

Binary linear forms over finite sets of integers

by

MELVYN B. NATHANSON (Bronx, NY),
KEVIN O'BRYANT (Staten Island, NY),
BROOKE OROSZ (New York, NY), IMRE RUZSA (Budapest),
and MANUEL SILVA (Lisboa)

1. Polynomials over finite sets of integers. Let $f = f(x_1, \dots, x_n)$ be a polynomial with integral coefficients. Let A be a nonempty finite set of integers or of congruence classes modulo m . We denote by $f(A)$ the image of the function f with domain A , that is,

$$f(A) = \{f(a_1, \dots, a_n) : a_i \in A \text{ for } i = 1, \dots, n\}.$$

If f is a polynomial in n variables, then $|f(A)| \leq |A|^n$.

The classical examples are the polynomials $x_1 + x_2$ and $x_1 - x_2$. The sumset $A + A$ is the set $s(A)$ for the polynomial $s(x_1, x_2) = x_1 + x_2$, and the difference set $A - A$ is the set $d(A)$ for the polynomial $d(x_1, x_2) = x_1 - x_2$. For any arithmetic progression A or, more generally, any symmetric set A of integers, we have $|d(A)| = |s(A)|$, but for “most” sets A the difference set contains more elements than the sumset. It had been conjectured (cf. Croft [1], Marica [3], Nathanson [7, 6]) that $|d(A)| \geq |s(A)|$ for every set A , but the set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$ is a counterexample, since

$$|A - A| = |d(A)| = 25$$

and

$$|A + A| = |s(A)| = 26.$$

Sets with more sums than differences have also been studied by Hegarty [2] and Martin and O'Bryant [4]. This suggests the following problem.

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PROBLEM 1. Let $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ be polynomials with integer coefficients. Determine if there exist finite sets A, B, C of positive integers with $|C| > 1$ such that

$$(1) \quad \begin{cases} |f(A)| > |g(A)|, \\ |f(B)| < |g(B)|, \\ |f(C)| = |g(C)|. \end{cases}$$

There is a stronger form of Problem 1.

PROBLEM 2. Let $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ be polynomials with integer coefficients. Does there exist a sequence $\{A_i\}_{i=1}^\infty$ of finite sets of integers such that

$$\lim_{i \rightarrow \infty} \frac{|f(A_i)|}{|g(A_i)|} = \infty?$$

Does there exist a sequence $\{C_i\}_{i=1}^\infty$ of finite sets of integers with $\lim_{i \rightarrow \infty} |C_i| = \infty$ and $|f(C_i)| = |g(C_i)|$ for all i ?

Linear polynomials constitute an important special case.

PROBLEM 3. Let $f(x_1, \dots, x_n) = u_1x_1 + \dots + u_nx_n$ and $g(x_1, \dots, x_n) = v_1x_1 + \dots + v_nx_n$ be linear forms with integer coefficients. Do there exist finite sets A, B, C of integers with $|C| > 1$ that satisfy (1)?

The *interval of integers* $[a, b]$ is the set of integers $\{a, a + 1, \dots, b\}$. For any integer u and sets A and B of integers, we define the *dilation*

$$u * A = \{ua : a \in A\}$$

and the *sumset*

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

If $f(x_1, \dots, x_n) = u_1x_1 + \dots + u_nx_n$ is a linear form, then

$$f(A) = u_1 * A + \dots + u_n * A.$$

Sets of integers A and B are *affinely equivalent* if there are rational numbers $u \neq 0$ and v such that $B = u * A + \{v\}$. In this case,

$$f(B) = f(u * A + \{v\}) = u * f(A) + \{f(v, \dots, v)\},$$

hence

$$|f(A)| = |f(B)|.$$

Note that every two-element set is affinely equivalent to the set $\{0, 1\}$, and that every set A of integers with $1 < |A| < \infty$ is affinely equivalent to a set A' such that $0 \in A'$ and $A' \setminus \{0\}$ is a set of relatively prime positive integers. The following theorem implies that Problem 3 is equivalent to Problem 2 in the case of linear forms.

THEOREM 1. Let $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ be linear forms with integer coefficients, and let A and C be finite sets of integers such that $|f(A)| > |g(A)|$, $|f(C)| = |g(C)|$, and $|C| > 1$. There exist sequences $\{A_i\}_{i=1}^\infty$ and $\{C_i\}_{i=1}^\infty$ of finite sets of integers with $\lim_{i \rightarrow \infty} |C_i| = \infty$ such that

$$\lim_{i \rightarrow \infty} \frac{|f(A_i)|}{|g(A_i)|} = \infty,$$

and $|f(C_i)| = |g(C_i)|$ for all i .

Proof. Let $f(x_1, \dots, x_n) = \sum_{i=1}^n u_i x_i$ and $g(x_1, \dots, x_n) = \sum_{i=1}^n v_i x_i$ be linear forms with integer coefficients, and let A be a finite set of integers. We define

$$m_{f,g}(A) = \max(|s| : s \in A \cup f(A) \cup g(A)),$$

and choose an integer

$$M > 2m_{f,g}(A).$$

Let

$$A_M = A + M * A = \{a + Ma' : a, a' \in A\}.$$

If $a_1, a'_1, a_2, a'_2 \in A$ and $a_1 + Ma'_1 = a_2 + Ma'_2$, then $a_1 - a_2 = M(a'_2 - a'_1)$. Since

$$M|a'_2 - a'_1| = |a_1 - a_2| \leq |a_1| + |a_2| \leq 2m_{f,g}(A) < M,$$

it follows that $a'_1 = a'_2$ and so $a_1 = a_2$ and

$$|A_M| = |A|^2.$$

The identity

$$\sum_{i=1}^n u_i (a_i + Ma'_i) = \sum_{i=1}^n u_i a_i + M \sum_{i=1}^n u_i a'_i$$

implies that

$$f(A_M) = f(A) + M * f(A).$$

If $s_1, s'_1, s_2, s'_2 \in f(A)$ and

$$s_1 + Ms'_1 = s_2 + Ms'_2,$$

it again follows that $s_1 = s_2$, $s'_1 = s'_2$, and

$$|f(A_M)| = |f(A)|^2.$$

Similarly,

$$|g(A_M)| = |g(A)|^2$$

and so

$$\frac{|f(A_M)|}{|g(A_M)|} = \left(\frac{|f(A)|}{|g(A)|} \right)^2.$$

The theorem follows by iterating this construction. ■

In the case of binary linear forms, we write $f(x, y) = ux + vy$ instead of $f(x_1, x_2) = u_1x_1 + u_2x_2$. We are interested only in the cardinality of the image of $f(x, y)$ on a finite set A of integers. We shall always assume that $uv \neq 0$. If $(u, v) = d > 1$ and $g(x, y) = (u/d)x + (v/d)y$, then $|f(A)| = |g(A)|$. Thus, we can assume that $(u, v) = 1$. Similarly, if $h(x, y) = vx + uy$, then $|f(A)| = |h(A)|$, and so we can assume that $|u| \geq |v|$. Finally, if $\ell(x, y) = -ux - vy$, then $|f(A)| = |\ell(A)|$, and we can assume that $u > 0$. Therefore, it suffices to consider only binary linear forms $f(x, y) = ux + vy$ that have been normalized so that

$$u \geq |v| \geq 1 \quad \text{and} \quad (u, v) = 1.$$

PROBLEM 4. *Let $f(x, y) = u_1x + v_1y$ and $g(x, y) = u_2x + v_2y$ be normalized binary linear forms with nonzero integer coefficients $(u_1, v_1) \neq (u_2, v_2)$. Do there exist finite sets of integers A and B such that $|f(A)| > |g(A)|$ and $|f(B)| < |g(B)|$?*

In this paper we shall prove that the answer to the question in Problem 4 is “yes”.

