A characterization of covering equivalence

by

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1. Introduction. For \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \) and \( a \in \{0, \ldots, n-1\} \), we write \( a(n) \) to denote the residue class \( \{x \in \mathbb{Z} : x \equiv a \pmod{n}\} \). For a finite system
\[
A = \left\{ a_s(n_s) \right\}_{s=1}^{k} \quad (0 \leq a_s < n_s)
\]
of residue classes, the \( n_1, \ldots, n_k \) are called its moduli, and its covering function \( w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, \ldots\} \) is given by
\[
w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}|.
\]
(The covering function \( w_\emptyset \) of an empty system is regarded as the zero function.) The periodic function \( w_A(x) \) has many surprising properties (cf. [S95], [S96]).

Let \( m \) be a positive integer. If \( w_A(x) = m \) for all \( x \in \mathbb{Z} \), then (1.1) is said to be an exact \( m \)-cover of \( \mathbb{Z} \) as in [S95] and [S96]. Recently Z. W. Sun (cf. [S04] and [S05b]) showed that (1.1) forms an exact \( m \)-cover of \( \mathbb{Z} \) if it covers \( |S(n_1, \ldots, n_k)| \) consecutive integers exactly \( m \) times, where
\[
S(n_1, \ldots, n_k) = \{r/n_s : r = 0, \ldots, n_s - 1; s = 1, \ldots, k\}.
\]
For problems and results on covers of \( \mathbb{Z} \) by residue classes, the reader is referred to [FFKPY], [G04] and [S03b].

For two finite systems \( A = \{a_s(n_s)\}_{s=1}^{k} \) and \( B = \{b_t(m_t)\}_{t=1}^{l} \), Sun [S89] called \( A \) and \( B \) covering equivalent (written \( A \sim B \)) if they have the same covering function (i.e., \( w_A = w_B \)). Thus (1.1) is an exact \( m \)-cover of \( \mathbb{Z} \) if and only if (1.1) is covering equivalent to the system consisting of \( m \) copies of \( 0(1) \).

In [S01] and [S02] Sun characterized the covering equivalence by various systems of equalities. In this paper we present a simple characterization involving roots of unity. Namely, we have the following result.

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Theorem 1.1. Let \( A = \{a_s(n_s)\}_{s=1}^k \) (\( 0 \leq a_s < n_s \)) and \( B = \{b_l(m_l)\}_{l=1}^l \) (\( 0 \leq b_l < m_l \)) be two finite systems of residue classes. Let \( p \) be a prime greater than \( |S(n_1, \ldots, n_k, m_1, \ldots, m_l)| \), and let \( \zeta_p \) be a primitive \( p \)th root of unity. Then \( A \) and \( B \) are covering equivalent if and only if

\[
\sum_{s=1}^k \frac{\zeta_{a_s}^n}{1 - \zeta_p^{n_s}} = \sum_{l=1}^l \frac{\zeta_{b_l}^{m_l}}{1 - \zeta_p^{m_l}}.
\]

Corollary 1.1. (1.1) forms an exact \( m \)-cover of \( \mathbb{Z} \) if and only if

\[
\sum_{s=1}^k \frac{e^{2\pi i a_s/p}}{1 - e^{2\pi i n_s/p}} = \frac{m}{1 - e^{2\pi i/p}},
\]

where \( p \) is any fixed prime greater than \( |S(n_1, \ldots, n_k)| \).

Proof. Simply apply Theorem 1.1 with \( B \) consisting of \( m \) copies of \( 0(1) \).

Remark 1.1. In 1975 Š. Znám [Z75a] used the transcendence of \( e \) to prove that (1.1) is a disjoint cover (i.e., exact 1-cover) of \( \mathbb{Z} \) if and only if

\[
\sum_{s=1}^k \frac{e^{a_s}}{1 - e^{n_s}} = \frac{1}{1 - e}.
\]

Corollary 1.2. Suppose that for a nonempty system (1.1) we have

\[
\sum_{s=1}^k \frac{e^{2\pi i a_s/p}}{1 - e^{2\pi i n_s/p}} = 0
\]

where \( p \) is a prime. Then

\[
n_1 + \cdots + n_k - k + 1 \geq |S(n_1, \ldots, n_k)| \geq p.
\]

Proof. Clearly \( |S(n_1, \ldots, n_k)| \leq n_1 + \cdots + n_k - k + 1 \). Since \( A \not\sim \emptyset \), applying Theorem 1.1 with \( B = \emptyset \) we find that \( |S(n_1, \ldots, n_k)| \) cannot be smaller than \( p \).

Corollary 1.2 partially confirms the following conjecture arising from the study of Fraenkel’s conjecture on disjoint covers of \( \mathbb{N} \) by Beatty sequences.

