ACTA ARITHMETICA
129.4 (2007)

A characterization of covering equivalence

by

Hao Pan (Shanghai) and Zhi-Wei Sun (Nanjing)

1. Introduction. For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $a \in \{0, \ldots, n-1\}$, we write $a(n)$ to denote the residue class $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \quad (0 \le a_s < n_s) \tag{1.1}$$

of residue classes, the $n_1, \ldots, n_k$ are called its moduli, and its *covering function* $w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, \ldots\}$ is given by

$$w_A(x) = |\{1 \le s \le k : x \in a_s(n_s)\}|. \tag{1.2}$$

(The covering function $w_\emptyset$ of an empty system is regarded as the zero function.) The periodic function $w_A(x)$ has many surprising properties (cf. [S03a], [S04] and [S05a]).

Let m be a positive integer. If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then (1.1) is said to be an *exact m-cover* of $\mathbb{Z}$ as in [S95] and [S96]. Recently Z. W. Sun (cf. [S04] and [S05b]) showed that (1.1) forms an exact m-cover of $\mathbb{Z}$ if it covers $|S(n_1, \ldots, n_k)|$ consecutive integers exactly m times, where

$$S(n_1, \ldots, n_k) = \{r/n_s : r = 0, \ldots, n_s - 1;\ s = 1, \ldots, k\}. \tag{1.3}$$

For problems and results on covers of $\mathbb{Z}$ by residue classes, the reader is referred to [FFKPY], [G04] and [S03b].

For two finite systems $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$, Sun [S89] called A and B *covering equivalent* (written $A \sim B$) if they have the same covering function (i.e., $w_A = w_B$). Thus (1.1) is an exact m-cover of $\mathbb{Z}$ if and only if (1.1) is covering equivalent to the system consisting of m copies of $0(1)$.

In [S01] and [S02] Sun characterized the covering equivalence by various systems of equalities. In this paper we present a simple characterization involving roots of unity. Namely, we have the following result.

2000 *Mathematics Subject Classification*: Primary 11B25; Secondary 11A07, 11B75.

Key words and phrases: covering equivalence, Graham–O'Bryant conjecture.

The second author is responsible for communications, and supported by the National Science Fund (Grant No. 10425103) for Distinguished Young Scholars in China.

THEOREM 1.1. *Let* $A=\{a_s(n_s)\}_{s=1}^k$ $(0\le a_s<n_s)$ *and* $B=\{b_t(m_t)\}_{t=1}^l$ $(0\le b_t<m_t)$ *be two finite systems of residue classes. Let* p *be a prime greater than* $|S(n_1,\ldots,n_k,m_1,\ldots,m_l)|$, *and let* ζ_p *be a primitive pth root of unity. Then* A *and* B *are covering equivalent if and only if*

$$\sum_{s=1}^k\frac{\zeta_p^{a_s}}{1-\zeta_p^{n_s}}=\sum_{t=1}^l\frac{\zeta_p^{b_t}}{1-\zeta_p^{m_t}}. \tag{1.4}$$

COROLLARY 1.1. (1.1) *forms an exact* m*-cover of* $\mathbb{Z}$ *if and only if*

$$\sum_{s=1}^k\frac{e^{2\pi i a_s/p}}{1-e^{2\pi i n_s/p}}=\frac{m}{1-e^{2\pi i/p}}, \tag{1.5}$$

where p *is any fixed prime greater than* $|S(n_1,\ldots,n_k)|$.

Proof. Simply apply Theorem 1.1 with B consisting of m copies of $0(1)$. ■

REMARK 1.1. In 1975 Š. Znám [Z75a] used the transcendence of e to prove that (1.1) is a disjoint cover (i.e., exact 1-cover) of $\mathbb{Z}$ if and only if

$$\sum_{s=1}^k\frac{e^{a_s}}{1-e^{n_s}}=\frac{1}{1-e}.$$

COROLLARY 1.2. *Suppose that for a nonempty system* (1.1) *we have*

$$\sum_{s=1}^k\frac{e^{2\pi i a_s/p}}{1-e^{2\pi i n_s/p}}=0$$

where p *is a prime. Then*

$$n_1+\cdots+n_k-k+1\ge|S(n_1,\ldots,n_k)|\ge p. \tag{1.6}$$

Proof. Clearly $|S(n_1,\ldots,n_k)|\le n_1+\cdots+n_k-k+1$. Since $A\not\sim\emptyset$, applying Theorem 1.1 with $B=\emptyset$ we find that $|S(n_1,\ldots,n_k)|$ cannot be smaller than p. ■

Corollary 1.2 partially confirms the following conjecture arising from the study of Fraenkel's conjecture on disjoint covers of $\mathbb{N}$ by Beatty sequences.

GRAHAM–O'BRYANT CONJECTURE ([GO]). *Let* $n_1,\ldots,n_k$ *be distinct positive integers less than and relatively prime to* $q\in\mathbb{Z}^+$. *If* $a_1,\ldots,a_k\in\mathbb{Z}$ *and*

$$\sum_{s=1}^k\frac{e^{2\pi i a_s/q}}{1-e^{2\pi i n_s/q}}=0,$$

then we must have $\sum_{s=1}^k n_s\ge q$.

The following example shows that we cannot replace the prime p in Corollary 1.2 or Theorem 1.1 by a composite number.