2. Pairs of binary linear forms with $u_1, u_2 \geq 2$. In this section we prove that if $f(x, y) = u_1x + v_1y$ and $g(x, y) = u_2x + v_2y$ are normalized binary linear forms with $u_1 \geq 2, u_2 \geq 2$, and $(u_1, v_1) \neq (u_2, v_2)$, then there exist finite sets A, B, C of integers such that $|f(A)| < |g(A)|, |f(B)| > |g(B)|$, and $|f(C)| = |g(C)|$.

THEOREM 2. *For $u > |v| \geq 1$ and $(u, v) = 1$, consider the normalized binary linear form*

$$f(x, y) = ux + vy.$$

- (i) *If $|A| = 2$, then $|f(A)| = 4$.*
- (ii) *If $u \geq 3$ and $|A| = 3$, then $|f(A)| = 8$ or 9 , and $|f(A)| = 8$ if and only if A is affinely equivalent to one of the two sets*

$$\{0, |v|, u\} \quad \text{and} \quad \{0, |v|, u + |v|\}.$$

- (iii) *If $u = 2$ and $|A| = 3$, then $|f(A)| < 9$ if and only if A is affinely equivalent to one of the two sets*

$$\{0, 1, 2\} \quad \text{and} \quad \{0, 1, 3\}.$$

Moreover, $|f(\{0, 1, 2\})| = 7$ and $|f(\{0, 1, 3\})| = 8$.

- (iv) *If $f(x, y) = ux + vy$, and $g(x, y) = ux - vy$, then $|f(A)| = |g(A)|$ for every set A with $|A| = 3$.*

Proof. If $f(x, y) = ux + vy$ is a normalized binary linear form, then $f(\{0, 1\}) = \{0, v, u, u + v\}$ and so $|f(\{0, 1\})| = 4$. Since every set A with $|A| = 2$ is affinely equivalent to $\{0, 1\}$, it follows that if $|A| = 2$, then $|f(A)| = 4$. This proves (i).

Let $|A| = 3$. The set A is affinely equivalent to a set A' such that $\min(A') = 0$ and $\gcd(A') = 1$. If $|f(A)| \leq 8$, then there exist $x_1, y_1, x_2, y_2 \in A'$ such that

$$ux_1 + vy_1 = ux_2 + vy_2 \quad \text{and} \quad (x_1, y_1) \neq (x_2, y_2).$$

It follows from (i) that $|\{x_1, y_1, x_2, y_2\}| > 2$ and so

$$\{x_1, y_1, x_2, y_2\} = \{a_1, a_2, a_3\} = A'.$$

There are three possibilities: Either

$$ua_1 + va_2 = ua_1 + va_3,$$

or

$$ua_1 + va_2 = ua_3 + va_1,$$

or

$$ua_1 + va_1 = ua_2 + va_3.$$

In the first case, $a_2 = a_3$, which is absurd.

In the second case, we have

$$u(a_1 - a_3) = v(a_1 - a_2).$$

Since $(u, v) = 1$, there exists an integer r such that

$$a_1 - a_2 = rv, \quad a_1 - a_3 = rv, \quad a_3 - a_2 = r(u - v).$$

Since $0 \in A'$, it follows that r divides each integer in A' , and so $r = \pm 1$. If $a_1 = 0$, then $r = -1$, $a_2 = u$, $a_3 = v = |v|$, and $A' = \{0, |v|, u\}$. If $a_2 = 0$, then $r = 1$, $a_1 = u$, and $a_3 = u - v$. If $v > 0$, then $A' = \{0, u - |v|, u\}$. If $v < 0$, then $A' = \{0, u, u + |v|\}$. If $a_3 = 0$, then $a_2 = -r(u - v)$ and so $r = -1$, $a_1 = -v = |v|$, $a_2 = u - v = u + |v|$, and $A' = \{0, |v|, u + |v|\}$.

In the third case,

$$u(a_1 - a_2) = v(a_3 - a_1)$$

and there is an integer $r = \pm 1$ such that

$$a_1 - a_2 = rv, \quad a_3 - a_1 = ru, \quad a_3 - a_2 = r(u + v).$$

If $a_1 = 0$, then $r = 1$, $a_3 = u$, $a_2 = -v = |v|$, and $A' = \{0, |v|, u\}$. If $a_2 = 0$, then $r = 1$, $a_1 = v = |v|$, $a_3 = u + v = u + |v|$, and $A' = \{0, |v|, u + |v|\}$. If $a_3 = 0$, then $r = -1$, $a_1 = u$, and $a_2 = u + v$. If $v > 0$, then $A' = \{0, u, u + |v|\}$. If $v < 0$, then $A' = \{0, u - |v|, u\}$.

Since the two sets $\{0, |v|, u\}$ and $\{0, u - |v|, u\}$ are affinely equivalent, and the two sets $\{0, u, u + |v|\}$ and $\{0, |v|, u + |v|\}$ are also affinely equivalent, it follows that the sets $\{0, |v|, u\}$ and $\{0, u, u + |v|\}$ are, up to affine equivalence, the only possible solutions of $|f(A)| \leq 8$ with $|A| = 3$.

We shall prove that if $u \geq 3$, then $|f(A)| = 8$ for both these sets. Let $v > 0$ and $f(x, y) = ux + vy$. If $A = \{0, v, u\}$, then

$$f(A) = \{0, v^2, uv, u^2, uv + v^2, 2uv, u^2 + v^2, uv + u^2\}.$$

Since $2uv < u^2 + v^2$ for $v < u$, we have

$$0 < v^2 < uv < uv + v^2 < 2uv < u^2 + v^2 < uv + u^2$$

and

$$uv < u^2 < u^2 + v^2.$$

If $u^2 = 2uv$, then $u = 2v = 2$ since $(u, v) = 1$. If $u^2 = uv + v^2$, then $u/v = (1 + \sqrt{5})/2$, which is impossible since u/v is rational. Therefore, $|f(A)| = 8$.

If $v > 0$ and $A = \{0, v, u + v\}$, then

$$f(A) = \{0, v^2, uv, uv + v^2, 2uv + v^2, u^2 + uv, u^2 + uv + v^2, u^2 + 2uv + v^2\}.$$

We have

$$0 < v^2 < uv < uv + v^2 < 2uv + v^2 < u^2 + uv + v^2 < u^2 + 2uv + u^2$$

and

$$uv + v^2 < u^2 + v^2 < u^2 + uv + v^2.$$

If $u^2 + v^2 = 2uv + v^2$, then $u = 2v = 2$, which is false, and so $|f(A)| = 8$. The case $u \geq 3$ and $v < 0$ is similar. This proves (ii).

