{\ln n}{\pi}\right) - 2\pi\int_0^{0.5} K_a(\pi t+\pi iv)\,dv \tag{2.4}$$

for any real t.

Proof. We take an arbitrary δ such that $0<\delta<10^{-6}$ and set $z=x+iy$, $f(z)=K_a((\pi-\delta)z)$, $\alpha=\delta/(4\pi)$. Since

$$\cos\{(\pi-\delta)y\}\ge\cos\{(\pi-\delta)(0.5+\alpha)\}>\sin\frac{\delta}{4}\ge 2\alpha$$

for any y such that $|y|\le 0.5+\alpha$, we have

$$|f(z)|=e^{-a\cosh\{(\pi-\delta)x\}\cos\{(\pi-\delta)y\}}\le e^{-2a\alpha\cosh\{(\pi-\delta)x\}}\le c(|z|+1)^{-(1+\beta)}$$

for suitable constants $\beta=\beta(\alpha)$, $c=c(\alpha)$ and for any x. The application of Lemma 2.5 yields

$$\int_{-\infty}^{\infty} K_a((\pi-\delta)u)\ln\zeta(0.5+i(t+u))\,du = \frac{1}{\pi-\delta}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\left(\frac{\ln n}{\pi-\delta}\right) - 2\pi\int_0^{0.5} K_a((\pi-\delta)(t+iv))\,dv. \tag{2.5}$$

Set

$$N=\left[\frac{1}{\delta^2}\left(\ln\frac{1}{\delta}\right)^{-1}\right]+1$$

and suppose δ is so small that $N>N_0=e^{\pi a\sqrt{2}}$. Now we split the sum in (2.5) into sums C_1,C_2 and C_3 corresponding to the intervals $n>N$, $N_0<n\le N$ and $n\le N_0$, respectively. Using Corollary 2.4 with $\lambda=(1/\pi)\ln n\ge a\sqrt{2}$, we obtain

$$\begin{aligned}|C_1| &\le \frac{1}{\pi-\delta}\sum_{n>N}\frac{\Lambda_1(n)}{\sqrt{n}}61.5\sqrt{\frac{\pi-\delta}{\ln n}}\exp\left(-\frac{\pi}{2}\frac{\ln n}{\pi-\delta}\right)\\ &\le \frac{61.5}{\sqrt{\pi-\delta}}\sum_{n>N}\frac{\Lambda(n)}{n(\ln n)^{3/2}}.\end{aligned}$$

The application of Abel's summation formula together with the bound

$$\psi(u)=\sum_{n\le u}\Lambda(n)\le c_1u,\quad c_1=1.03883 \tag{2.6}$$

(see [RS, Th. 12]), which is valid for any $u > 0$, yields

$$\sum_{n>N} \frac{\Lambda(n)}{n(\ln n)^{3/2}} = -\int_N^\infty (\psi(u) - \psi(N))\, d\frac{1}{(\ln u)^{3/2}}$$
$$\le -c_1 \int_N^\infty u\, d\frac{1}{(\ln u)^{3/2}} = c_1\left(\frac{2}{\sqrt{\ln N}} + \frac{1}{(\ln N)^{3/2}}\right).$$

Using the inequalities $\ln N \ge \ln(1/\delta)$ and $0 < \delta < 10^{-6}$, we get the estimate

$$|C_1| \le \frac{123c_1}{\sqrt{\pi - \delta}} \frac{1}{\sqrt{\ln(1/\delta)}}\left(1 + \frac{1}{2\ln(1/\delta)}\right) < \frac{75}{\sqrt{\ln(1/\delta)}}.$$

Similarly,

$$\left|\frac{1}{\pi}\sum_{n>N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right)\right| < \frac{74.9}{\sqrt{\ln(1/\delta)}}.$$

Thus we get

$$C_1 = \frac{1}{\pi}\sum_{n>N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) + \frac{149.9}{\sqrt{\ln(1/\delta)}}.$$

Further, we represent C_2 as

$$\frac{1}{\pi - \delta}\sum_{N_0<n\le N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - \frac{1}{\pi}\sum_{N_0<n\le N} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} d_n,$$

where

$$d_n = \widehat{K}_a\left(\frac{\ln n}{\pi}\right) - \widehat{K}_a\left(\frac{\ln n}{\pi - \delta}\right) = \int_{-\infty}^{\infty} K_a(u)(e^{-i\varphi_1} - e^{-i\varphi_2})\, du,$$
$$\varphi_1 = \frac{u\ln n}{\pi}, \quad \varphi_2 = \frac{u\ln n}{\pi - \delta}.$$

Since

$$|e^{-i\varphi_1} - e^{-i\varphi_2}| = 2\left|\sin\frac{\varphi_1 - \varphi_2}{2}\right| \le |\varphi_1 - \varphi_2| = \frac{\delta|u|\ln n}{\pi(\pi - \delta)},$$

we obtain

$$|d_n| \le \frac{\delta|u|\ln n}{\pi(\pi - \delta)} \int_{-\infty}^{\infty} |u| e^{-\cosh(\pi u)}\, du < 0.01\delta\ln n.$$

Using the bound (2.6) again, we get

$$\Big|\sum_{N_0<n\le N}\frac{\Lambda_1(n)}{\sqrt{n}}n^{it}d_n\Big|\le 0.01\delta\sum_{N_0<n\le N}\frac{\Lambda(n)}{\sqrt{n}}$$

$$\le 0.01\delta\bigg(\frac{\psi(N)}{\sqrt{N}}+\frac{1}{2}\int_1^N\frac{\psi(u)}{u^{3/2}}\,du\bigg)\le 0.02c_1\delta\sqrt{N}<\frac{0.1}{\sqrt{\ln(1/\delta)}},$$

and hence

$$C_2=\frac{1}{\pi-\delta}\sum_{N_0<n\le N}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)+\frac{0.1\theta}{\sqrt{\ln(1/\delta)}}.$$

Finally, the error arising from the replacement of $\pi-\delta$ by π in the last expression does not exceed

$$\frac{\delta}{\pi(\pi-\delta)}\sum_{N_0<n\le N}\frac{\Lambda_1(n)}{\sqrt{n}}\Big|\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)\Big|\le\frac{61.5\delta\sqrt{\pi}}{\pi(\pi-\delta)}\sum_{n\le N}\frac{\Lambda(n)}{n(\ln n)^{3/2}}<25\delta$$

in modulus. Therefore,

$$C_2=\frac{1}{\pi}\sum_{N_0<n\le N}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)+\theta\bigg(25\delta+\frac{0.1}{\sqrt{\ln(1/\delta)}}\bigg).$$

Thus, (2.5) takes the form

$$(2.7)\qquad \int_{-\infty}^{\infty}K_a((\pi-\delta)u)\ln\zeta(0.5+i(t+u))\,du$$

$$=\frac{1}{\pi-\delta}\sum_{n\le N_0}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\Big(\frac{\ln n}{\pi-\delta}\Big)+\frac{1}{\pi}\sum_{n>N_0}\frac{\Lambda_1(n)}{\sqrt{n}}n^{-it}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)$$

$$-2\pi\int_0^{0.5}K_a((\pi-\delta)(t+iv))\,dv+\theta\bigg(25\delta+\frac{150}{\sqrt{\ln(1/\delta)}}\bigg).$$

The integrals on both sides of (2.7) and the sum C_3 over $n\le N_0$ are continuous functions of δ, $0\le\delta\le 10^{-6}$. Letting δ tend to zero leads to the desired statement. Lemma 2.6 is proved. ∎

3. Basic lemma. The classical 'Dirichlet approximation theorem' asserts that for any fixed vector $(\alpha_1,\dots,\alpha_m)$ with real components and for an arbitrarily small ε, $0<\varepsilon<0.5$, the interval $(1,c)$, $c=\varepsilon^{-m}$, contains a number t such that $\|t\alpha_j\|<\varepsilon$, $j=1,\dots,m$.

Its standard proof (see, for example, [VK, Appendix, §9, Th. 4]) does not yield the existence of a t with the above property in every interval of the type $(T,T+c_1)$, where $c_1>0$ is a constant depending only on $(\alpha_1,\dots,\alpha_m)$ and ε.

In this section, we prove the analogue of Dirichlet's theorem which is free of the above disadvantage ([2]). However, the replacement of $(1, c)$ by an arbitrary interval $(T, T + c_1)$ leads to a loss of generality (the condition of linear independence of $1, \alpha_1, \ldots, \alpha_m$ over $\mathbb{Q}$ appears) and to non-effectiveness of the constant $c_1 = c_1(\alpha_1, \ldots, \alpha_m; \varepsilon)$. The last fact is a reason for the non-effectiveness of the constants c_0 and T_0 in Theorems 4.1–5.2 (c_0 and N_0 in Theorem 5.3, respectively) and for the impossibility of replacing A in Theorem 4.1 by some increasing function of T.

LEMMA 3.1. *For any vector* $\overline{\alpha} = (1, \alpha_1, \ldots, \alpha_n)$ *whose components are linearly independent over the rationals and for any* $0 < \varepsilon < 0.5$, *there exists a constant* $c = c(\overline{\alpha}, \varepsilon)$ *such that each interval of length* c *contains a* t *such that* $\|t\alpha_j\| < \varepsilon$, $j = 1, \ldots, n$.

Proof. The proof is preceded by some remarks.

REMARK 3.2. Let l be the line in $\mathbb{R}^{n+1}$ parallel to $\overline{\alpha}$ and passing through the origin, and let $X = (x_0, x_1, \ldots, x_n)$ be a point. Then the distance $d = d(X)$ between X and l is given by

$$d = \frac{1}{|\overline{\alpha}|}\sqrt{\sum_{0\le i<j\le n} \Delta_{ij}^2}, \quad \text{where} \quad |\overline{\alpha}| = \sqrt{1 + \sum_{1\le j\le n} \alpha_j^2}, \tag{3.1}$$

and Δ_{ij} is the minor of the matrix

$$\begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix},$$

formed by columns i and j. Suppose that the lattice point $M = (m_0, m_1, \ldots, m_n)$ satisfies $d(M) < \varepsilon_1 = \varepsilon|\alpha|^{-1}$. Then

$$\sum_{0\le i<j\le n} \Delta_{ij}^2 < \varepsilon^2$$

and therefore

$$|\Delta_{01}| = |\alpha_1 m_0 - m_1| < \varepsilon, \ \ldots, \ |\Delta_{0n}| = |\alpha_1 m_0 - m_n| < \varepsilon. \tag{3.2}$$

Since $0 < \varepsilon < 0.5$, the inequalities (3.2) imply that $\|\alpha_j t\| < \varepsilon$ for any $1 \le j \le n$, and for $t = m_0$.

Thus, it suffices to prove the existence of an infinite sequence of points M_j of the lattice $\mathbb{Z}^{n+1}$ lying in the ε-neighborhood of l and such that the distance between any neighbouring points M_j and M_{j+1} is bounded from above by some constant depending only on $\overline{\alpha}$ and ε.

([2]) The author sincerely thanks O. N. German and N. G. Moshchevitin who kindly communicated to him the idea of the proof of Lemma 3.1.

Remark 3.3. Set

$$\delta = \frac{\varepsilon_1}{n+1} = \frac{\varepsilon|\overline{\alpha}|^{-1}}{n+1}$$

and denote by C_δ the infinite cylinder of radius δ with axis l in $\mathbb{R}^{n+1}$. Suppose that there exist $K_1, \ldots, K_{n+1} \in \mathbb{Z}^{n+1}$ inside C_δ such that the vectors $\overline{v}_j = \overrightarrow{OK_j}$, $j = 1, \ldots, n+1$ are linearly independent. Then $\overline{v}_1, \ldots, \overline{v}_{n+1}$ generate an integer lattice $\mathcal{L}$ in $\mathbb{R}^{n+1}$ with fundamental domain Π, where Π is the parallelepiped spanned by $\overline{v}_1, \ldots, \overline{v}_{n+1}$.