Graham–O’Bryant Conjecture ([GO]). Let \( n_1, \ldots, n_k \) be distinct positive integers less than and relatively prime to \( q \in \mathbb{Z}^+ \). If \( a_1, \ldots, a_k \in \mathbb{Z} \) and

\[
\sum_{s=1}^k \frac{e^{2\pi i a_s/q}}{1 - e^{2\pi i n_s/q}} = 0,
\]

then we must have \( \sum_{s=1}^k n_s \geq q \).

The following example shows that we cannot replace the prime \( p \) in Corollary 1.2 or Theorem 1.1 by a composite number.
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Example 1.1. Let $q > 1$ be an integer and let $p$ be a prime divisor of $q$. Then, for any $n = 1, \ldots, q - 1$, we have

$$\sum_{s=0}^{p-1} \frac{e^{2\pi i (sq/p)/q}}{1 - e^{2\pi i n/q}} = \sum_{s=0}^{p-1} \frac{e^{2\pi i s/p}}{1 - e^{2\pi i n/q}} = 0$$

but $|S(n, \ldots, n)| = n < q$. Thus the conditions $0 \leq a_s < n_s$ $(s = 1, \ldots, k)$ in Corollary 1.2 cannot be cancelled. If $q$ is composite, then there are $q/p - 1 > 0$ integers in the interval $((p-1)/p, q-1]$. So we cannot substitute a composite number for the prime $p$ in Corollary 1.2.

\[\text{Corollary 1.3.} \quad \text{Let } A = \{a_s(n_s)\}_{s=1}^{k} \text{ and } B = \{b_t(m_t)\}_{t=1}^{l} \quad \text{both have distinct moduli. Let } p \text{ be a prime greater than } |S(n_1, \ldots, n_k)|, \text{ and let } \zeta_p \text{ be a primitive } p\text{th root of unity. Then } A \sim B \text{ if and only if } (1.4) \text{ holds.} \]

Proof. By a result of Znám [Z75b], $A$ and $B$ are identical if they have the same covering function. Combining this with Theorem 1.1 we immediately get the desired result. ■

Observe that $A = \{a_s(n_s)\}_{s=1}^{k}$ and $B = \{b_t(m_t)\}_{t=1}^{l}$ are covering equivalent if and only if

\[\sum_{s=1}^{k} \sum_{x \in a_s(n_s)} 1 + \sum_{t=1}^{l} (-1) = 0 \text{ for every } x \in \mathbb{Z}.\]

Thus Theorem 1.1 has the following equivalent form which will be proved in the next section.

\[\text{Theorem 1.2.} \quad \text{Let } A = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^{k} \text{ where } \lambda_s, a_s, n_s \in \mathbb{Z} \text{ and } 0 \leq a_s < n_s. \text{ Let } p > \max\{n_1, \ldots, n_k\} \text{ be a prime, and let } \zeta_p \text{ be any primitive } p\text{th root of unity. Then } A \sim \emptyset \text{ (i.e., } w_A(x) = \sum_{1 \leq s \leq k, x \in a_s(n_s)} \lambda_s = 0 \text{ for all } x \in \mathbb{Z}) \text{ if and only if } (1.7) \]

\[\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0.\]

2. Proof of Theorem 1.2. Let $S = S(n_1, \ldots, n_k)$. As $p > |S| \geq \max\{n_1, \ldots, n_k\}$, there is a common multiple $N \in \mathbb{Z}^+$ of the moduli $n_1, \ldots, n_k$ such that $N \equiv 1 \pmod{p}$. Just as in [S05a], we have

$$\sum_{r=0}^{N-1} w_A(r) z^r = \sum_{r=0}^{N-1} \sum_{1 \leq s \leq k} \lambda_s z^{r} = \sum_{s=1}^{k} \lambda_s \sum_{0 \leq r < N} \sum_{n_s | a_s - r} z^r$$
\[= \sum_{s=1}^{k} \lambda_s z^{a_s} \sum_{0 \leq q < N/n_s} (z^{n_s})^q\]
\[= N \sum_{1 \leq s \leq k} \frac{\lambda_s}{n_s} z^{a_s} + (1 - z^N) \sum_{1 \leq s \leq k} \frac{\lambda_s}{1 - z^{n_s}}.\]

Thus
\[N^{-1} \sum_{r=0}^{N-1} w_A(r) \zeta_p^r = (1 - \zeta_p^N) \sum_{s=1}^{k} \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}}.\]

It follows that
\[\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0 \iff \sum_{l=0}^{p-1} c_l \zeta_p^l = 0,\]
where
\[c_l = \sum_{x=0}^{N-1} \frac{w_A(x)}{x \in l(p)} \in \mathbb{Z}.\]

If \(w_A(x) = 0\) for all \(x \in \mathbb{Z}\), then (1.7) holds by the above.