If $u = 2$, then $v = \pm 1$ and $f(x, y) = 2x \pm y$. Up to affine equivalence, the sets A with $|f(A)| \leq 8$ are $\{0, 1, 2\}$ and $\{0, 1, 3\}$. For these sets, $|f(\{0, 1, 2\})| = 7$ and $|f(\{0, 1, 3\})| = 8$. This proves (iii).

To obtain (iv), we observe that the binary linear forms $f(x, y) = ux + vy$ and $g(x, y) = ux - vy$ generate the same exceptional sets, and so $|f(A)| = |g(A)|$ whenever $|A| = 3$. This completes the proof. ■

LEMMA 1. *Let*

$$F(x, y) = c_0x^k + c_1x^{k-1}y + \cdots + c_{k-1}xy^{k-1} + c_ky^k$$

be a nonzero homogeneous polynomial with integer coefficients. Let j be the largest integer such that $c_j \neq 0$. If u and v are relatively prime nonzero integers such that $F(u, v) = 0$, then $|c_j| \geq |u|$.

We call c_j the *last coefficient* in the polynomial $F(x, y)$.

Proof. If

$$F(u, v) = \sum_{i=0}^k c_i u^{k-i} v^i = u^{k-j} \sum_{i=0}^j c_i u^{j-i} v^i = 0$$

then

$$u \sum_{i=0}^{j-1} c_i u^{j-i-1} v^i = -c_j v^j$$

and so u divides $c_j v^j$. Since $(u, v) = 1$ and $c_j \neq 0$, it follows that u divides c_j and so $|c_j| \geq |u|$. ■

THEOREM 3. *Let*

$$f(x, y) = u_1x + v_1y \quad \text{and} \quad g(x, y) = u_2x + v_2y$$

be normalized binary linear forms with

$$u_1 \geq 2, \quad u_2 \geq 2,$$

and

$$(u_1, |v_1|) \neq (u_2, |v_2|).$$

There exist sets A and B with $|A| = |B| = 3$ such that

$$|f(A)| < |g(A)| \quad \text{and} \quad |f(B)| > |g(B)|.$$

Proof. If $u_1 < u_2$ and $u_2 \neq u_1 + |v_1|$, then the sets $A = \{0, |v_1|, u_1\}$ and $B = \{0, |v_2|, u_2\}$ satisfy $|f(A)| = |g(B)| \leq 8$ and $|f(B)| = |g(A)| = 9$. If $u_1 < u_2$ and $u_2 = u_1 + |v_1|$, then $u_2 + |v_2| > u_1 + |v_1|$ and the sets $A = \{0, |v_1|, u_1\}$ and $B = \{0, |v_2|, u_2 + |v_2|\}$ satisfy $|f(A)| = |g(B)| = 8$ and $|f(B)| = |g(A)| = 9$.

If $u_1 = u_2$ and $|v_1| < |v_2|$, then sets $A = \{0, |v_1|, u_1 + |v_1|\}$ and $B = \{0, |v_2|, u_2 + |v_2|\}$ satisfy $|f(A)| = |g(B)| = 8$ and $|f(B)| = |g(A)| = 9$. This completes the proof. ■

THEOREM 4. *Let*

$$f(x, y) = ux + vy \quad \text{and} \quad g(x, y) = ux - vy$$

be normalized binary linear forms with $u > v \geq 1$. For $u = 2$, if

$$A = \{0, 3, 4, 6\}, \quad B = \{0, 4, 6, 7\}$$

then

$$|f(A)| = 13 > 12 = |g(A)|, \quad |f(B)| = 13 < 14 = |g(B)|.$$

For $u \geq 3$, if

$$A = \{0, u^2 - v^2, u^2, u^2 + uv\}, \quad B = \{0, u^2 - uv, u^2 - v^2, u^2\}$$

then

$$|f(A)| = 14 > 13 = |g(A)|, \quad |f(B)| = 13 < 14 = |g(B)|.$$

Proof. For $u = 2$ and sets $A = \{0, 3, 4, 6\}$ and $B = \{0, 4, 6, 7\}$, we have

$$\begin{aligned} f(A) &= \{0, 3, 4, 6, 8, 9, 10, 11, 12, 14, 15, 16, 18\}, \\ g(A) &= \{-6, -4, -3, 0, 2, 3, 4, 5, 6, 8, 9, 12\}, \\ f(B) &= \{0, 4, 6, 7, 8, 12, 14, 15, 16, 18, 19, 20, 21\}, \\ g(B) &= \{-7, -6, -4, 0, 1, 2, 4, 5, 6, 7, 8, 10, 12, 14\} \end{aligned}$$

with $|f(A)| = |f(B)| = 13$, $|g(A)| = 12$, and $|g(B)| = 14$.

Let $u \geq 3$ and $A = \{0, u^2 - v^2, u^2, u^2 + uv\}$. We list the elements of the set $f(A) = \{ux + vy : x, y \in A\}$ in the following table:

$f(A)$	0	$u^2 - v^2$	u^2	$u^2 + uv$
0	0	$u^2v - v^3$	u^2v	$u^2v + uv^2$
$u^2 - v^2$	$u^3 - uv^2$	$u^3 + u^2v - uv^2 - v^3$	$u^3 + u^2v - uv^2$	$u^3 + u^2v$
u^2	u^3	$u^3 + u^2v - v^3$	$u^3 + u^2v$	$u^3 + u^2v + uv^2$
$u^2 + uv$	$u^3 + u^2v$	$u^3 + 2u^2v - v^3$	$u^3 + 2u^2v$	$u^3 + 2u^2v + uv^2$

The number $u^3 + u^2v$ occurs three times in this table, and so $|f(A)| \leq 14$. By Lemma 1, if two numbers in the table are equal for positive integers u and v with $u \geq 3$ and $(u, v) = 1$, then the difference of the two numbers is an expression of the form $F(u, v)$, where $F(x, y)$ is a homogeneous polynomial of degree 3 with last coefficient at least 3. Since $1 \leq v < u$, the numbers in the table are increasing from left to right in each row and from top to bottom in each column. The following ten numbers in the set $f(A)$ are strictly increasing:

$$\begin{aligned}
 0 &< u^2v - v^3 < u^3 - uv^2 < u^3 + u^2v - uv^2 - v^3 \\
 &< u^3 + u^2v - uv^2 < u^3 + u^2v - v^3 < u^3 + u^2v \\
 &< u^3 + u^2v + uv^2 < u^3 + 2u^2v < u^3 + 2u^2 + uv^2.
 \end{aligned}$$

The other four numbers in $f(A)$ satisfy

$$u^2v < u^3 < u^3 + 2u^2v - v^3$$

and

$$u^2v < u^2v + uv^2 < u^3 + 2u^2v - v^3.$$

Comparing numbers among the three chains of inequalities, we see that there is no difference with last coefficient greater than 2, and so $|f(A)| = 14$.