It is known that any shift $\Pi + \overline{\xi}$ of Π by a vector $\overline{\xi} \in \mathbb{R}^{n+1}$ contains a point of $\mathcal{L}$ which is also a point of $\mathbb{Z}^{n+1}$. Further, Π is obviously contained in the cylinder $C_{\varepsilon_1} = (n+1)C_\delta$ of radius $(n+1)\delta = \varepsilon_1$, coaxial to C_δ.

Hence, any shift $\Pi + \overline{\xi}$ with $\overline{\xi}$ parallel to $\overline{\alpha}$ is fully contained inside C_{ε_1}. At the same time, this shift contains some lattice point $M(\overline{\xi})$.

Choosing vectors $\overline{\xi}_j$ in such a way that the shifts $\Pi + \overline{\xi}_j$ are pairwise disjoint, we find the desired infinite sequence $M_j = M(\overline{\xi}_j)$ (see Fig. 1).

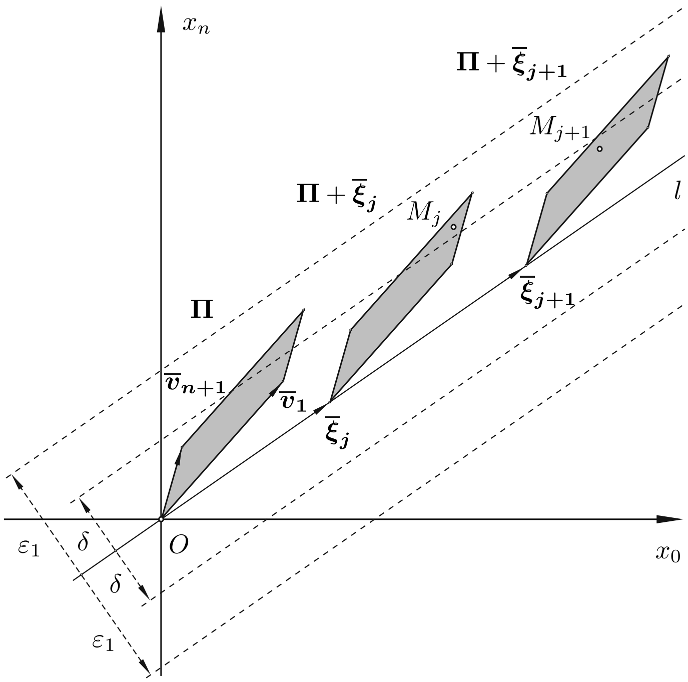

Fig. 1. Any shift $\Pi + \overline{\xi}_j$ of the parallelepiped Π contains a point M_j of $\mathbb{Z}^{n+1}$.

Thus, taking $\overline{\xi}_j = jc_0\overline{\alpha}$, $j = 0, \pm 1, \pm 2, \ldots$, where $c_0 = 2(|\overline{v}_1| + \cdots + |\overline{v}_{n+1}|)$ is twice the sum of the lengths of the edges of Π issuing from the same vertex, one can check that the first coordinate of $\overline{\xi}_j$, which is equal to jc_0, differs from the first coordinate $m_0^{(j)}$ of M_j by at most $|\overline{v}_1| + \cdots + |\overline{v}_{n+1}| = 0.5c_0$. In view of Remark 3.2, each of these first coordinates satisfies $\|\alpha_i m_0^{(j)}\| < \varepsilon$, $i = 1, \ldots, n+1$. Since

$$|m_0^{(j)} - m_0^{(j+1)}| \le (j+1)c_0 + 0.5c_0 - (jc_0 - 0.5c_0) = 2c_0,$$

any interval of the type $(\tau, \tau + 3c_0)$ contains a point of the sequence $m_0^{(j)}$, $j = 0, \pm 1, \pm 2, \ldots$.

Thus, it suffices to prove that any cylinder C_δ with axis l contains $n+1$ linearly independent vectors of $\mathbb{Z}^{n+1}$.

Now let us prove Lemma 3.1. First we show that C_δ contains an infinite set of lattice points.

The line l does not contain lattice points different from the origin O. Indeed, otherwise $d(K) = 0$, $k_0 \ne 0$, for such a point $K = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$. Hence $\Delta_{0j} = \alpha_j k_0 - k_j = 0$ for any $j = 1, \ldots, n$, and therefore $\alpha_j = k_j/k_0 \in \mathbb{Q}$. But this contradicts the linear independence of $1, \alpha_1, \ldots, \alpha_n$ over the rationals.

Let Ω_n be the n-dimensional hyperplane passing through O and perpendicular to l. Then the n-dimensional volume V_1 of the ball $C_\delta \cap \Omega_n$ is $V_1 = c(n)\delta^n$, where $c(n) = \pi^{n/2}\Gamma^{-1}(n/2+1)$. Now define H_1 by the relation $H_1 V_1 = 2^{n-1}$ and consider the $(n+1)$-dimensional cylinder T_1 of height $2H_1$ cut out from C_δ by two hyperplanes parallel to Ω_n and at distance H_1 from the origin.

Since the volume of T_1 is $2H_1V_1 = 2^n$, Minkowski's convex body theorem (see for example [GL, §5]) implies that T_1 contains a lattice point N_1 different from O.

Without loss of generality, we can assume that N_1 is closest to l among all such lattice points of T_1. In view of the above remark, $N_1 \notin l$, so $d(N_1) > 0$.

Further, set $\delta_2 = 0.5d(N_1)$ and define H_2 by $H_2 V_2 = 2^{n-1}$, $V_2 = c(n)\delta_2^n$. Applying the same arguments to the cylinder T_2 of radius δ_2 and height $2H_2$, symmetrical with respect to the origin and coaxial to T_1, we find a lattice point $N_2 \ne O$ inside it, closest to l. Since $d(N_2) \le \delta_2 < d(N_1)$, we have $N_2 \ne N_1$. In view of symmetry of both T_1 and T_2 with respect to O, we can assume that N_1 and N_2 lie in the same half-space with boundary Ω_n.

Taking $\delta_3 = 0.5d(N_2)$, $H_3V_3 = 2^{n-1}$, $V_3 = c(n)\delta_3^n$, we construct in the same way a cylinder T_3 of radius δ_3 and height $2H_3$ and find a lattice point N_3 inside it, which differs from O, N_1, N_2 and lies in the same half-space with boundary Ω_n.

Continuing, we get an infinite sequence of different points N_j of $\mathbb{Z}^{n+1}$ in the same half of the cylinder C_δ with respect to Ω_n and satisfying $0 < d(N_{j+1}) \le 0.5d(N_j)$, $j = 1, 2, \ldots$ (see Fig. 2).

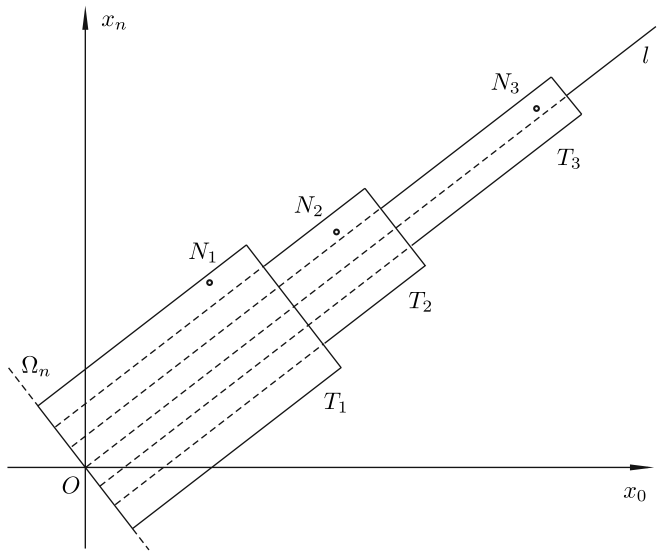

Fig. 2. An infinite sequence of lattice points N_j

Now we prove the existence of $n+1$ linearly independent vectors among the infinite set $\overrightarrow{ON_j}$, $j = 1, 2, \ldots$.

Assume to the contrary that the maximal number s of linearly independent vectors from this set does not exceed n. Let $\overline{u}_1, \ldots, \overline{u}_s \in \mathbb{Z}^{n+1}$ be such vectors and let ω_s be the s-dimensional hyperplane they span.

Then $\omega_s \cap C_\delta$ contains an infinite sequence of points N_j of $\mathbb{Z}^{n+1}$. Hence, this intersection is unbounded. But $\omega_s \cap C_\delta$ is unbounded if and only if ω_s is parallel to the line l or contains it (see Fig. 3).

In the first case, all the distances between N_j and l are bounded from below by some positive constant (the distance between ω_s and l). But this is impossible since $d(N_j) \to 0$ as $j \to \infty$.

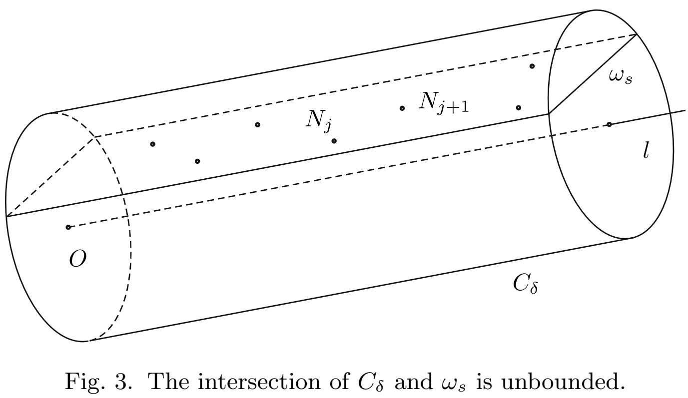

Fig. 3. The intersection of C_δ and ω_s is unbounded.

Now, if $l \subset \omega_s$ then $\overline{\alpha}$ is a linear combination of the form $\overline{\alpha} = t_1\overline{u}_1 + \cdots + t_s\overline{u}_s$. Denoting the components of $\overline{u}_j$ by $u_{0j}, u_{1j}, \ldots, u_{nj}$, we get

$$\begin{cases} t_1u_{01} + \cdots + t_su_{0s} = 1, \\ t_1u_{11} + \cdots + t_su_{1s} = \alpha_1, \\ \cdots \\ t_1u_{n1} + \cdots + t_su_{ns} = \alpha_n. \end{cases} \tag{3.3}$$

Since $\overline{u}_1, \ldots, \overline{u}_s$ are linearly independent, the $(n+1) \times s$-matrix of their components has rank s. Hence, it has s linearly independent rows; let $0 \le i_1 < \cdots < i_s \le n$ be their indices. If necessary, we put $\alpha_0 = 1$ and consider the corresponding subsystem of (3.3),

$$\begin{cases} t_1u_{i_1 1} + \cdots + t_su_{i_1 s} = \alpha_{i_1}, \\ \cdots \\ t_1u_{i_s 1} + \cdots + t_su_{i_s s} = \alpha_{i_s}. \end{cases}$$

Its determinant is a non-zero integer. Cramer's formulas imply that the unique solution of this system has the form

$$\begin{cases} t_1 = r_{11}\alpha_{i_1} + \cdots + r_{1s}\alpha_{i_s}, \\ \cdots \\ t_s = r_{s1}\alpha_{i_1} + \cdots + r_{ss}\alpha_{i_s}, \end{cases}$$

where r_{ij} are some rationals. Since $s \le n$, there exist at least one equation in (3.3) whose index j differs from $i_1, \ldots, i_s$. Thus we get

$$\begin{aligned} \alpha_j &= t_1u_{j1} + \cdots + t_su_{js} \\ &= (r_{11}\alpha_{i_1} + \cdots + r_{1s}\alpha_{i_s})u_{j1} + \ldots + (r_{s1}\alpha_{i_1} + \cdots + r_{ss}\alpha_{i_s})u_{js} \\ &= q_1\alpha_{i_1} + \cdots + q_s\alpha_s, \end{aligned}$$

where $q_1, \ldots, q_s \in \mathbb{Q}$, contrary to linear independence.