Below we assume (1.7). Then \(\sum_{l=0}^{p-1} c_l \zeta_p^l = \sum_{r=0}^{N-1} w_A(r) \zeta_p^r = 0\). In the case \(N = 1\), it follows that \(w_A(x) = w_A(0) = 0\) for all \(x \in \mathbb{Z}\). Now suppose \(N > 1\). Clearly \(N > p\) as \(N \equiv 1 \pmod{p}\). Since \(1 + x + \cdots + x^{p-1} = (x^p - 1)/(x - 1)\) is the minimal polynomial of \(\zeta_p\) over the field of rational numbers, we must have \(c_0 = c_1 = \cdots = c_{p-1}\). (See also M. Newman [N71].)

Observe that if \(x \in \mathbb{Z}\) then
\[w_A(x) = \sum_{s=1}^{k} \lambda_s \frac{n_s^{-1}}{a_s - x} e^{2\pi i a_s x/n_s} = \sum_{\alpha \in S} e^{-2\pi i \alpha x} \sum_{s=1}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s}.\]

(This trick appeared in [S91] and [S04].) Since \(|S| < p\), for each \(l = 0, \ldots, |S|\) we have
\[c_l = \sum_{x=0}^{N-1} \frac{w_A(x)}{x \in l(p)} = \sum_{\alpha \in S} \sum_{s=1}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \sum_{x=0}^{N-1} e^{-2\pi i \alpha x} \sum_{x \in l(p)} \frac{1}{x} \in \mathbb{Z}.
\]

\[\sum_{\alpha \in S} \sum_{s=1}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \sum_{j=0}^{[(N-1-l)/p]} e^{-2\pi i \alpha p j} ,\]

where \([ \cdot ]\) is the greatest integer function. If \(l \in \{1, \ldots, |S|\}\) then
\[\left\lceil \frac{N - 1 - l}{p} \right\rceil = \frac{N - 1}{p} + \left\lfloor \frac{-l}{p} \right\rfloor = \frac{N - 1}{p} - 1;\]
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if $\alpha \in S \setminus \{0\}$ then

$$C(\alpha) := \sum_{j=0}^{(N-1)/p-1} e^{-2\pi i \alpha j} = \frac{1 - (e^{-2\pi i \alpha})(N-1)/p}{1 - e^{-2\pi i \alpha}} = \frac{1 - e^{2\pi i \alpha}}{1 - e^{-2\pi i \alpha}} \neq 0.$$ 

Let $c = c_0 = \cdots = c_{p-1}$. By the above,

$$\sum_{\alpha \in S} e^{-2\pi i \alpha j} f(\alpha) = c$$

for every $j = 0, \ldots, |S| - 1$, where

$$f(0) = \frac{N - 1}{p} \sum_{s=1}^{k} \frac{\lambda_s}{n_s}$$

and

$$f(\alpha) = e^{-2\pi i \alpha} C(\alpha) \sum_{\alpha \in S \setminus \{0\}} e^{2\pi i \alpha a_s} \text{ for } \alpha \in S \setminus \{0\}.$$ 

Let $\alpha_0 = 0$, $\alpha_1, \ldots, \alpha_{|S|-1}$ be all the distinct elements of $S$. Now that

$$\sum_{t=0}^{|S|-1} e^{-2\pi i \alpha_t j} f(\alpha_t) = c \text{ for each } j = 0, \ldots, |S| - 1,$$

by Cramer’s rule $D_t = D f(\alpha_t)$ vanishes for every $t = 1, \ldots, |S| - 1$, where

$$D = \det ((e^{-2\pi i \alpha_t j})_{0 \leq j, t < |S|})$$

is of Vandermonde’s type and hence nonzero. Therefore

$$\sum_{\alpha \in S \setminus \{0\}} \sum_{s=1}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \text{ for all } \alpha \in S \setminus \{0\}$$

and hence $w_A(x) = \sum_{s=1}^{k} \lambda_s/n_s$ for all $x \in \mathbb{Z}$ by (2.2). It follows that

$$0 = \sum_{r=0}^{N-1} w_A(r) \zeta_p^r = \sum_{s=1}^{k} \frac{\lambda_s}{n_s} \left(1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{N-1}\right) = \sum_{s=1}^{k} \frac{\lambda_s}{n_s} \cdot \frac{1 - \zeta_p^N}{1 - \zeta_p}.$$ 

So $\sum_{s=1}^{k} \lambda_s/n_s = 0$ and hence $A \sim \emptyset$. We are done.

References


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