Consider now the set $g(A) = \{ux - vy : x, y \in A\}$, whose elements are listed in the following table:

$g(A)$	0	$u^2 - v^2$	u^2	$u^2 + uv$
0	0	$-u^2v + v^3$	$-u^2v$	$-u^2v - uv^2$
$u^2 - v^2$	$u^3 - uv^2$	$u^3 - u^2v - uv^2 + v^3$	$u^3 - u^2v - uv^2$	$u^3 - u^2v - 2uv^2$
u^2	u^3	$u^3 - u^2v + v^3$	$u^3 - u^2v$	$u^3 - u^2v - uv^2$
$u^2 + uv$	$u^3 + u^2v$	$u^3 + v^3$	u^3	$u^3 - uv^2$

The numbers $u^3, u^3 - uv^2$, and $u^3 - u^2v - uv^2$ occur twice in the table, and so $|g(A)| \leq 13$. The numbers in the table are decreasing from left to right in each row and increasing from top to bottom in each column. The following nine numbers in $g(A)$ form a strictly increasing sequence:

$$\begin{aligned}
 -u^2v - uv^2 &< -u^2v < -u^2v + v^3 < 0 < u^3 - u^2v \\
 &< u^3 - u^2v + v^3 < u^3 < u^3 + v^3 < u^3 + 3u^2v.
 \end{aligned}$$

The other four numbers satisfy

$$u^3 - u^2v - 2uv^2 < u^3 - u^2v - uv^2 < u^3 - u^2v - uv^2 + v^3 < u^3 - uv^2.$$

Indeed, there is no pair of expressions in the table whose difference has last coefficient greater than 2, and so $|g(A)| = 13$.

Finally, we consider the sets $f(B)$ and $g(B)$:

$f(B)$	0	$u^2 - uv$	$u^2 - v^2$	u^2
0	0	$u^2v - uv^2$	$u^2v - v^3$	u^2v
$u^2 - uv$	$u^3 - u^2v$	$u^3 - uv^2$	$u^3 - v^3$	u^3
$u^2 - v^2$	$u^3 - uv^2$	$u^3 + u^2v - 2uv^2$	$u^3 + u^2v - uv^2 - v^3$	$u^3 + u^2v - uv^2$
u^2	u^3	$u^3 + u^2v - uv^2$	$u^3 + u^2v - v^3$	$u^3 + u^2v$
$g(B)$	0	$u^2 - uv$	$u^2 - v^2$	u^2
0	0	$-u^2v + uv^2$	$-u^2v + v^3$	$-u^2v$
$u^2 - uv$	$u^3 - u^2v$	$u^3 - 2u^2v + uv^2$	$u^3 - 2u^2v + v^3$	$u^3 - 2u^2v$
$u^2 - v^2$	$u^3 - uv^2$	$u^3 - u^2v$	$u^3 - u^2v - uv^2 + v^3$	$u^3 - u^2v - uv^2$
u^2	u^3	$u^3 - u^2v + uv^2$	$u^3 - u^2v + v^3$	$u^3 - u^2v$

In the table for $f(B)$, the numbers u^3 , $u^3 - uv^2$, and $u^3 + u^2v - uv^2$ occur twice, and so $|f(B)| \leq 13$. In the table for $g(B)$, the number $u^3 - u^2v$ occurs three times, so $|g(B)| \leq 14$. In neither table is there a pair of numbers whose difference has last coefficient greater than 2, and so $|f(B)| = 13$ and $|g(B)| = 14$. This completes the proof. ■

THEOREM 5. *Let u and v be relatively prime positive integers with $u > v$, and consider the forms*

$$f = ux + vy \quad \text{and} \quad g = ux - vy.$$

If A is an arithmetic progression of length $t \leq u$, then $|f(A)| = |g(A)| = t^2$.

Proof. Since an arithmetic progression of length t is affinely equivalent to the interval $[0, t - 1]$, it suffices to consider the sets $A_t = [0, t - 1]$ for $t = 1, \dots, u$.

If $x_1, x_2, y_1, y_2 \in A_t$ and $ux_1 + vy_1 = ux_2 + vy_2$, then we have $u(x_1 - x_2) = v(y_2 - y_1)$. Since $(u, v) = 1$, it follows that u divides $y_2 - y_1$. Since $|y_2 - y_1| < t \leq u$, it follows that $y_1 = y_2$, and so $x_1 = x_2$. Thus, every element in $f(A_t)$ has a unique representation in the form $ux + vy$, and $|f(A_t)| = t^2$. The proof that $|g(A_t)| = t^2$ is similar. ■

3. The pair of linear forms $ux + vy$ and $x - y$

THEOREM 6. *Let u and v be relatively prime positive integers with $u > v$, and consider the linear forms*

$$f(x, y) = ux + vy, \quad d(x, y) = x - y.$$

Let

$$A = \{0, v^3, v^3 + v^2u, v^3 + v^2u + vu^2, v^3 + v^2u + vu^2 + u^3\}.$$

Then

$$|f(A)| \leq 19, \quad |d(A)| = 21.$$

Proof. Let

$$\begin{aligned} a_0 &= 0, \\ a_1 &= v^3, \\ a_2 &= v^3 + v^2u, \\ a_3 &= v^3 + v^2u + vu^2, \\ a_4 &= v^3 + v^2u + vu^2 + u^3. \end{aligned}$$

Then $A = \{a_0, a_1, a_2, a_3, a_4\}$, and

$$a_0 < a_1 < a_2 < a_3 < a_4.$$

Since $|A| = 5$, we have $|f(A)| \leq 25$ and $|d(A)| \leq 21$. To show that $|f(A)| \leq 19$, it suffices to give six different integers, each of which has two distinct representations in $f(A)$. Here they are:

$$\begin{aligned} n_1 &= ua_1 + va_1 = ua_0 + va_2, \\ n_2 &= ua_2 + va_1 = ua_0 + va_3, \\ n_3 &= ua_2 + va_2 = ua_1 + va_3, \\ n_4 &= ua_3 + va_1 = ua_0 + va_4, \\ n_5 &= ua_3 + va_2 = ua_1 + va_4, \\ n_6 &= ua_3 + va_3 = ua_2 + va_4. \end{aligned}$$

A straightforward calculation shows that

$$n_1 < n_2 < n_3 < n_4 < n_5 < n_6$$

and so $|f(A)| \leq 19$.

Next we prove that $|d(A)| = 21$. Let $D = \{a_j - a_i : 0 \leq i < j \leq 4\}$. It suffices to prove that $|D| = 10$. If $v = 1$, then $A = \{1, 1 + u, 1 + u + u^2, 1 + u + u^2 + u^3\}$ is a Sidon set and $|D| = 10$.

Suppose that $v \geq 2$. Since $u > v$ and $a_0 = 0$, we have

$$a_1 < a_2 - a_1 < a_3 - a_2 < a_4 - a_3 < a_4 - a_2 < a_4 - a_1 < a_4$$

and

$$a_2 < a_3 - a_1 < a_3.$$

Let

$$D_1 = \{a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3, a_4 - a_2, a_4 - a_1, a_4\},$$

$$D_2 = \{a_2, a_3 - a_1, a_3\}.$$

We must show that $D_1 \cap D_2 = \emptyset$. There are three cases.

CASE OF a_2 . Since

$$a_2 - a_1 < a_2 < a_4 - a_2,$$

it follows that if $a_2 \in D_1$, then $a_2 = a_3 - a_2$ or $a_2 = a_4 - a_3$. If $a_2 = a_3 - a_2$, then $v^2(v + u) = v^3 + v^2u = vu^2$, and $v(v + u) = u^2$. Since $(u, v) = 1$, it follows that $v = 1$. If $a_2 = a_4 - a_3$, then $v^2(v + u) = v^3 + v^2u = u^3$ and $v = 1$.