This contradiction implies that the hyperplane ω_s does not contain the line l. By Remark 3.3, this proves the lemma. ∎

Corollary 3.4. *For any vector $\overline{\alpha} = (1, \alpha_1, \ldots, \alpha_n)$ whose components are linearly independent over the rationals, for any tuple of real numbers $\beta_1, \ldots, \beta_n$ and for any $0 < \varepsilon < 0.5$, there exists a constant $c = c(\overline{\alpha}, \varepsilon)$ such that each interval of length c contains a t such that $\|t\alpha_j + \beta_j\| < \varepsilon$, $j = 1, \ldots, n$.*

Proof. We use the notation of Lemma 3.1. The above arguments imply that the cylinder C with radius $\varepsilon_1 = \varepsilon|\overline{\alpha}|^{-1}$ and axis l passing through the origin and parallel to $\overline{\alpha}$ contains an $(n+1)$-dimensional parallelepiped Π with vertices in $\mathbb{Z}^{n+1}$.

Then the cylinder $C_0 = C + \overline{\beta}$, where $\overline{\beta} = (1, \beta_1, \dots, \beta_n)$, contains the parallelepiped $\Pi_0 = \Pi + \overline{\beta}$. Any shift of Π contains a lattice point. Hence, both Π_0 and any parallelepiped Π_j which is the shift of Π_0 by a vector $\overline{\xi}_j = c_0 j\overline{\alpha}$, $j = \pm 1, \pm 2, \dots$, parallel to the axis of C_0, contain points of $\mathbb{Z}^{n+1}$. It is easy to note that the parallelepipeds Π_j have no common points.

Finally, let $M_j = (m_0, \dots, m_n)$ be a lattice point in Π_j. The distance between M_j and the axis of C_0 does not exceed ε_1. At the same time, this distance is expressed by (3.1), where Δ_{ij} is the minor of

$$\begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_n \\ m_0 & m_1 - \beta_1 & \dots & m_n - \beta_n \end{pmatrix}$$

formed by columns i, j. Hence,

$$|\Delta_{ij}| = |m_0\alpha_j - (m_j - \beta_j)| = |m_0\alpha_j + \beta_j - m_j| < \varepsilon$$

for any j, $1 \le j \le n$. Since $\varepsilon < 0.5$, we obtain $\|m_0\alpha_j + \beta_j\| < \varepsilon$. To end the proof, we note that the first coordinates m_0 of the points M_j form an increasing sequence, whose neighbouring elements differ by at most to $3c_0$. ∎

4. Large values of the Riemann zeta-function on the critical line. In this section, we give a conditional solution of Karatsuba's problem based on the Riemann hypothesis. We also prove a series of statements concerning the existence of large values of the function $S(t)$ on short intervals of the real axis.

THEOREM 4.1. *Suppose that the Riemann hypothesis is true, and let A be an arbitrarily large fixed constant. Then there exist constants $c_0 = c_0(A) > 0$ and $T_0 = T_0(A)$ such that each interval of the form $(T - H, T + H)$, $H = (1/\pi)\ln\ln\ln T + c_0$, $T > T_0$, contains a point t such that $|\zeta(0.5 + it)| > A$.*

Proof. Fix $a > 1$ satisfying

$$e^a\sqrt{\frac{\pi}{2a}} \ge \ln A. \tag{4.1}$$

Taking real parts in (2.4), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5 + i(t+u))|\,du \\ = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\cos(t\ln n) - 2\pi\int_0^{0.5}\Re K_a(\pi t + \pi i v)\,dv.\end{aligned} \tag{4.2}$$

Taking $t = 0$ in (4.2) and noting that $K_a(\pi iv) = e^{-a\cos\pi v}$, we have

$$(4.3)\qquad \int\limits_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5+iu)|\,du = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big) - 2\pi\int\limits_0^{0.5} e^{-a\cos(\pi v)}\,dv.$$

Further, the relation $|K_a(\pi t+\pi iv)| = e^{-a\cosh(\pi t)\cos(\pi v)}$ implies that the modulus of the last integral in (4.2) does not exceed

$$(4.4)\quad 2\pi\int\limits_0^{0.5} e^{-a\cosh(\pi t)\cos(\pi v)}\,dv = 2\pi\int\limits_0^{0.5} e^{-a\cosh(\pi t)\sin(\pi v)}\,dv < \frac{\pi}{a\cosh(\pi t)}$$

in modulus. Subtracting (4.3) from (4.2) and using (4.4), we find

$$(4.5)\qquad \int\limits_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\,du - \int\limits_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5+iu)|\,du$$
$$= 2\pi\int\limits_0^{0.5} e^{-a\cos(\pi v)}\,dv - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)\sin^2\Big(\frac{t}{2}\ln n\Big) + \frac{\pi\theta_1}{\cosh(\pi t)}.$$

Let ε, N satisfy $0 < \varepsilon < 0.5$, $N > N_0 = e^{\pi a\sqrt{2}}$ and depend only on a; their precise values will be chosen below. Applying Lemmas 2.1 and 3.1, we find a constant $c_0 = c_0(a)$ such that each interval of the real axis of length c_0 contains a point τ such that $\|(\tau/(2\pi))\ln p\| < \varepsilon$ for all primes $p \le N$. Let t in (4.5) to such a τ from $(T, T+c_0)$.

Given a prime $p \le N$, we define an integer n_p and a real ε_p satisfying $|\varepsilon_p| < \varepsilon$ such that $(t/(2\pi))\ln p = n_p + \varepsilon_p$. Then

$$\sin^2\Big(\frac{t}{2}\ln n\Big) = \sin^2(\pi k n_p + \pi k\varepsilon_p) = \sin^2(\pi k\varepsilon_p) < (\pi k\varepsilon)^2$$

for any $k \ge 1$ and $n = p^k$.

Let C be the sum on the right-hand side of (4.5). Denote by C_1 and C_2 the contributions to C from the terms corresponding to $n = p^k$, $k \ge 1$, $p \le N$ and from all other terms, respectively. Then

$$|C_1| \le \frac{2}{\pi}(\pi\varepsilon)^2 \sum_{\substack{n=p^k\\ k\ge1,\, p\le N}} \frac{k}{\sqrt{n}}\Big|\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)\Big|.$$

We split the domain of n into the intervals $n \le N_0$, $N_0 < n \le N$ and $n > N$ and denote the corresponding parts of the sum by C_3, C_4, C_5. The estimate

$|\widehat{K}_a((1/\pi)\ln n)| \le \widehat{K}_a(0)$ implies

$$|C_3| \le 2\pi\varepsilon^2 \widehat{K}_a(0) \sum_{p\le N_0} \sum_{k=1}^{\infty} kp^{-k/2} = 2\pi\varepsilon^2 \widehat{K}_a(0) \sum_{p\le N_0} \frac{1}{\sqrt{p}}\left(1-\frac{1}{\sqrt{p}}\right)^{-2} \le 2\pi\varepsilon^2\left(1-\frac{1}{\sqrt{2}}\right)^{-2} \widehat{K}_a(0) \sum_{p\le N_0} \frac{1}{\sqrt{p}}.$$

Let us use the inequality

$$\sum_{p\le x} \frac{1}{\sqrt{p}} \le \frac{2.784\sqrt{x}}{\ln x},$$

which can be verified for $2 \le x \le 1.5\cdot 10^6$ by using Wolfram Mathematica 7.0 and follows from [RS, Th. 2, Corollary 1, (3.6)] by Abel's summation formula for $x > 1.5\cdot 10^6$. Thus we get

$$|C_3| < 45.9\varepsilon^2 \widehat{K}_a(0) \frac{e^{\pi a/\sqrt{2}}}{a} \le (7\varepsilon)^2 e^{\pi a/\sqrt{2}} \widehat{K}_a(0).$$

Further, Corollary 2.4 implies

$$|C_4| \le 2\pi\varepsilon^2 \sum_{\substack{N_0<n\le N\\ n=p^k}} kp^{-k/2}\cdot \frac{61.5\sqrt{\pi}}{\sqrt{k\ln p}} \exp\left(-\frac{\pi}{2}\frac{1}{\pi}\ln p^k\right)$$

$$= 123\pi\sqrt{\pi}\,\varepsilon^2 \sum_{\substack{N_0<n\le N\\ n=p^k}} \frac{\sqrt{k}}{p^k\sqrt{\ln p}} \le 123\pi\sqrt{\pi}\,\varepsilon^2 \sum_{p\le N} \frac{1}{\sqrt{\ln p}} \sum_{k=1}^{\infty} kp^{-k}$$

$$< 123\pi\sqrt{\pi}\,\varepsilon^2 \sum_{p\le N} \frac{1}{p\sqrt{\ln p}}\left(1-\frac{1}{p}\right)^{-2} < 123\pi\sqrt{\pi}\,\varepsilon^2 \sum_{p} \frac{p}{(p-1)^2\sqrt{\ln p}}$$

$$< 3000\varepsilon^2.$$

Applying Corollary 2.4 together with (2.6) and noting that $\ln N \ge \pi a\sqrt{2} \ge \pi\sqrt{2}$, we find

$$|C_5| \le \frac{2}{\pi} \sum_{n>N} \frac{\Lambda_1(n)}{\sqrt{n}} \frac{61.5\sqrt{\pi}}{\sqrt{n\ln n}} = \frac{123}{\sqrt{\pi}} \sum_{n>N} \frac{\Lambda(n)}{n(\ln n)^{3/2}}.$$

Abel's summation formula together with the bound

$$\psi(u) = \sum_{n\le u} \Lambda(n) \le c_1 u, \quad c_1 = 1.03883$$

(see [RS, Th. 12]), which is valid for any $u > 0$, implies

$$\sum_{n>N}\frac{\Lambda(n)}{n(\ln n)^{3/2}} = -\int_N^\infty (\psi(u)-\psi(N))\,d\frac{1}{(\ln u)^{3/2}} \le -c_1\int_N^\infty u\,d\frac{1}{(\ln u)^{3/2}} = c_1\biggl(\frac{2}{\sqrt{\ln N}}+\frac{1}{(\ln N)^{3/2}}\biggr).$$

Since $\ln N \ge \pi a\sqrt{2} \ge \pi\sqrt{2}$, we finally get

$$|C_5| \le \frac{123}{\sqrt{\pi}}\,\frac{2c_1}{\sqrt{\ln N}}\biggl(1+\frac{1}{2\pi\sqrt{2}}\biggr) < \frac{160.5}{\sqrt{\ln N}},$$

and so

$$|C_1| \le |C_3|+|C_4|+|C_5| < (7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{160.5}{\sqrt{\ln N}}.$$

Applying the same arguments to C_2, we obtain

$$|C_2| \le \frac{2}{\pi}\sum_{n>N}\frac{\Lambda_1(n)}{\sqrt{n}}\,\frac{61.5\sqrt{\pi}}{\sqrt{n\ln n}} < \frac{160.5}{\sqrt{\ln N}}.$$

Thus

$$|C| \le |C_1|+|C_2| < (7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}},$$

and therefore

$$\int_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\,du \ge 2\int_0^{\pi/2} e^{-a\sin v}\,dv + 2\int_0^\infty K_a(\pi u)\ln|\zeta(0.5+iu)|\,du - \biggl((7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{\pi}{\cosh \pi t}\biggr). \tag{4.6}$$

Now we estimate the modulus of the improper integral on the right-hand side of (4.6). We split it into integrals j_1 and j_2 over $0 \le u \le 10$ and $u > 10$, respectively. Since $|\ln|\zeta(0.5+iu)|| \le 0.641973\ldots < 2/3 - 1/50$ for $0 \le u \le 10$, we find

$$|j_1| < \biggl(\frac{2}{3}-\frac{1}{50}\biggr)\int_0^{10} K_a(\pi u)\,du < \frac{1}{\pi}\biggl(\frac{1}{3}-\frac{1}{100}\biggr)\widehat{K}_a(0).$$