CASE OF $a_3 - a_1$. Since

$$a_3 - a_2 < a_3 - a_1 < a_4 - a_2,$$

it follows that if $a_3 - a_1 \in D_1$, then $a_3 - a_1 = a_4 - a_3$ and so $v(v + u) = u^2$. This implies that $v = 1$.

CASE OF a_3 . Since

$$a_3 - a_2 < a_3 < a_4 - a_1,$$

it follows that if $a_3 \in D_1$, then $a_3 = a_4 - a_3$ or $a_3 = a_4 - a_2$. If $a_3 = a_4 - a_3$, then $v(v^2 + vu + u^2) = v^3 + v^2u + vu^2 = u^3$, and $v = 1$. If $a_3 = a_4 - a_2$, then $v^3 + v^2u + vu^2 = vu^2 + u^3$, hence $v^2(v + u) = u^3$ and $v = 1$. This completes the proof. ■

4. The pair of linear forms $ux + vy$ and $x + y$. In Section 2 we solved Problem 4 for pairs of normalized binary linear forms $f(x, y) = u_1x + v_1y$ and $g(x, y) = u_2x + v_2y$ with $u_1, u_2 \geq 2$. In Section 3 we solved the case $f(x, y) = ux + vy$ with $u \geq 2$ and $d(x, y) = x - y$. The remaining case is $f(x, y) = ux + vy$ with $u \geq 2$ and $s(x, y) = x + y$.

For example, consider the form $f(x, y) = 2x + y$. We have

$$4 = |f(\{0, 1\})| > |s(\{0, 1\})| = 3.$$

We shall construct a set A with $|f(A)| < |s(A)|$. Start by defining the four sets

$$R_{13} = \{0, 1, 6, 7, 9, 11\},$$

$$R_{15} = \{0, 1, 5, 6, 10, 11, 13\},$$

$$R_{16} = \{0, 1, 3, 5, 7, 9, 11, 13, 15\},$$

$$R_{19} = \{0, 1, 11, 12, 14, 16, 18\}.$$

Note that

$$13 \cdot 15 \cdot 16 \cdot 19 = 59280$$

and

$$|R_{13}| \cdot |R_{15}| \cdot |R_{16}| \cdot |R_{19}| = 6 \cdot 7 \cdot 9 \cdot 7 = 2646.$$

Let $x \bmod m$ denote the least nonnegative integer that is congruent to x modulo m . We define

$$A = \{x \in [1, 59280] : x \bmod m \in R_m \text{ for all } m \in \{13, 15, 16, 19\}\}.$$

The set A contains 2646 elements. By direct calculation, we have $|f(A)| = 108014$ and $|s(A)| = 114575$.

The linear form $f(x, y) = 2x + y$ is a special case. In general, we do not have an algorithm to construct finite sets A of integers such that $|f(A)| < |s(A)|$ for an arbitrary normalized bilinear form $f(x, y) = ux + vy$ with $u \geq 2$. However, such sets do exist. In the following sections we shall show that, associated to the form $f(x, y) = ux + vy$, there is an infinite set M of positive integers with the property that, for each $m \in M$, there is a set of congruence classes $R_m \subseteq \mathbb{Z}/m\mathbb{Z}$ such that $s(R_m) = \mathbb{Z}/m\mathbb{Z}$ and $f(R_m) \subsetneq \mathbb{Z}/m\mathbb{Z}$. From the sets R_m we construct a finite set A of nonnegative integers such that $|f(A)| < |s(A)|$. Thus, we combine local solutions of the inequality to construct a global solution.

5. A local to global criterion for pairs of linear forms in n variables

LEMMA 2. *Let $f(x_1, \dots, x_n)$ be a polynomial with integer coefficients. Let m_1, \dots, m_r be pairwise relatively prime positive integers, and $m = m_1 \cdots m_r$. Let R_{m_i} be a set of congruence classes modulo m_i for $i = 1, \dots, r$. Let R_m be the set of all congruence classes $a + m\mathbb{Z}$ such that $a + m_i\mathbb{Z} \in R_{m_i}$ for $i = 1, \dots, r$. Then*

$$|R_m| = \prod_{i=1}^r |R_{m_i}| \quad \text{and} \quad |f(R_m)| = \prod_{i=1}^r |f(R_{m_i})|.$$

Proof. This follows from the Chinese remainder theorem. ■

LEMMA 3. *Let $f(x_1, \dots, x_n) = u_1x_1 + \cdots + u_nx_n$ be a linear form with integer coefficients, and let*

$$h_f = |u_1| + \cdots + |u_n|.$$

Let R_m be a set of congruence classes in $\mathbb{Z}/m\mathbb{Z}$, and let A be the set of integers that consists of the least nonnegative element of each congruence class in R_m . Then

$$(2) \quad |f(R_m)| \leq |f(A)| \leq 2h_f|f(R_m)|.$$

Proof. The triangle inequality implies that

$$|f(a_1, \dots, a_n)| \leq h_f \max(|a_i| : i = 1, \dots, n)$$

for all integers a_1, \dots, a_n . Since $A \subseteq [0, m - 1]$, it follows that $f(A) \subseteq [-h_f(m - 1), h_f(m - 1)]$. If $a \in f(A)$, then $a + m\mathbb{Z} \in f(R_m)$. The lower bound in (2) follows from the fact that $f(A)$ contains at least one element of every congruence class in $f(R_m)$. The upper bound in (2) follows from the fact that the interval $[-h_f(m - 1), h_f(m - 1)]$ contains at most $2h_f$ members of any congruence class modulo m . ■

THEOREM 7. *Let $f(x_1, \dots, x_n) = u_1x_1 + \dots + u_nx_n$ and $g(x_1, \dots, x_n) = v_1x_1 + \dots + v_nx_n$ be binary linear forms. Let $M = \{m_i\}_{i=1}^\infty$ be a set of pairwise relatively prime integers such that $m_i \geq 2$ for all $m_i \in M$. If for every $m_i \in M$ there exists a nonempty set R_{m_i} of congruence classes in $\mathbb{Z}/m_i\mathbb{Z}$ such that*

$$(3) \quad \prod_{i=1}^\infty \frac{|f(R_{m_i})|}{|g(R_{m_i})|} = 0,$$

then there is a finite set A of integers such that

$$|f(A)| < |g(A)|.$$

Proof. Let

$$h_f = |u_1| + \dots + |u_n|.$$

Since the infinite product (3) diverges to 0, there is an integer r such that

$$\prod_{i=1}^r \frac{|f(R_{m_i})|}{|g(R_{m_i})|} < \frac{1}{2h_f}.$$

Let $m = m_1 \cdots m_r$ and let R_m be the set of all congruence classes $a + m\mathbb{Z}$ such that $a + m_i\mathbb{Z} \in R_{m_i}$ for $i = 1, \dots, r$. By Lemma 2,

$$\frac{|f(R_m)|}{|g(R_m)|} = \prod_{i=1}^r \frac{|f(R_{m_i})|}{|g(R_{m_i})|}.$$

Let A be the set of integers that consists of the least nonnegative element in each congruence class in R_m . By Lemma 3 we have

$$|f(A)| \leq 2h_f|f(R_m)| < |g(R_m)| \leq |g(A)|.$$

This completes the proof. ■

THEOREM 8. *Let $f(x_1, \dots, x_n) = u_1x_1 + \dots + u_nx_n$ and $g(x_1, \dots, x_n) = v_1x_1 + \dots + v_nx_n$ be binary linear forms. Let $M = \{m_i\}_{i=1}^\infty$ be a set of pairwise relatively prime positive integers such that $m_i \geq 2$ for all $m_i \in M$ and*

$$\sum_{i=1}^\infty \frac{1}{m_i} = \infty.$$

If for every $m_i \in M$ there exists a nonempty set R_{m_i} of congruence classes in $\mathbb{Z}/m_i\mathbb{Z}$ such that

$$f(R_{m_i}) \neq \mathbb{Z}/m_i\mathbb{Z} \quad \text{and} \quad g(R_{m_i}) = \mathbb{Z}/m_i\mathbb{Z},$$

then there is a finite set A of integers such that

$$|f(A)| < |g(A)|.$$

Proof. Since $|f(R_{m_i})| \leq m_i - 1$ and $|g(R_{m_i})| = m_i$ for all $m_i \in M$, we have

$$\frac{|f(R_{m_i})|}{|g(R_{m_i})|} \leq 1 - \frac{1}{m_i}.$$

The divergence of the infinite series $\sum_{i=1}^{\infty} m_i^{-1}$ implies that

$$\prod_{i=1}^{\infty} \frac{|f(R_{m_i})|}{|g(R_{m_i})|} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{m_i}\right) = 0,$$

and the result follows immediately from Theorem 7. ■

We can restate Theorem 8 as follows.

THEOREM 9. *Let $f(x_1, \dots, x_n) = u_1x_1 + \dots + u_nx_n$ and $g(x_1, \dots, x_n) = v_1x_1 + \dots + v_nx_n$ be binary linear forms. Let $M = \{m_i\}_{i=1}^{\infty}$ be a set of pairwise relatively prime positive integers such that $m_i \geq 2$ for all $m_i \in M$ and*

$$\sum_{i=1}^{\infty} \frac{1}{m_i} = \infty.$$

If for every $m_i \in M$ there exists an integer q_{m_i} and a finite set A_{m_i} of integers such that

- (i) $f(a_1, \dots, a_n) \not\equiv q_{m_i} \pmod{m_i}$ for all $a_1, \dots, a_n \in A_{m_i}$, and
- (ii) for every integer q the congruence $g(a_1, \dots, a_n) \equiv q \pmod{m_i}$ is solvable with $a_1, \dots, a_n \in A_{m_i}$,

then there is a finite set A of integers such that

$$|f(A)| < |g(A)|.$$

6. An application of quadratic reciprocity

THEOREM 10. *Let p be a prime number such that $p \equiv 1 \pmod{4}$ and $p > 5$, and let*

$$R_p = \{k^2 + p\mathbb{Z} : k = 1, \dots, (p-1)/2\}$$

be the set of quadratic residues modulo p . Let $s(x, y) = x + y$ and $d(x, y) = x - y$. Then

$$s(R_p) = d(R_p) = \mathbb{Z}/p\mathbb{Z}.$$

Proof. Since $p \equiv 1 \pmod{4}$ there is an integer $s \not\equiv 0 \pmod{p}$ with $s^2 \equiv -1 \pmod{p}$. For all integers a and u with $u \not\equiv 0 \pmod{p}$ there is an integer v (unique modulo p) such that $uv \equiv a \pmod{p}$. Let $x \equiv (u - v)/2 \pmod{p}$ and $y \equiv (u + v)/2 \pmod{p}$. Then $y^2 + (sx)^2 \equiv y^2 - x^2 \equiv a \pmod{p}$, and this congruence has at least $p - 1$ solutions. Since there are at most two solutions with $x \equiv 0 \pmod{p}$ and at most two solutions with $y \equiv 0 \pmod{p}$, it follows that if $p > 5$ then there is at least one solution with $x^2 + p\mathbb{Z} \in R_p$ and $y^2 + p\mathbb{Z} \in R_p$. Therefore, $s(R_p) = d(R_p) = \mathbb{Z}/p\mathbb{Z}$. ■

THEOREM 11. *Consider the binary linear form*

$$f(x, y) = ux + vy$$

where u and v are integers not divisible by p . Let p be a prime number and let R_p be the set of quadratic residues modulo p . Then $p\mathbb{Z} \in f(R_p)$ if and only if $-uv$ is a quadratic residue modulo p .

Proof. If $p\mathbb{Z} \in f(R_p)$, then there are integers k_1 and k_2 not divisible by p such that

$$uk_1^2 + vk_2^2 \equiv 0 \pmod{p}.$$

Then

$$uvk_1^2 + (vk_2)^2 \equiv 0 \pmod{p}$$

and so

$$-uv \equiv (vk_2k_1^{-1})^2 \pmod{p},$$

that is, $-uv$ is a quadratic residue modulo p .

Conversely, if $-uv$ is a quadratic residue modulo p , then there is an integer $z \not\equiv 0 \pmod{p}$ such that $-uv \equiv z^2 \pmod{p}$ and so

$$f(v^2, z^2) = uv^2 + vz^2 \equiv 0 \pmod{p}.$$

Thus, $p\mathbb{Z} = f(v^2 + p\mathbb{Z}, z^2 + p\mathbb{Z}) \in f(R_p)$. ■

THEOREM 12. *Let*

$$f(x, y) = ux + vy$$

be a normalized bilinear form such that $|uv|$ is not a perfect square. Let

$$s(x, y) = x + y \quad \text{and} \quad d(x, y) = x - y.$$

There exist finite sets A and A' of integers such that

$$|f(A)| < |s(A)|, \quad |f(A')| < |d(A')|.$$

Proof. Since $f(x, y)$ is normalized and $|uv|$ is not a square, we can write

$$|uv| = w^2 2^\varepsilon \prod_{j=1}^t q_j$$

where w is a positive integer, $\varepsilon \in \{0, 1\}$, q_1, \dots, q_t are distinct odd primes, and $q_i \neq p$ for $i = 1, \dots, t$. For $p \equiv 1 \pmod{4}$, we have

$$\left(\frac{-uv}{p}\right) = \left(\frac{\pm w^2 2^\varepsilon \prod_{j=1}^t q_j}{p}\right) = \left(\frac{2}{p}\right)^\varepsilon \prod_{j=1}^t \left(\frac{q_j}{p}\right) = \left(\frac{2}{p}\right)^\varepsilon \prod_{j=1}^t \left(\frac{p}{q_j}\right).$$

If $\varepsilon = 1$, we choose p so that $p \equiv 5 \pmod{8}$ and $p \equiv 1 \pmod{q_j}$ for $j = 1, \dots, t$. Then $(-uv|p) = -1$. If $\varepsilon = 0$, then $t \geq 1$ and we choose p so that $p \equiv 1 \pmod{4}$, $p \equiv 1 \pmod{q_j}$ for $j = 2, \dots, t$, and $(p|q_1) = -1$. Again, $(-uv|p) = -1$. In both cases, there is at least one infinite arithmetic progression $P(u, v)$ such that if p is a prime and $p \in P(u, v)$, then $p \equiv 1 \pmod{4}$ and $(-uv|p) = -1$. By Dirichlet’s theorem, the arithmetic progression $P(u, v)$ contains infinitely many primes and

$$\sum_{\substack{p \in P(u, v) \\ p \text{ prime}}} \frac{1}{p} = \infty.$$

By Theorems 10 and 11, for each prime $p \in P(u, v)$, the set

$$R_p = \{k^2 + p\mathbb{Z} : k = 1, \dots, (p - 1)/2\}$$

satisfies

$$f(R_p) \neq \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad s(R_p) = d(R_p) = \mathbb{Z}/p\mathbb{Z}.$$