Further, the formula for $\widehat{K}_a(0)$ from [O, Ex. 9.1] implies that

$$\tfrac{7}{8}e^{-a}\sqrt{2\pi/a} < \widehat{K}_a(0) < e^{-a}\sqrt{2\pi/a} \tag{4.7}$$

for $a > 1$. Hence,

$$|j_2| \le \frac{\widehat{K}_a(0)}{\widehat{K}_a(0)} \int_{10}^{\infty} e^{-a\cosh(\pi u)} \big|\ln|\zeta(0.5+iu)|\big|\, du$$

$$\le \widehat{K}_a(0)\frac{8}{7}e^{a}\sqrt{\frac{a}{2\pi}} \int_{10}^{\infty} \exp(-0.5ae^{\pi u})\big|\ln|\zeta(0.5+iu)|\big|\, du$$

$$= \widehat{K}_a(0)\frac{8}{7}e^{-a}\sqrt{\frac{a}{2\pi}} \int_{10}^{\infty} \exp\big(-0.5a(e^{\pi u}-4)\big)\big|\ln|\zeta(0.5+iu)|\big|\, du.$$

Since $0.5(e^{\pi u}-4) > 2u^2$ for $u \ge 10$, we find

$$|j_2| \le \frac{\widehat{K}_a(0)}{\widehat{K}_a(0)} \int_{10}^{\infty} e^{-2u^2} \big|\ln|\zeta(0.5+iu)|\big|\, du$$

$$\le \widehat{K}_a(0)\frac{8}{7}e^{-a}\sqrt{\frac{a}{2\pi}} \cdot 1.52\cdot 10^{-89} < 1.5\cdot 10^{-90}\widehat{K}_a(0).$$

Thus we get

$$|j_1| + |j_2| < \frac{1}{\pi}\left(\frac{1}{3} - \frac{1}{100}\right)\widehat{K}_a(0) + 1.5\cdot 10^{-90}\widehat{K}_a(0) < \frac{\widehat{K}_a(0)}{3\pi}.$$

Obviously,

$$\int_0^{\pi/2} e^{-a\sin v}\, dv \ge \int_0^{\pi/2} e^{-av}\, dv = \frac{1}{a}(1-e^{-\pi a/2}).$$

Therefore, (4.6) implies

$$\int_{-\infty}^{\infty} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\, du \ge \frac{2}{a}(1-e^{-\pi a/2}) - \left((7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}} + 3000\varepsilon^2 + \frac{321}{\sqrt{\ln N}} + \frac{\widehat{K}_a(0)}{3\pi} + \frac{\pi}{\cosh \pi t}\right). \tag{4.8}$$

Further, we set $h = (1/\pi)(\ln\ln\ln T - \ln(a/2))$ and split the integral

$$\left(\int_{-h}^{h} + \int_{h}^{\infty} + \int_{-\infty}^{-h}\right) K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\, du = j_3 + j_4 + j_5.$$

The formula

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s\int_1^{\infty} \frac{\varrho(u)}{u^{s+1}}\, du,$$

where $\varrho(u) = 0.5 - \{u\}$, $\Re s > 0$, $s \ne 1$ (see [K1, Ch. II, Lemma 2]), implies that $0 \le |\zeta(0.5+iv)| \le |v| + 3$ for any real v. Hence,

$$-\infty \le \ln|\zeta(0.5+iv)| < \ln(|v|+3).$$

Passing to the estimate of j_4, we get

$$-\infty \le j_4 = \int_h^{\infty} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\,du$$
$$< \int_h^{\infty} K_a(\pi u)\ln(|t+u|+3)\,du$$
$$= \biggl(\int_h^{t} + \int_t^{\infty}\biggr) K_a(\pi u)\ln(|t+u|+3)\,du = j_6 + j_7.$$

Estimating the integrals j_6 and j_7 separately, we find

$$j_6 \le \ln(2t+3)\int_h^{\infty}\exp(-0.5ae^{\pi u})\,du = \frac{1}{\pi}\ln(2t+3)\int_{0.5ae^{\pi h}}^{\infty} e^{-w}\,\frac{dw}{w}$$
$$= \frac{1}{\pi}\ln(2t+3)\int_{\ln\ln T}^{\infty} e^{-w}\,\frac{dw}{w} < \frac{\ln(2t+3)}{\pi\ln T}\,\frac{1}{\ln\ln T}.$$

Similarly,

$$j_7 \le \int_t^{\infty}\exp(-0.5ae^{\pi u})\ln(2u+3)\,du$$
$$\le 2\int_t^{\infty}\exp(-0.5ae^{\pi u})(\ln u)\,du$$
$$< \frac{2}{\pi}\ln(\pi t/2)e^{-\pi t/2}\exp(-e^{\pi t/2}).$$

Therefore,

$$-\infty \le j_4 = j_6 + j_7$$
$$< \frac{\ln(2t+3)}{\pi\ln T}\frac{1}{\ln\ln T} + \frac{2}{\pi}\ln(\pi t/2)e^{-\pi t/2}\exp(-e^{\pi t/2}) < \frac{1}{3\ln\ln T}.$$

The integral j_5 is estimated in the same way:

$$j_5 = \int_h^{\infty} K_a(\pi u)\ln|\zeta(0.5+i(t-u))|\,du$$
$$< \int_h^{\infty} K_a(\pi u)\ln(|t-u|+3)\,du$$
$$= \biggl(\int_h^{2t} + \int_{2t}^{\infty}\biggr) K_a(\pi u)\ln(|t-u|+3)\,du = j_8 + j_9,$$

where

$$j_8 \le \ln(t+3)\int\limits_h^\infty K_a(\pi u)\,du < \frac{\ln(t+3)}{\pi\ln T}\,\frac{1}{\ln\ln T},$$

$$j_9 < \int\limits_{2t}^\infty K_a(\pi u)\ln(u+3)\,du < \frac{2}{\pi}\ln(\pi t)e^{-\pi t}\exp(-e^{\pi t}),$$

and hence $j_5 < (3\ln\ln T)^{-1}$.

Going back to (4.8), we obtain

$$\begin{aligned}(4.9)\qquad &\int\limits_{-h}^{h} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\,du\\ &> \frac{2}{a} - \left(\frac{2}{a}e^{-\pi a/2}+(7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+3000\varepsilon^2+\frac{321}{\sqrt{\ln N}}+\frac{2\widehat{K}_a(0)}{3\pi}+\frac{1}{\ln\ln T}\right)\\ &> \frac{2}{a} - \left(\frac{2}{a}\cdot\frac{1}{4}+(7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+3000\varepsilon^2+\frac{321}{\sqrt{\ln N}}+\frac{2\widehat{K}_a(0)}{3\pi}\right).\end{aligned}$$

In view of (4.7), the last expression in brackets does not exceed

$$\begin{aligned}&\frac{1}{2a}+(7\varepsilon)^2\sqrt{\frac{2\pi}{a}}\,e^{(\pi/\sqrt{2}-1)a}+3000\varepsilon^2\frac{321}{\sqrt{\ln N}}+\frac{2}{3\pi}e^{-a}\sqrt{\frac{2\pi}{a}}\\ &\qquad < \frac{1}{2a}\left(\frac{3}{2}+2(7\varepsilon)^2\sqrt{2\pi a}\,e^{(\pi/\sqrt{2}-1)a}+6000a\varepsilon^2+\frac{642a}{\sqrt{\ln N}}\right).\end{aligned}$$

Now we set

$$\varepsilon = \frac{e^{-2a/3}}{100\sqrt{a}},\qquad N = e^{(3852a)^2}.$$

Then the left-hand side of the last inequality does not exceed

$$\frac{1}{2a}\left(\frac{3}{2}+\frac{\sqrt{2\pi}}{100}e^{-0.1a}+\frac{3}{5}e^{-4a/3}+\frac{1}{6}\right) < \frac{1}{2a}\left(\frac{5}{3}+\frac{1}{6}+\frac{1}{6}\right) = \frac{1}{a}.$$

Now (4.9) implies that

$$(4.10)\qquad \int\limits_{-h}^{h} K_a(\pi u)\ln|\zeta(0.5+i(t+u))|\,du > \frac{2}{a}-\frac{1}{a}=\frac{1}{a}.$$

Denote by M the maximum of $\ln|\zeta(0.5+i(t+u))|$ on $|u|\le h$. Then (4.10) implies that $M>0$. Hence, the integral in (4.10) is less than

$$M\int\limits_{-h}^{h} K_a(\pi u)\,du < \frac{M}{\pi}\int\limits_{-\infty}^{\infty} K_a(u)\,du = \frac{M}{\pi}\widehat{K}_a(0).$$

Using (4.7) and (4.1), we find

$$\frac{M}{\pi}\widehat{K}_a(0) > \frac{1}{a}, \qquad M > \frac{\pi}{a}\widehat{K}_a^{-1}(0) > e^a\sqrt{\frac{\pi}{2a}} \geq \ln A.$$

To end the proof, we note that the distance between T and the point u where the maximum M is attained, does not exceed

$$c_0 + h = \frac{1}{\pi}(\ln\ln\ln T - \ln(a/2)) + c_0.$$

The proof of Theorem 4.1 is complete. ■

REMARK 4.2. In [FK], the following conjecture is stated: the probability density of the random variable $\sigma(T)$ with the values

$$-2\ln F(t;2\pi) + 2\ln\ln\frac{t}{2\pi} - \frac{3}{2}\ln\ln\ln\frac{t}{2\pi}, \quad t_0 \leq t \leq T,$$

tends to $p(x) = 2e^x\mathcal{K}_0(2e^{x/2})$ as $T \to \infty$, where $\mathcal{K}_\nu(z)$ denotes the modified Bessel function of the second kind. In [H], there are some arguments that support the hypothesis that the inequalities

$$\frac{\ln t}{(\ln\ln t)^{2+\varepsilon}} \leq F(t;2\pi) \leq \frac{\ln t}{(\ln\ln t)^{0.25-\varepsilon}}$$

hold for "almost all" t in $(T, 2T)$, for $T \to +\infty$ and for any $\varepsilon > 0$.

THEOREM 4.3. *Suppose that the quantity*

$$S_0 = \frac{1}{\pi}\sum_{n=p^{2k+1}}(-1)^k\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right) = \frac{1}{\pi}\Im\sum_{n=2}^{\infty} i^{\Omega(n)}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right)$$

is positive for some $a \geq 1$. Then for any fixed $\varepsilon > 0$ satisfying $0 < \varepsilon <$ $\min(0.5, S_0)$ there exist constants c_0 and T_0 depending on a and ε only and such that

$$\max_{|t-T|\leq H}(\pm S(t)) > \frac{S_0 - \varepsilon}{\pi\widehat{K}_a(0)}$$

for any $T \geq T_0$ and $H = (1/\pi)\ln\ln\ln T + c_0$.

Proof. Taking real parts in (2.4), we obtain

$$\pi\int_{-\infty}^{\infty} K_a(\pi u)S(t+u)\,du = -\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\sin(t\ln n) + \frac{\pi\theta_1}{\cosh \pi t}. \tag{4.11}$$

Let ε_1, N be numbers depending on a, ε and such that $0 < \varepsilon_1 < 0.5$, $N \geq e^{\pi a\sqrt{2}}$; their explicit values will be chosen later. By Lemma 3.1, there exists a constant $c = c(a, \varepsilon)$ such that each interval of length c contains a point τ such that

$$\left\| \frac{\tau}{2\pi} \ln p + \frac{1}{4} \right\| < \varepsilon_1 \tag{4.12}$$

for any prime $p \le N$. Take the parameter t in (4.11) equal to such a τ from the interval $(T, T+c)$.