The result now follows from Theorem 8. ■

7. An exponential sum. After Theorem 12, we are left to consider only normalized bilinear forms $f(x, y) = ux + vy$ such that $|uv|$ is a square. Since u and v are relatively prime, it follows that there are positive integers U and V not divisible by p such that $u = U^2$ and $v = \pm V^2$. Let R_p denote the set of quadratic residues modulo the prime p . If $f(x, y) = U^2x + V^2y$, then $f(R_p) = s(R_p)$. If $f(x, y) = U^2x - V^2y$, then $f(R_p) = d(R_p)$. This suggests that considering only squares mod p will not suffice to resolve Problem 4 in this remaining case. We shall generalize our method to k th powers. We begin by applying elementary harmonic analysis on finite fields to binary linear forms. A general reference is Nathanson [5, Chapter 4].

Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field of congruence classes modulo p . We denote the multiplicative group of the field by \mathbb{F}_p^\times and define

$$e_p(t) = e^{2\pi it/p}.$$

For all integers a, b , and t we have

$$\sum_{t=0}^{p-1} e_p((a - b)t) = \begin{cases} 0 & \text{if } a \not\equiv b \pmod{p}, \\ p & \text{if } a \equiv b \pmod{p}. \end{cases}$$

If x is a congruence class modulo p , that is, if $x = t + p\mathbb{Z}$ for some integer t , then we define $e_p(x) = e_p(t)$. This function is well-defined on \mathbb{F}_p .

Let γ be a complex-valued function on \mathbb{F}_p . We define the *Fourier transform* $\hat{\gamma} : \mathbb{F}_p \rightarrow \mathbb{C}$ by

$$\hat{\gamma}(x) = \sum_{y \in \mathbb{F}_p} \gamma(y)e_p(-xy).$$

We have

$$\hat{\gamma}(0) = \sum_{y \in \mathbb{F}_p} \gamma(y)$$

and Plancherel's formula [5, Theorem 4.9]

$$(4) \quad \sum_{x \in \mathbb{F}_p} |\hat{\gamma}(x)|^2 = p \sum_{x \in \mathbb{F}_p} |\gamma(x)|^2.$$

Let H be a subset of \mathbb{F}_p of cardinality n . We also use H to denote the characteristic function of H , that is, $H : \mathbb{F}_p \rightarrow \mathbb{C}$ is the function defined by

$$H(x) = \begin{cases} 1 & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

Then

$$\hat{H}(x) = \sum_{y \in \mathbb{F}_p} H(y)e_p(-xy) = \sum_{h \in H} e_p(-xh), \quad \hat{H}(0) = \text{card}(H) = n.$$

Applying Plancherel's formula to the function H , we obtain

$$\sum_{x \in \mathbb{F}_p} |\hat{H}(x)|^2 = p \sum_{x \in \mathbb{F}_p} |H(x)|^2 = p \text{card}(H) = pn$$

and so

$$(5) \quad \sum_{x \in \mathbb{F}_p^\times} |\hat{H}(x)|^2 = \sum_{x \in \mathbb{F}_p} |\hat{H}(x)|^2 - |\hat{H}(0)|^2 = (p - n)n.$$

THEOREM 13. *Let $f(x, y) = ux + vy$, where u and v are integers not divisible by p . Let H be a subgroup of order $n \geq 2$ of the multiplicative group \mathbb{F}_p^\times and let*

$$k = [\mathbb{F}_p^\times : H] = \frac{p - 1}{n}.$$

If $p > k^4$, then $\mathbb{F}_p^\times \subseteq f(H)$, that is, every element of \mathbb{F}_p^\times can be represented in the form $f(h_1, h_2)$ for some $h_1, h_2 \in H$.

Proof. Define the representation function $r : \mathbb{F}_p \rightarrow \mathbb{N}_0$ as follows. For every $x \in \mathbb{F}_p$, let $r(x)$ denote the number of ordered pairs $(h_1, h_2) \in H \times H$ such that $f(h_1, h_2) = x$. Then

$$\sum_{x \in \mathbb{F}_p} r(x) = |H|^2 = n^2$$

and

$$\begin{aligned} \widehat{r}(x) &= \sum_{y \in \mathbb{F}_p} r(y)e_p(-xy) = \sum_{y \in \mathbb{F}_p} \left(\sum_{\substack{h_1, h_2 \in H \\ uh_1 + vh_2 = y}} 1 \right) e_p(-xy) \\ &= \sum_{h_1 \in H} \sum_{h_2 \in H} e_p(-(uh_1 + vh_2)x) = \sum_{h_1 \in H} e_p(-uxh_1) \sum_{h_2 \in H} e_p(-vxh_2) \\ &= \widehat{H}(ux)\widehat{H}(vx). \end{aligned}$$

Applying Plancherel’s formula (4) to the function $r(x)$, we obtain

$$\begin{aligned} \sum_{x \in \mathbb{F}_p^\times} |\widehat{H}(ux)\widehat{H}(vx)|^2 &= \sum_{x \in \mathbb{F}_p} |\widehat{H}(ux)\widehat{H}(vx)|^2 - |\widehat{H}(0)|^4 = \sum_{x \in \mathbb{F}_p} \widehat{r}(x)^2 - n^4 \\ &= p \sum_{x \in \mathbb{F}_p} r(x)^2 - n^4 = p \sum_{x \in \mathbb{F}_p} \left(r(x) - \frac{n^2}{p} \right)^2. \end{aligned}$$

Let $x, x' \in \mathbb{F}_p^\times$. If x and x' belong to the same coset of \mathbb{F}_p/H , then there exists $h' \in H$ such that $x = x'h'$. It follows that

$$\widehat{H}(x) = \sum_{h \in H} e_p(-xh) = \sum_{h \in H} e_p(-x'h'h) = \sum_{h \in H} e_p(-x'h) = \widehat{H}(x')$$

and so the Fourier transform $\widehat{H}(x)$ is constant on the cosets of \mathbb{F}_p^\times/H . Similarly, $uh_1 + vh_2 = x'$ if and only if $uh_1h' + vh_2h' = x'h' = x$, and so $r(x) = r(x')$, that is, the representation function $r(x)$ is also constant on the cosets of \mathbb{F}_p^\times/H .