Given a prime $p \le N$, we define an integer n_p and a real ε_p satisfying $|\varepsilon_p| < \varepsilon_1$ and such that

$$\frac{t}{2\pi} \ln p = n_p + \varepsilon_p - \frac{1}{4}.$$

Then

$$\sin(t \ln n) = -\sin(\pi k/2) \cos(2\pi k \varepsilon_p) + \cos(\pi k/2) \sin(2\pi k \varepsilon_p)$$

for any $k \ge 1$ and $n = p^k$, $p \le N$. If k is even then $|\sin(t \ln n)| = |\sin(2\pi k \varepsilon_p)| \le 2\pi k \varepsilon_1$; otherwise, we have

$$\sin(t \ln n) = (-1)^{(k+1)/2} \cos(2\pi k \varepsilon_p) = (-1)^{(k+1)/2} - 2\theta_2 (\pi k \varepsilon_1)^2.$$

Let S be the sum on the right-hand side of (4.11). Denote by S_1, S_2 and S_3 the contributions to this sum from the terms corresponding to the following conditions: $n = p^k$, $p \le N$, k odd; $n = p^k$, $p \le N$, k even; $n = p^k$, $p > N$, respectively. Then

$$\begin{aligned} S_1 &= -\frac{1}{\pi} \sum_{\substack{n=p^{2k+1} \\ p \le N,\, k \ge 0}} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \left((-1)^{k+1} - 2\theta_2 (\pi(2k+1)\varepsilon_1)^2\right) \\ &= \frac{1}{\pi} \sum_{\substack{n=p^{2k+1} \\ p \le N,\, k \ge 0}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \\ &\quad + 2\theta_3 \pi \varepsilon_1^2 \sum_{\substack{n=p^{2k+1} \\ p \le N,\, k \ge 0}} (2k+1)^2 \frac{\Lambda_1(n)}{\sqrt{n}} \left| \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \right|. \end{aligned} \tag{4.13}$$

Obviously, the last sum in (4.13) is less than

$$\begin{aligned} 2\pi \varepsilon_1^2 \widehat{K}_a(0) \sum_{p \le N} \sum_{k=0}^{\infty} \frac{2k+1}{p^k \sqrt{p}} &= 2\pi \varepsilon_1^2 \widehat{K}_a(0) \sum_{p \le N} \frac{1}{\sqrt{p}} \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{-2} \\ &\le 12\pi \varepsilon_1^2 \widehat{K}_a(0) \sum_{p \le N} \frac{1}{\sqrt{p}} < \frac{105 \varepsilon_1^2 \widehat{K}_a(0) \sqrt{N}}{\ln N}. \end{aligned}$$

Further, we replace the interval $p \le N$ in the first sum on the right-hand side of (4.13) by an infinite one. The resulting error does not exceed (in modulus)

$$\frac{1}{\pi} \sum_{n > N} \frac{\Lambda_1(n)}{\sqrt{n}} \left| \widehat{K}_a\left(\frac{\ln n}{\pi}\right) \right| < \frac{81}{\sqrt{\ln N}}. \tag{4.14}$$

Hence, the difference between S_1 and S_0 is less than

$$\frac{105\varepsilon_1^2\widehat{K}_a(0)\sqrt{N}}{\ln N}+\frac{81}{\sqrt{\ln N}}.$$

Further,

$$|S_2|\le\frac{1}{\pi}\sum_{\substack{n=p^{2k}\\ p\le N,\,k\ge 1}}\frac{\Lambda_1(n)}{\sqrt{n}}|\widehat{K}_a(0)|\cdot 4\pi k\varepsilon_1\le 2\varepsilon_1\widehat{K}_a(0)\sum_{p\le N}\sum_{k=1}^{\infty}p^{-k}$$
$$=2\varepsilon_1\widehat{K}_a(0)\sum_{p\le N}\frac{1}{p-1}<3\varepsilon_1\widehat{K}_a(0)\ln\ln N.$$

Obviously, the modulus of S_3 does not exceed the right-hand side of (4.14).

Therefore, S and S_0 differ by at most

$$\frac{105\varepsilon_1^2\widehat{K}_a(0)\sqrt{N}}{\ln N}+\frac{162}{\sqrt{\ln N}}+3\varepsilon_1\widehat{K}_a(0)\ln\ln N.$$

We set $h=(1/\pi)(\ln\ln\ln T-\ln(a/2))$ and split the improper integral in (4.11) into integrals j_1, j_2 and j_3 corresponding to $|u|\le h$, $u>h$ and $u<-h$ respectively. If $|v|\ge 280$, Backlund's classical estimate [B] implies that

$$\text{(4.15)}\quad |S(v)|<0.1361\ln|v|+0.4422\ln\ln|v|+4.3451$$
$$\le\left(0.1361+0.4422\frac{\ln\ln 280}{\ln 280}+\frac{4.3451}{\ln 280}\right)\ln|v|<1.05\ln|v|.$$

Otherwise, we have $|S(v)|\le 1$ (see [L, Tab. 1]). From these estimates, it follows that $|j_2|+|j_3|<2(\ln\ln T)^{-1}$. Hence,

$$\text{(4.16)}\quad \pi\int_{-h}^{h}K_a(\pi u)S(t+u)\,du$$
$$>S_0-\left(\frac{105\varepsilon_1^2\widehat{K}_a(0)\sqrt{N}}{\ln N}+\frac{162}{\sqrt{\ln N}}+3\widehat{K}_a(0)\varepsilon_1\ln\ln N+\frac{3}{\ln\ln T}\right).$$

The expression in brackets is less than

$$\text{(4.17)}\quad \frac{105\varepsilon_1^2\sqrt{N}}{\ln N}e^{-a}\sqrt{\frac{2\pi}{a}}+\frac{162}{\sqrt{\ln N}}+3e^{-a}\sqrt{\frac{2\pi}{a}}\,\varepsilon_1\ln\ln N+\frac{3}{\ln\ln T}$$
$$<\frac{(10\varepsilon_1)^2\sqrt{N}}{\ln N}+\frac{162}{\sqrt{\ln N}}+3\varepsilon_1\ln\ln N.$$

Now we take

$$\varepsilon_1=\exp\left(-\left(\frac{162}{\varepsilon}\right)^2\right),\qquad N=\exp\left(\left(\frac{324}{\varepsilon}\right)^2\right).$$

Then the right-hand side of (4.17) is bounded from above by

$$\left(\frac{\varepsilon}{30}\right)^2 + 6\exp\left(-\left(\frac{162}{\varepsilon}\right)^2\right)\ln\left(\frac{324}{\varepsilon}\right) + \frac{\varepsilon}{2} < \varepsilon.$$

Since $0 < \varepsilon < S_0$, the right-hand side of (4.16) is positive. Denoting $M_1 = \max_{|u|\le h} S(t+u)$, we therefore have

$$M_1 > 0, \qquad S_0 - \varepsilon < \pi M_1 \int_{-h}^{h} K_a(\pi u)\, du < M_1 \widehat{K}_a(0).$$

Thus, $M_1 > (S_0 - \varepsilon)\widehat{K}_a^{-1}(0)$. Since the distance between T and the point $t+u$ where the maximum is attained, is less than $H = h + c = (1/\pi)(\ln\ln\ln T - \ln(a/2)) + c$, the first statement of theorem is proved. The proof of the second one is similar. The only difference is that t is chosen now in $(T, T+c)$ to satisfy

$$\left\|\frac{t}{2\pi}\ln p - \frac{1}{4}\right\| < \varepsilon_1$$

for all primes $p \le N$. ∎

The very slow convergence of the series S_0 and the absence of the analogue of the identity (4.3) make the verification of the condition $S_0 > 0$ very difficult. However, a small modification of the above proof allows one to obtain a series of numerical results.

THEOREM 4.4. *Let a, b, τ be any positive numbers satisfying $0 < b < \pi/2$, $b\tau > 0.5$, $\gamma = b\tau + 0.5$, let $N \ge 2$ be an integer, and let*

$$S_N(u) = \sum_{p \le N} \arctan\left(\frac{2\sqrt{p}}{p-1}\cos(u\tau \ln p)\right).$$

Further, let

$$\kappa = \kappa(a,b) = 2\int_0^\infty e^{-a\cos(b)\cosh(u)}\, du,$$

$$\zeta_N(\gamma) = \prod_{p>N}(1-p^{-\gamma})^{-1} = \zeta(\gamma)\prod_{p\le N}(1-p^{-\gamma}),$$

$$I = \frac{1}{\pi}\left(\int_0^\infty K_a(u) S_N(u)\, du - \kappa \ln \zeta_N(\gamma)\right) > 0.$$

Then, for any fixed $0 < \varepsilon < \varepsilon_0(a,b,\tau)$, there exists a constant $c_0 = c_0(\varepsilon; a, b, \tau)$ such that

$$\max_{|T-t|\le H}(\pm S(t)) > \frac{I-\varepsilon}{\widehat{K}_a(0)}$$

for any $T \ge T_0(\varepsilon; a, b, \tau)$ and $H = \tau \ln\ln\ln T + c_0$.

Proof. Setting $f(u) = (1/\tau)K_a(u/\tau)$ in Lemma 2.5 and taking imaginary parts, we get

$$\frac{1}{\tau}\int_{-\infty}^{\infty} K_a\Big(\frac{u}{\tau}\Big)S(t+u)\,du = C + \frac{\pi\theta_1}{a\cosh(t/\tau)}, \tag{4.18}$$

where

$$C = -\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a(\tau\ln n)\sin(\tau\ln n).$$

We denote

$$c = \kappa(4/3 + 7\ln\zeta(\gamma)), \qquad \varepsilon_0 = \min(0.5, I/c, \varepsilon/c)$$

and fix ε_1 and N satisfying $0 < \varepsilon_1 < \varepsilon_0$, $N > 2$. By Corollary 3.4, there exists a constant c_0 depending only on ε_1, N and such that each interval of length c_0 contains a point τ such that (4.12) holds for any prime $p \le N$. Suppose that t is such a value from $(T, T + c_0)$. Similarly to the proof of Theorem 4.3, we split the sum C into C_1, C_2 and C_3. We get $C_1 = C_0 + \theta_2 C_4$, where

$$C_0 = \frac{1}{\pi}\sum_{\substack{n=p^{2k+1},\, k\ge 0\\ p\le N}}(-1)^k\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a(\tau\ln n),$$

$$C_4 = 2\pi\varepsilon_1^2\sum_{\substack{n=p^{2k+1},\, k\ge 0\\ p\le N}}(2k+1)^2\frac{\Lambda_1(n)}{\sqrt{n}}|\widehat{K}_a(\tau\ln n)|.$$

Moreover,

$$|C_2| \le 4\varepsilon_1\sum_{\substack{n=p^{2k},\, k\ge 1\\ p\le N}}k\frac{\Lambda_1(n)}{\sqrt{n}}|\widehat{K}_a(\tau\ln n)|,$$

$$|C_3| \le \frac{1}{\pi}\sum_{n=p^k,\, p>N}\frac{\Lambda_1(n)}{\sqrt{n}}|\widehat{K}_a(\tau\ln n)|.$$

Lemma 2.2 yields

$$\begin{aligned}|C_2| &\le 4\varepsilon_1\sum_{\substack{n=p^{2k},\, k\ge 1\\ p\le N}}\frac{k}{2k\sqrt{n}}\kappa e^{-b\tau\ln n} = 2\kappa\varepsilon_1\sum_{p\le N}\sum_{k=1}^{\infty}p^{-2k\gamma}\\ &= 2\kappa\varepsilon_1\sum_{p\le N}p^{-2\gamma}(1-p^{-2\gamma})^{-1} \le \frac{2\kappa\varepsilon_1}{1-2^{-2\gamma}}\ln\zeta(2\gamma)\\ &< \frac{2\kappa\varepsilon_1}{1-2^{-2}}\ln\zeta(2) < \frac{4}{3}\kappa\varepsilon_1,\end{aligned}$$

$$|C_3| \le \frac{\kappa}{\pi} \sum_{\substack{n=p^k,\, k\ge 1\\ p>N}} \frac{\Lambda_1(n)}{n^\gamma} = \frac{\kappa}{\pi} \sum_{p>N} \ln(1-p^{-\gamma})^{-1}$$
$$= \frac{\kappa}{\pi} \ln\Big(\zeta(\gamma) \prod_{p\le N} (1-p^{-\gamma})\Big) = \frac{\kappa}{\pi} \ln \zeta_N(\gamma),$$

and finally

$$|C_4| \le 2\pi\kappa\varepsilon_1^2 \sum_{\substack{n=p^{2k+1},\, k\ge 0\\ p\le N}} \frac{2k+1}{n^\gamma} = 2\pi\kappa\varepsilon_1^2 \sum_{p\le N} \sum_{k=0}^{\infty} \frac{2k+1}{p^{(2k+1)\gamma}}$$
$$= 2\pi\kappa\varepsilon_1^2 \sum_{p\le N} \frac{1}{p^\gamma} \frac{1+p^{-2\gamma}}{(1-p^{-2\gamma})^2} \le 2\pi\kappa\varepsilon_1^2 \frac{1+2^{-2\gamma}}{(1-2^{-2\gamma})^2} \sum_{p\le N} \frac{1}{p^\gamma}$$
$$< \frac{40\pi}{9} \kappa\varepsilon_1^2 \ln \zeta(\gamma).$$

Transforming the sum C_0, we obtain

$$C_0 = \frac{1}{\pi} \sum_{\substack{n=p^{2k+1},\, k\ge 0\\ p\le N}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} \int_{-\infty}^{\infty} K_a(u) e^{-iu\tau \ln n}\, du \tag{4.19}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} K_a(u) \Big(\sum_{\substack{n=p^{2k+1},\, k\ge 0\\ p\le N}} (-1)^k \frac{\Lambda_1(n)}{\sqrt{n}} n^{-iu\tau} \Big)\, du$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} K_a(u) \sum_{p\le N} \Big(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \Big(\frac{p^{-iu\tau}}{\sqrt{p}}\Big)^{2k+1} \Big)\, du$$
$$= \frac{1}{2\pi i} \int_0^{\infty} K_a(u) \sum_{p\le N} \Big\{ \ln\Big(1+\frac{ip^{-iu\tau}}{\sqrt{p}}\Big) - \ln\Big(1-\frac{ip^{-iu\tau}}{\sqrt{p}}\Big)$$
$$+ \ln\Big(1+\frac{ip^{iu\tau}}{\sqrt{p}}\Big) - \ln\Big(1-\frac{ip^{iu\tau}}{\sqrt{p}}\Big) \Big\}\, du.$$

For fixed $p \le N$, we denote

$$z_1 = 1 + \frac{ip^{iu\tau}}{\sqrt{p}} = |z_1| e^{i\varphi_1}, \qquad z_2 = 1 - \frac{ip^{iu\tau}}{\sqrt{p}} = |z_2| e^{i\varphi_2},$$

where $-\pi < \varphi_1, \varphi_2 \le \pi$. Then the summands in braces in (4.19) take the form

$$\ln \overline{z}_2 - \ln \overline{z}_1 + \ln z_1 - \ln z_2 = \ln \frac{\overline{z}_2}{z_2} - \ln \frac{\overline{z}_1}{z_1} = 2i(\varphi_1 - \varphi_2).$$

Writing $\alpha_p = u\tau \ln p$ and noting that

$$z_1 = 1 - \frac{\sin\alpha_p}{\sqrt{p}} + \frac{i\cos\alpha_p}{\sqrt{p}},$$

we find

$$\tan\varphi_1 = \tan(\arg z_1) = \frac{(\cos\alpha_p)/\sqrt{p}}{1-(\sin\alpha_p)/\sqrt{p}} = \frac{\cos\alpha_p}{\sqrt{p}-\sin\alpha_p}.$$

Similarly,

$$\tan\varphi_2 = \tan(\arg z_2) = -\frac{\cos\alpha_p}{\sqrt{p}+\sin\alpha_p}.$$

Hence

$$\tan(\varphi_1-\varphi_2) = \frac{\tan\varphi_1-\tan\varphi_2}{1+\tan\varphi_1\tan\varphi_2} = \frac{2\sqrt{p}}{p-1}\cos\alpha_p,$$

and therefore

$$\varphi_1-\varphi_2 = \arctan\left(\frac{2\sqrt{p}}{p-1}\cos\alpha_p\right),$$

$$C_0 = \frac{1}{\pi}\int_0^\infty K_a(u)\sum_{p\le N}\arctan\left(\frac{2\sqrt{p}}{p-1}\cos\alpha_p\right)du = \frac{1}{\pi}\int_0^\infty K_a(u)S_N(u)\,du.$$

Summing the above bounds, we conclude that the difference between C_0 and the right-hand side of (4.18) does not exceed (in modulus)

$$\frac{\kappa}{\pi}\ln\zeta_N(\gamma) + \frac{4}{3}\kappa\varepsilon_1 + \frac{40\pi}{9}\kappa\varepsilon_1^2\ln\zeta(\gamma) + \frac{\pi}{a\cosh(t/\tau)}$$
$$< \frac{\kappa}{\pi}\ln\zeta_N(\gamma) + \kappa\varepsilon_1\left(\frac{4}{3}+7\ln\zeta(\gamma)\right) - \frac{3}{\ln\ln T} = \frac{\kappa}{\pi}\ln\zeta_N(\gamma) + c\varepsilon_1 - \frac{3}{\ln\ln T}.$$

Let $h = \tau(\ln\ln\ln T - \ln(a/2))$. Splitting the integral in (4.18) as

$$j_1+j_2+j_3 = \frac{1}{\tau}\left(\int_{-h}^{h} + \int_{h}^{\infty} + \int_{-\infty}^{-h}\right)K_a\left(\frac{u}{\tau}\right)S(t+u)\,du$$

and using the same bounds for $S(u)$ as in the proof of Theorem 4.3, we find $|j_2|+|j_3| < 3(\ln\ln T)^{-1}$. Hence,

$$j_1 = \frac{1}{\tau}\int_{-h}^{h} K_a\left(\frac{u}{\tau}\right)S(t+u)\,du > C_0 - \frac{\kappa}{\pi}\ln\zeta_N(\gamma) - c\varepsilon_1 = I - c\varepsilon_1.$$

Since $0 < \varepsilon_1 < I/c$, the right-hand side above is strictly positive, and so is $M_1 = \max_{|u|\le h} S(t+u)$. Obviously, we have $j_1 < M_1\widehat{K}_a(0)$, and therefore $M_1 > (I-\varepsilon)/\widehat{K}_a(0)$. The lower bound of $M_2 = \max_{|u|\le h}(-S(t+u))$ is established by similar arguments. ∎

The condition $I > 0$ can be checked without significant difficulties. Let

$$\mu = \frac{I}{\widehat{K}_a(0)} = \frac{1}{\pi\widehat{K}_a(0)}\Big(\int_0^\infty K_a(u)S_N(u)\,du - \kappa\ln\zeta_N(\gamma)\Big).$$

Taking $a = 3$, $b = 7/5$, $\tau = 2/5$ and choosing $N = p_n$, we obtain: $n = 16500$, $\mu = 1.00507513\ldots$; $n = 78000$, $\mu = 2.00632298\ldots$; $n = 2500000$; $\mu = 3.00126370\ldots > 3 + 10^{-3}$. Thus we get

COROLLARY 4.5. *If the Riemann hypothesis is true, then there exist constants c_0 and T_0 such that*

$$\max_{|t-T|\le H}(\pm S(t)) > 3 + 10^{-3}$$

for any $T \ge T_0$ and $H = 0.4\ln\ln\ln T + c_0$.

THEOREM 4.6. *Suppose that the Riemann hypothesis is true. Then for an arbitrarily large fixed $A \ge 1$, there exist constants T_0, c_0 and h depending only on A and such that*

$$\min_{|t-T|\le H}\big(S(t+h) - S(t-h)\big) < -A$$

for all $T \ge T_0$ and $h = (1/\pi)\ln\ln\ln T + c_0$.

Proof. Fix $a > 1$ and $0 < h < 1$. Replacing t in (2.4) by $t + h$ and $t - h$ and subtracting the corresponding relations, we obtain

$$\int_{-\infty}^{\infty} K_a(\pi u)\big(\ln\zeta(0.5 + i(t+h)) - \ln\zeta(0.5 + i(t-h))\big)\,du$$
$$= \frac{2}{\pi i}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\sin(h\ln n)n^{-it}$$
$$- 2\pi\int_0^{0.5}\big(K_a(\pi(t+h+iv)) - K_a(\pi(t-h+iv))\big)\,dv. \tag{4.20}$$

Taking imaginary parts in (4.20), we get

$$\pi\int_{-\infty}^{\infty} K_a(\pi u)\big(S(t+h+u) - S(t-h+u)\big)\,du$$
$$= -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\left(\frac{\ln n}{\pi}\right)\sin(h\ln n)\cos(t\ln n)$$
$$- 2\pi\Im\int_0^{0.5}\big(K_a(\pi(t+h+iv)) - K_a(\pi(t-h+iv))\big)\,dv. \tag{4.21}$$

If $t = 0$ then the integral on the right-hand side in (4.21) has the form

$$- 2\pi\Im\int\limits_0^{0.5} e^{-a\cosh(\pi h)\cos(\pi v)}\big(e^{-ia\sinh(\pi h)\sin(\pi v)} - e^{ia\sinh(\pi h)\sin(\pi v)}\big)\,dv$$
$$= 4\pi\int\limits_0^{0.5} e^{-a\cosh(\pi h)\cos(\pi v)}\sin(a\sinh(\pi h)\sin(\pi v))\,dv.$$

Hence,

$$(4.22)\qquad -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)\sin(h\ln n)$$
$$= -4\pi\int\limits_0^{0.5} e^{-a\cosh(\pi h)\cos(\pi v)}\sin(a\sinh(\pi h)\sin(\pi v))\,dv$$
$$+\pi\int\limits_{-\infty}^{\infty} K_a(\pi u)\big(S(u+h)-S(u-h)\big)\,du.$$

Let ε, N satisfy $0<\varepsilon<0.5$, $N>e^{\pi a\sqrt{2}}$ and depend only on a; their precise values will be chosen below.

By Lemma 3.1, given ε, N satisfying $0<\varepsilon<0.5$, $N>e^{\pi a\sqrt{2}}$, there exists a constant c such that each interval of length c contains a point τ such that $\|(\tau/(2\pi))\ln p\|<\varepsilon$ for any prime $p\le N$. Taking t in (4.21) to be such a τ in $(T, T+c)$, estimating the integral on the right-hand side of (4.21) by $2\pi(a\cosh\pi(t-h))^{-1}$ and using (4.22), we transform the right-hand side of (4.21) to

$$(4.23)\qquad -4\pi\int\limits_0^{0.5} e^{-a\cosh(\pi h)\cos(\pi v)}\sin(a\sinh(\pi h)\sin(\pi v))\,dv$$
$$+\pi\int\limits_{-\infty}^{\infty} K_a(\pi u)(S(u+h)-S(u-h))\,du$$
$$+\frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{\sqrt{n}}\widehat{K}_a\Big(\frac{\ln n}{\pi}\Big)\sin(h\ln n)\sin^2\Big(\frac{t}{2}\ln n\Big)+\frac{2\pi\theta_1}{a\cosh\pi(t-h)}.$$

The sum over n on the right-hand side of (4.23) is estimated in the same way as the sum C in Theorem 4.1 and does not exceed

$$2\bigg((7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+3000\varepsilon^2+\frac{321}{\sqrt{\ln N}}\bigg)$$

in modulus. In view of (4.15), the improper integral in (4.22) does not exceed

$$2\pi\int\limits_{-279}^{279} K_a(\pi u)\,du+\pi\Big(\int\limits_{279}^{\infty}+\int\limits_{-\infty}^{-279}\Big)K_a(\pi u)\cdot 2.1\ln(|u|+1)\,du$$
$$< 2\widehat{K}_a(0)+10^{-100}\widehat{K}_a(0)<2.1\widehat{K}_a(0)$$

in absolute value. Hence, changing the signs in (4.23), we get

$$(4.24)\qquad \pi\int_{-\infty}^{\infty} K_a(\pi u)\big(S(u+h)-S(u-h)\big)\,du$$
$$> 4\pi\int_{0}^{0.5} e^{-a\cosh(\pi h)\cos(\pi v)}\sin(a\sinh(\pi h)\sin(\pi v))\,dv$$
$$-2\bigg((7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+3000\varepsilon^2+\frac{3210}{\sqrt{\ln N}}+2.1\widehat{K}_a(0)+\frac{2\pi}{a\cosh\pi(t-h)}\bigg).$$