Let $\{x_1, \dots, x_k\} \subseteq \mathbb{F}_p^\times$ be a set of coset representatives of H , that is,

$$\mathbb{F}_p^\times/H = \{x_1H, \dots, x_kH\}.$$

Applying (5), we obtain

$$(p - n)n = \sum_{x \in \mathbb{F}_p^\times} |\widehat{H}(x)|^2 = \sum_{i=1}^k \sum_{x \in x_iH} |\widehat{H}(x)|^2 = n \sum_{i=1}^k |\widehat{H}(x_i)|^2$$

and so

$$\sum_{i=1}^k |\widehat{H}(x_i)|^2 = p - n.$$

For every $x \in \mathbb{F}_p^\times$, there is an integer $j \in \{1, \dots, k\}$ such that $x \in x_jH$ and $\widehat{H}(x) = \widehat{H}(x_j)$. It follows that

$$|\widehat{H}(x)|^2 = |\widehat{H}(x_j)|^2 \leq \sum_{i=1}^k |\widehat{H}(x_i)|^2 = p - n.$$

Since $(u, p) = (v, p) = 1$, we have

$$\begin{aligned} p \sum_{x \in \mathbb{F}_p} \left(r(x) - \frac{n^2}{p} \right)^2 &= \sum_{x \in \mathbb{F}_p^\times} |\widehat{H}(ux)\widehat{H}(vx)|^2 \leq (p-n) \sum_{x \in \mathbb{F}_p^\times} |\widehat{H}(ux)|^2 \\ &= (p-n) \sum_{x \in \mathbb{F}_p^\times} |\widehat{H}(x)|^2 = (p-n)^2 n. \end{aligned}$$

Since the representation function $r(x)$ is constant on cosets of H , we have

$$\begin{aligned} \sum_{i=1}^k \left(r(x_i) - \frac{n^2}{p} \right)^2 &= \frac{1}{n} \sum_{i=1}^k \sum_{x \in x_i H} \left(r(x) - \frac{n^2}{p} \right)^2 = \frac{1}{n} \sum_{x \in \mathbb{F}_p} \left(r(x) - \frac{n^2}{p} \right)^2 \\ &\leq \frac{(p-n)^2}{p}. \end{aligned}$$

For every $x \in \mathbb{F}_p^\times$, we have $x \in x_j H$ for some j , and so

$$\left(r(x) - \frac{n^2}{p} \right)^2 = \left(r(x_j) - \frac{n^2}{p} \right)^2 \leq \frac{(p-n)^2}{p}.$$

If $r(x) = 0$ for some $x \in \mathbb{F}_p^\times$, then

$$\frac{n^4}{p^2} \leq \frac{(p-n)^2}{p}.$$

Since $|H| = n \geq 2$ and $p(p-n) \leq p(p-2) < (p-1)^2$, we have

$$\left(\frac{p-1}{k} \right)^4 = n^4 \leq p(p-n)^2 < (p-1)^3 \quad \text{and} \quad p \leq k^4.$$

This proves that if $p > k^4$, then $r(x) \geq 1$ for all $x \in \mathbb{F}_p^\times$ and so $\mathbb{F}_p^\times \subseteq f(H)$. ■

A finite cyclic group G of order N has a unique subgroup H of order n for every positive divisor n of N . If $k = [G : H] = N/n$ and if g is a generator of G , then $H = \{g^{ik} : i = 0, 1, \dots, n-1\} = \{x^k : x \in G\}$ is the set of k th powers in G .

Let H be a subgroup of order n of \mathbb{F}_p^\times and let $k = [\mathbb{F}_p^\times : H] = (p-1)/n$. Since the multiplicative group of a finite field is cyclic, it follows that H is the subgroup of k th powers mod p , that is, $H = \{x^k : x \in \mathbb{F}_p^\times\}$. We can restate Theorem 13 as follows.

THEOREM 14. *Let $f(x, y) = ux + vy$ be a binary linear form with nonzero integral coefficients u and v . For $k \geq 2$, let p be a prime number such that $p \equiv 1 \pmod{k}$, $p > k^4$, and $(p, uv) = 1$. If H is the set of k th powers in \mathbb{F}_p^\times , then*

$$\mathbb{F}_p^\times \subseteq f(H).$$

We shall prove that if $u \geq 2$, then there are infinitely many primes p such that $0 \notin f(H)$ and so $f(H) = \mathbb{F}_p^\times$. We use a standard result about irreducible polynomials: If a is a nonzero rational number and q is a prime number such that a is not a q th power, then the polynomial $g(x) = x^q - a$ is irreducible in $\mathbb{Q}[x]$.

THEOREM 15. *Let u and v be relatively prime integers such that $u > |v| \geq 1$. Let*

$$f(x, y) = ux + vy.$$

There exist finite sets A and A' of integers such that

$$|f(A)| < |s(A)| \quad \text{and} \quad |f(A')| < |d(A')|.$$

Proof. Let k be a positive integer and let p be a prime such that $p \equiv 1 \pmod{k}$. Let H be the set of k th powers of elements of \mathbb{F}_p^\times . Then H is a multiplicative subgroup of \mathbb{F}_p^\times of order $|H| = (p - 1)/k$ and $[\mathbb{F}_p^\times : H] = k$.

Let u and v be integers relatively prime to p , and consider the binary linear form

$$f(x, y) = ux + vy.$$

By Theorem 13, if $p > k^4$, then $\mathbb{F}_p^\times \subseteq f(H)$ and so

$$|f(H)| \geq p - 1.$$

Let $u = 1$ and $v = \pm 1$. For the polynomial $d(x, y) = x - y$, we have

$$d(1^k, 1^k) = 0 \in d(H) \quad \text{and so} \quad d(H) = \mathbb{F}_p.$$

If k is odd, then for the polynomial $s(x, y) = x + y$ we have

$$s(1^k, (-1)^k) = 0 \in s(H) \quad \text{and} \quad s(H) = \mathbb{F}_p.$$

In both cases,

$$|d(H)| = |s(H)| = p.$$

Let $u > |v| \geq 1$ and $(u, v) = (uv, p) = 1$. Let $a = -uv^{k-1}$. If there exist $h_1 = \ell_1^k \in H$ and $h_2 = \ell_2^k \in H$ such that

$$f(h_1, h_2) = uh_1 + vh_2 = u\ell_1^k + v\ell_2^k = 0,$$

then

$$\left(\frac{u\ell_1}{\ell_2}\right)^k + u^{k-1}v = 0$$

and the polynomial $g(x) = x^k - a$ has a root in \mathbb{F}_p , and is, therefore, reducible. It follows that if $g(x)$ is irreducible in $\mathbb{F}_p[x]$, then $0 \notin f(H)$ and so

$$|f(H)| = p - 1.$$

The rational integer $a = -u^{k-1}v$ is not a q th power for all sufficiently large primes q , and so the polynomial $g(x) = x^q - a$ is irreducible over \mathbb{Q} . Let \mathcal{P} be the set of primes $p > q^4$ such that $p \equiv 1 \pmod{q}$ and $g(x)$ is

irreducible in $\mathbb{F}_p[x]$. The Chebotarev density theorem implies that the series $\sum_{p \in \mathcal{P}} 1/p$ diverges, and Theorem 15 now follows directly from Theorem 8. This completes the proof. ■

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Department of Mathematics
Lehman College (CUNY)
Bronx, NY 10468, U.S.A.
E-mail: melvyn.nathanson@lehman.cuny.edu

Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary
E-mail: ruzsa@renyi.hu

Department of Mathematics
College of Staten Island (CUNY)
Staten Island, NY 10314, U.S.A.
E-mail: kevin@member.ams.org

Department of Mathematics
New University of Lisbon
Lisbon, Portugal
E-mail: mnasilva@gmail.com

Department of Mathematics
CUNY Graduate Center
New York, NY 10036, U.S.A.
E-mail: borosz@gc.cuny.edu

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