Now we take $h=(2\pi a)^{-1}$ and estimate the integral on the right-hand side of (4.24) from below. Since

$$\sin\big(a\sinh(\pi h)\sin(\pi v)\big)\ge\sin\bigg(a\pi h\cdot\frac{2}{\pi}\pi v\bigg)=\sin v\ge\frac{2}{\pi}v,$$
$$\cosh\pi h<\cosh\frac{1}{2}<\frac{8}{7},$$

the integral under consideration is greater than

$$4\pi\int_{0}^{0.5} e^{-(8a/7)\cos(\pi v)}\frac{2}{\pi}v\,dv=\frac{8}{\pi^2}\int_{0}^{\pi/2} e^{-(8a/7)\cos w}w\,dw$$
$$=\frac{8}{\pi^2}\int_{0}^{\pi/2} e^{-(8a/7)\sin w}\bigg(\frac{\pi}{2}-w\bigg)\,dw\ge\frac{2}{\pi}\int_{0}^{\pi/4} e^{-(8a/7)\sin w}\,dw$$
$$\ge\frac{2}{\pi}\int_{0}^{\pi/4} e^{-(8a/7)w}\,dw=\frac{7}{4\pi a}(1-e^{-2\pi a/7})>\frac{7}{4\pi a}(1-e^{-2\pi/7})>\frac{0.33}{a}.$$

Therefore,

$$\pi\int_{-\infty}^{\infty} K_a(\pi u)\big(S(t+u-h)-S(t+u+h)\big)\,du>\frac{0.33}{a}$$
$$-\bigg(2(7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+3000\varepsilon^2+\frac{642}{\sqrt{\ln N}}+2.1\widehat{K}_a(0)+\frac{2\pi}{a\cosh\pi(t-h)}\bigg).$$

Let $H_0=(1/\pi)(\ln\ln\ln T-\ln(a/2))$. Then the sum of the integrals over $(-\infty,-H_0)$ and (H_0,∞) on the right-hand side is less than $(\ln\ln T)^{-1}$ in modulus. Thus we get

$$(4.25)\qquad \pi\int_{-H_0}^{H_0} K_a(\pi u)\big(S(t+u-h)-S(t+u+h)\big)\,du>\frac{0.33}{a}$$
$$-\bigg(2(7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}}+6000\varepsilon^2+\frac{642}{\sqrt{\ln N}}+2.1\widehat{K}_a(0)+\frac{2}{\ln\ln T}\bigg).$$

Suppose now that $a > 8$ and take $\varepsilon = e^{-2a/3}/(65\sqrt{a})$, $N = e^{(c_1 a)^2}$, $c_1 = 2^{16}$. Then

$$(2(7\varepsilon)^2\widehat{K}_a(0)e^{\pi a/\sqrt{2}} + 6000\varepsilon^2 < 98\varepsilon^2 e^{-a}\sqrt{2\pi/a}\, e^{\pi a/\sqrt{2}} + 6000\varepsilon^2$$
$$< \frac{98\sqrt{2\pi}}{65^2}\frac{e^{-0.1a}}{\sqrt{a}}\frac{1}{a} + \frac{6000}{65^2}\frac{e^{-4a/3}}{a} < \frac{10^{-2}}{a},$$

$$\frac{642}{\sqrt{\ln N}} = \frac{642}{2^{16}a} < \frac{10^{-2}}{a}, \quad 2.1\widehat{K}_a(0) + \frac{2}{\ln\ln T} < 2.1e^{-a}\sqrt{2\pi a}\,\frac{1}{a} < \frac{5\cdot 10^{-3}}{a}.$$

Thus, the right-hand side of (4.25) is bounded from below by

$$\frac{0.33}{a} - \left(\frac{2\cdot 10^{-2}}{a} + \frac{5\cdot 10^{-3}}{a}\right) > \frac{0.3}{a}.$$

Hence,

$$M_0 = \max_{|u|\le H_0}\bigl(S(t+u-h) - S(t+u+h)\bigr) > 0,$$

and the left-hand side of (4.25) does not exceed $M_0\widehat{K}_a(0)$. Therefore,

$$M_0 > \frac{3\widehat{K}_a^{-1}(0)}{10a} > \frac{3e^a}{10\sqrt{2\pi a}} > \frac{e^a}{10\sqrt{a}}.$$

Choosing $a > 8$ such that

$$\frac{e^a}{10\sqrt{a}} > A,$$

we arrive at the assertion of the theorem. ∎

In [M], [Ba], [Ko1], [K5] and [Bo1], one can find some other examples of application of the function $K_a(z)$ to the theory of $\zeta(s)$.

The key ingredient in the proof of the unboundedness of $|\zeta(0.5+it)|$ on the interval $|t-T| \ll \ln\ln\ln T$ is the presence of the term

$$2\pi\int_0^{0.5} e^{-a\cos(\pi v)}\,dv$$

on the right-hand side of (4.3). It follows from the proof of (2.3) that the pole of $\zeta(s)$ at $s=1$ is the reason for the appearance of that term. Hence, it would be of interest to prove an analogue of Theorem 4.1 for functions that are “similar” to $\zeta(s)$ but have no pole at $s=1$ (for example, for Dirichlet’s L-function $L(s,\chi_4)$, where χ_4 is a non-principal character modulo 4).

5. The distribution of zeros of the zeta-function. The above theorems allow one to establish some new statements concerning the distribution of zeros of the Riemann zeta-function. Here we also suppose that the Riemann hypothesis is true.

Let $N(t)$ be the number of zeros of $\zeta(s)$ whose ordinate is positive and does not exceed t. It is known that

$$N(t) = \frac{1}{\pi}\vartheta(t) + 1 + S(t) = \frac{t}{2\pi}\ln\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(t^{-1}),$$

where $\vartheta(t)$ denotes the increment of a continuous branch of the argument of $\pi^{-s/2}\Gamma(s/2)$ along the line segment joining $s = 0.5$ and $s = 0.5 + it$. Then the Gram point t_n $(n \geq 0)$ is defined as the unique solution of the equation $\vartheta(t_n) = (n-1)\pi$ with $\vartheta'(t_n) > 0$. It is easy to check that the number of zeros of $\zeta(0.5 + it)$ lying in the Gram interval $G_n = (t_{n-1}, t_n]$ is equal to

$$N(t_n + 0) - N(t_{n-1} + 0) = 1 + \Delta(n) - \Delta(n-1), \tag{5.1}$$

where $\Delta(n) = S(t_n + 0)$. Since the interval $[0, T]$ contains

$$\frac{1}{\pi}\vartheta(T) + O(1) = N(T) + O(\ln T)$$

Gram intervals G_n, there is precisely one zero of $\zeta(0.5 + it)$ per one Gram interval G_n "in the mean". That is why the difference $\Delta(n) - \Delta(n-1)$ in (5.1) is the deviation of the number of zeros of $\zeta(0.5+it)$ in the interval G_n from its mean value, that is, 1.

In 1946, A. Selberg [S2] proved that the interval G_n contains no zeros of $\zeta(0.5+it)$ for a positive proportion of n, and contains at least two zeros for a positive proportion of n at the same time. These facts show the evident irregularity in the distribution of zeta zeros.

However, nothing is known about the distribution of the Gram intervals G_n which are free of zeros of $\zeta(0.5+it)$. The theorem below establishes an upper bound for the length $h = h(t)$ of the interval $(t, t+h)$ which certainly contains an "empty" Gram interval G_n.

THEOREM 5.1. *Suppose that the Riemann hypothesis is true and let ε be any fixed positive constant. Then there exist constants $T_0 = T_0(\varepsilon)$ and $c_0 = c_0(\varepsilon)$ such that any interval $[T - H, T + H]$, where $T \geq T_0$ and $H = (1/\pi)\ln\ln\ln T + c_0$, contains at least $N = [0.1\sqrt{\varepsilon}\exp((\pi\varepsilon)^{-1})]$ Gram intervals $G_n = (t_{n-1}, t_n]$ that do not contain zeros of $\zeta(0.5+it)$. Moreover, there exist at least N among the above "empty" Gram intervals that lie in the same interval of length ε.*

Proof. Let $a = (\pi\varepsilon)^{-1}$, $h = (2\pi a)^{-1} = 0.5\varepsilon$ and suppose ε is so small that $M = e^a/(10\sqrt{a}) \geq 5$. By Theorem 4.6, there exist constants $T_0 = T_0(\varepsilon)$ and $c_1 = c_1(\varepsilon)$ such that

$$\min_{|t-T|\leq H}\bigl(S(t+h) - S(t-h)\bigr) \leq -M$$

for any $T \geq T_0$ with $H = (1/\pi)\ln\ln\ln T + c_1$.

Let k be sufficiently large and suppose that $t_{k-1} \le a < b \le t_k$. If $S(t)$ has no discontinuities in (a, b), then the Riemann–von Mangoldt formula together with Lagrange's mean value theorem imply that

$$(5.2)\qquad \begin{aligned} S(b) - S(a) &= (b-a)S'(c) = (b-a)\Big(-\frac{1}{2\pi}\ln\frac{c}{2\pi} + o(1)\Big) \\ &= -(b-a)(L_k + o(1)), \qquad L_k = \frac{1}{2\pi}\ln\frac{t_k}{2\pi}, \end{aligned}$$

for some $a < c < b$. The relation (5.2) also holds true if a or b coincides with the ordinate of a zeta zero. In those cases, one should replace $S(a), S(b)$ by $S(a+0), S(b-0)$, respectively.

Suppose that $\gamma_{(1)} < \cdots < \gamma_{(r)}$ are all the ordinates of zeros of $\zeta(s)$ lying in $[a, b]$, and let $\kappa_{(1)}, \ldots, \kappa_{(k)}$ be their multiplicities. Then (see Fig. 4)

$$(5.3)\qquad \begin{aligned} & S(b-0) - S(a+0) \\ &= \big(S(b-0) - S(\gamma_{(k)}+0)\big) + \big(S(\gamma_{(k)}+0) - S(\gamma_{(k)}-0)\big) \\ &\quad + \big(S(\gamma_{(k)}-0) - S(\gamma_{(k-1)}+0)\big) + \cdots + \big(S(\gamma_{(1)}+0) - S(\gamma_{(1)}-0)\big) \\ &\quad + \big(S(\gamma_{(1)}-0) - S(a+0)\big) \\ &= \kappa_{(1)} + \ldots + \kappa_{(k)} - (b-a)(L_k + o(1)) \ge -(b-a)(L_k + o(1)) \\ &\quad \ge -(t_k - t_{k-1})(L_k + o(1)) = -1 - o(1). \end{aligned}$$

Now we define m and n from the relations $t_{m-1} < \tau - h \le t_m$, $t_n \le \tau + h < t_{n+1}$. Suppose first that neither of the two points $\tau \pm h$ is the ordinate of a zeta zero. By (5.3), we have

$$S(t_m - 0) - S(\tau - h) \ge -1 - o(1), \quad S(\tau + h) - S(t_n + 0) \ge -1 - o(1),$$

and hence

$$(5.4)\qquad \Delta(m) = S(t_m + 0) \ge S(t_m - 0) \ge S(\tau - h) - 1 - o(1),$$

$$(5.5)\qquad \Delta(n) = S(t_n + 0) \le S(\tau + h) + 1 + o(1).$$

Subtracting (5.4) from (5.5), we find

$$\Delta(n) - \Delta(m) \le M + 2 + o(1) < M + 3.$$

Suppose now that $\tau + h$ is the ordinate of a zero of $\zeta(s)$ of multiplicity $\kappa \ge 1$. Then (5.3) implies

$$S(\tau + h - 0) - S(t_{n-1} + 0) \ge -2 - o(1),$$

and therefore

$$(5.6)\qquad \begin{aligned} \Delta(n-1) &\le S(\tau + h) + 2 + o(1) = S(\tau + h) - 0.5\kappa + 2 + o(1) \\ &\le S(\tau + h) + 1.5 + o(1). \end{aligned}$$

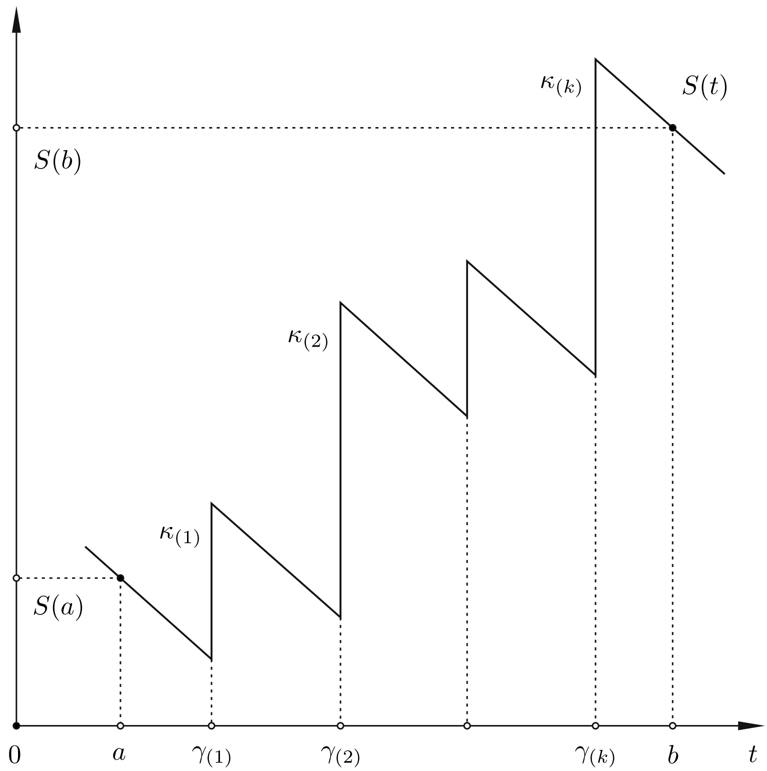

Fig. 4. At each point $\gamma_{(r)}$ of discontinuity, the function $S(t)$ makes a jump equal to the multiplicity of the ordinate $\gamma_{(r)}$, that is, to the sum of the multiplicities of all zeta zeros with this point as ordinate.

In view of (5.4), we get

$$\Delta(n-1) - \Delta(m) \le M + 2.5 + o(1) < M + 3.$$

Similarly, if $\tau - h$ is an ordinate of a zero of $\zeta(s)$, then

$$S(t_{m+1} - 0) - S(\tau - h) \ge -2 - o(1),$$

and hence

$$\begin{aligned} \Delta(m+1) &= S(t_{m+1} + 0) \ge S(t_{m+1} - 0) \\ &\ge S(\tau + h + 0) - 2 - o(1) \ge S(\tau - h) - 1.5 - o(1). \end{aligned} \tag{5.7}$$

Taking (5.5) into account, we find

$$\Delta(n) - \Delta(m+1) \le M + 2.5 + o(1) < M + 3.$$

Finally, suppose both $\tau \pm h$ are the ordinates of some zeros. By (5.6) and (5.7), we then have

$$\Delta(n-1) - \Delta(m+1) \le M + 3 + o(1) < M + 3 + 10^{-4}.$$

The above estimates imply that the smallest difference among $\Delta(n-i)-\Delta(m+j)$, $0\le i,j\le 1$, does not exceed $M+3+10^{-4}$ in any case. Denote by n_1 and m_1 the corresponding values of $n-i$ and $m+j$ and set $N=[-(M+3+10^{-4})]$. Since $N\ge 1$, we get

$$(5.8)\qquad (\Delta(n_1)-\Delta(n_1-1))+(\Delta(n_1-1)-\Delta(n_1-2)) \\ +\cdots+(\Delta(m_1+1)-\Delta(m_1))\le -N.$$

Formula (5.1) implies that $\Delta(k)-\Delta(k-1)\ge -1$, with equality if and only if the Gram interval G_k is free of zeros of $\zeta(0.5+it)$. Thus, (5.8) means that there are at least N negative differences (i.e. equal to -1) among $\Delta(k)-\Delta(k-1)$, $k=m+1,\ldots,n$. Hence, there are at least N intervals among G_k, $k=m+1,\ldots,n$, which are free of zeros of $\zeta(0.5+it)$.

To end the proof, we note that

$$N\ge \frac{e^a}{10\sqrt{a}}-4>\frac{e^a}{16\sqrt{a}}=\frac{\sqrt{\pi\varepsilon}}{16}\exp((\pi\varepsilon)^{-1})>0.1\sqrt{\varepsilon}\exp((\pi\varepsilon)^{-1}),$$

and that all the G_k, $k=m+1,\ldots,n$, are contained in the interval $[\tau-h,\tau+h]$ of length $2h=\varepsilon$. ∎

Corollary 4.5 implies a similar (but weaker) result for the distribution of the intervals G_n containing at least two zeros of $\zeta(s)$.

THEOREM 5.2. *Suppose that the Riemann hypothesis is true. Then there exist constants $T_0=T_0(\varepsilon)$ and $c_0=c_0(\varepsilon)$ such that each interval $[T-H,T+H]$, where $T\ge T_0$ and $H=0.8\ln\ln\ln T+c_0$, contains an interval G_k with at least two zeros of $\zeta(s)$.*

Proof. By Corollary 4.5, for sufficiently large c and $H_1=0.4\ln\ln\ln T_1+c$, the interval (T_1-H_1,T_1+H_1) contains a point τ_1 such that $S(\tau_1)<-3-10^{-3}$, and (T_1+H_1,T_1+3H_1) contains τ_2 such that $S(\tau_2)>3+10^{-3}$.

We define m, n by $t_m<\tau_1\le t_{m+1}$, $t_{n-1}<\tau_2\le t_n$. Using the same arguments as in the proof of Theorem 4.6 together with the inequalities $\tau_1<\tau_2$, $S(\tau_2)-S(\tau_1)>6+2\cdot 10^{-3}$, we find

$$S(\tau_1-0)-S(t_m+0)\ge -1-o(1),$$

and hence

$$-\Delta(m)\ge -S(\tau_1-0)-1-o(1)\ge -S(\tau_1)-1-o(1).$$

Similarly,

$$S(t_n+0)-S(\tau_2)=\big(S(t_n+0)-S(t_n-0)\big)+\big(S(t_n-0)-S(\tau_2+0)\big) \\ +\big(S(\tau_2+0)-S(\tau_2)\big)\ge -1-o(1),$$

so $\Delta(n)\ge S(\tau_2)-1-o(1)$. Therefore,

$$\Delta(n)-\Delta(m)\ge S(\tau_2)-S(\tau_1)-2-o(1)>4.$$

Thus, $\Delta(k) - \Delta(k-1) \geq 1$ for at least one $k = m+1, \ldots, n$. In view of (5.1), the corresponding Gram interval G_k contains at least two zeros of $\zeta(0.5+it)$. This interval lies in $[T_1 - H_1, T_1 + 3H_1 + t_n - t_{n-1}]$ whose length is less than $1.6 \ln\ln\ln T_1 + 4c + 10^{-3}$. Setting $c_0 = 2c + 10^{-3}$, we arrive at the desired assertion. ∎

Let $\gamma_n > 0$ be the ordinate of a zero of $\zeta(s)$. Given n, we consider the unique number $m = m(n)$ such that $t_{m-1} < \gamma_n \leq t_m$. Following Selberg [S2], we denote $\Delta_n = m - n$. It is known (see [S3, p. 355, Remark 1] and [Ko3]) that $\Delta_n \neq 0$ for "almost all" n. Moreover, one can show that the number of $n \leq N$ satisfying

$$\Delta_n \leq \frac{x}{\pi\sqrt{2}}\sqrt{\ln\ln N}$$

is

$$N\biggl(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du + O\biggl(\frac{\ln\ln\ln N}{\sqrt{\ln\ln N}}\biggr)\biggr)$$

for any x (see [Bo4, Th. 5] and [Ko4, Th. 4-6]). Given $N \geq N_0$, Theorem 4.4 allows us to point out an $M = M(N)$ such that the interval $(N, N+M]$ certainly contains an n with $\Delta_n \neq 0$. Moreover, the following assertion holds.

Theorem 5.3. *Suppose that the Riemann hypothesis is true. Then there exist constants N_0 and $c_0 = c_0(\varepsilon)$ such that the interval $(N, N+M]$, where $N \geq N_0$ and*

$$M = \biggl[\frac{31}{5\pi}(\ln N + c_0)\ln\ln\ln N\biggr],$$

contains n, m with $\Delta_n = 3$, $\Delta_m = -3$.

Proof. We precede the proof by some remarks.

Firstly, the analogue of the intermediate value theorem holds true for the function $S(t)$. Namely, if $\tau_1 < \tau_2$ and $S(\tau_1) > S(\tau_2)$ then for any α with $S(\tau_2) < \alpha < S(\tau_1)$, there exists $\tau \in (\tau_1, \tau_2)$ such that $S(t)$ is continuous at τ and $S(\tau) = \alpha$ (see [Ko5, proof of Th. 3]).

Secondly, $S(t)$ is an integer if and only if t is a Gram point (see [Ko5, proof of Th. 1]).

Suppose now that T is sufficiently large. By Corollary 4.5, for sufficiently large $c_1 > 0$ and $h = 0.4\ln\ln\ln T + c_1$, the interval $(T, T+3h)$ contains points $\tau_1 < \tau_2$ such that $S(\tau_1) > 3 + 10^{-3}$, $S(\tau_2) < -3 - 10^{-3}$. By the first remark above, there exist $t \in (\tau_1, \tau_2)$ such that $S(t) = S(t+0) = -3$. By the second remark, this point is a Gram point, that is, $t = t_{\nu_0}$, $S(t_{\nu_0}+0) = \Delta(\nu_0) = -3$ for some ν_0.

Similarly, we prove that each of intervals $(T + (4j-1)h, T + (4j+3)h)$, $j = 1, \ldots, 5$, contains a Gram point t_{ν_j} such that $S(t_{\nu_j}+0) = \Delta(\nu_j) = -3$.

Now we take $T = t_N$. Since

$$h = 0.4 \ln\ln\ln t_N + c_1 < 0.4 \ln\ln\ln N + c_1,$$

the index ν defined by $t_{N+\nu} < T + 23h \le t_{N+\nu+1}$ satisfies

$$\nu = \frac{1}{\pi}(\vartheta(t_{N+\nu}) - \vartheta(t_N)) < \frac{23h}{\pi}\vartheta'(t_{N+\nu}) < \frac{23h}{2\pi}\ln N < M.$$

Hence, $(N, N+\nu]$ contains at least six indices ν_j, $j = 0, \ldots, 5$, such that $\Delta(\nu_j) = -3$. It is known (see [Ko4, Lemma 2]) that the number of indices in the same interval satisfying $\Delta_n = 3$ differs from the above quantity by at most $3 + (3-1) = 5$ in modulus. Hence, it is positive.

The proof of the second assertion of the theorem is similar. It uses the fact that the difference between the number of indices n satisfying $\Delta_n = -3$ and the number of indices with $\Delta(\nu) = 3$ lying in the same interval, does not exceed $|-3| + |-3-1| = 7$. ∎

Acknowledgements. The author is grateful to the anonymous referee for the careful reading of the paper and for his remarks.

The author is supported by Russian Scientific Fund (grant no. 14-11-00433).

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Received on 1.5.2014
and in revised form on 11.7.2014 (7787)