EXAMPLE 1.1. Let $q > 1$ be an integer and let p be a prime divisor of q. Then, for any $n = 1, \ldots, q-1$, we have

$$\sum_{s=0}^{p-1} \frac{e^{2\pi i (sq/p)/q}}{1 - e^{2\pi i n/q}} = \frac{\sum_{s=0}^{p-1} e^{2\pi i s/p}}{1 - e^{2\pi i n/q}} = 0$$

but $|S(n, \ldots, n)| = n < q$. Thus the conditions $0 \le a_s < n_s$ $(s = 1, \ldots, k)$ in Corollary 1.2 cannot be cancelled. If q is composite, then there are $q/p - 1 > 0$ integers in the interval $((p-1)q/p,\ q-1]$. So we cannot substitute a composite number for the prime p in Corollary 1.2.

COROLLARY 1.3. *Let $A = \{a_s(n_s)\}_{s=1}^k$ $(0 \le a_s < n_s)$ and $B = \{b_t(m_t)\}_{t=1}^l$ $(0 \le b_t < m_t)$ both have distinct moduli. Let p be a prime greater than $|S(n_1, \ldots, n_k, m_1, \ldots, m_l)|$, and let ζ_p be a primitive pth root of unity. Then A and B are identical if and only if* (1.4) *holds.*

Proof. By a result of Znám [Z75b], A and B are identical if they have the same covering function. Combining this with Theorem 1.1 we immediately get the desired result. ∎

Observe that $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ are covering equivalent if and only if

$$\sum_{\substack{s=1 \\ x \in a_s(n_s)}}^{k} 1 + \sum_{\substack{t=1 \\ x \in b_t(m_t)}}^{l} (-1) = 0 \quad \text{for every } x \in \mathbb{Z}.$$

Thus Theorem 1.1 has the following equivalent form which will be proved in the next section.

THEOREM 1.2. *Let $\mathcal{A} = \{\langle \lambda_s, a_s, n_s \rangle\}_{s=1}^k$ where $\lambda_s, a_s, n_s \in \mathbb{Z}$ and $0 \le a_s < n_s$. Let $p > |S(n_1, \ldots, n_k)|$ be a prime, and let ζ_p be any primitive pth root of unity. Then $\mathcal{A} \sim \emptyset$ (i.e., $w_{\mathcal{A}}(x) = \sum_{1 \le s \le k,\, x \in a_s(n_s)} \lambda_s = 0$ for all $x \in \mathbb{Z}$) if and only if*

$$\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0. \tag{1.7}$$

2. Proof of Theorem 1.2. Let $S = S(n_1, \ldots, n_k)$. As $p > |S| \ge \max\{n_1, \ldots, n_k\}$, there is a common multiple $N \in \mathbb{Z}^+$ of the moduli $n_1, \ldots, n_k$ such that $N \equiv 1 \pmod p$. Just as in [S05a], we have

$$\sum_{r=0}^{N-1} w_{\mathcal{A}}(r) z^r = \sum_{r=0}^{N-1} \sum_{\substack{1 \le s \le k \\ n_s \mid a_s - r}} \lambda_s z^r = \sum_{s=1}^{k} \lambda_s \sum_{\substack{0 \le r < N \\ r \in a_s(n_s)}} z^r$$

$$= \sum_{s=1}^{k} \lambda_s z^{a_s} \sum_{0 \le q < N/n_s} (z^{n_s})^q$$
$$= N \sum_{\substack{1\le s\le k \\ z^{n_s}=1}} \frac{\lambda_s}{n_s} z^{a_s} + (1 - z^N) \sum_{\substack{1\le s\le k \\ z^{n_s}\ne 1}} \lambda_s \frac{z^{a_s}}{1 - z^{n_s}}.$$

Thus
$$\sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = (1 - \zeta_p^N) \sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}}.$$

It follows that
$$\sum_{s=1}^{k} \lambda_s \frac{\zeta_p^{a_s}}{1 - \zeta_p^{n_s}} = 0 \iff \sum_{l=0}^{p-1} c_l \zeta_p^l = 0, \tag{2.1}$$

where
$$c_l = \sum_{\substack{x=0 \\ x\in l(p)}}^{N-1} w_{\mathcal{A}}(x) \in \mathbb{Z}.$$

If $w_{\mathcal{A}}(x) = 0$ for all $x \in \mathbb{Z}$, then (1.7) holds by the above.

Below we assume (1.7). Then $\sum_{l=0}^{p-1} c_l\zeta_p^l = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = 0$. In the case $N = 1$, it follows that $w_{\mathcal{A}}(x) = w_{\mathcal{A}}(0) = 0$ for all $x \in \mathbb{Z}$. Now suppose $N > 1$. Clearly $N > p$ as $N \equiv 1 \pmod p$. Since $1 + x + \cdots + x^{p-1} = (x^p - 1)/(x - 1)$ is the minimal polynomial of ζ_p over the field of rational numbers, we must have $c_0 = c_1 = \cdots = c_{p-1}$. (See also M. Newman [N71].) Observe that if $x \in \mathbb{Z}$ then
$$w_{\mathcal{A}}(x) = \sum_{s=1}^{k} \frac{\lambda_s}{n_s} \sum_{r=0}^{n_s-1} e^{2\pi i \frac{a_s - x}{n_s} r} = \sum_{\alpha\in S} e^{-2\pi i\alpha x} \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s}. \tag{2.2}$$

(This trick appeared in [S91] and [S04].) Since $|S| < p$, for each $l = 0, \ldots, |S|$ we have
$$c_l = \sum_{\substack{x=0 \\ x\in l(p)}}^{N-1} w_{\mathcal{A}}(x) = \sum_{\alpha\in S} \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} \sum_{\substack{x=0 \\ x\in l(p)}}^{N-1} e^{-2\pi i\alpha x}$$
$$= \sum_{\alpha\in S} e^{-2\pi i\alpha l} \sum_{\substack{s=1 \\ \alpha n_s\in\mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i\alpha a_s} \sum_{j=0}^{\lfloor (N-1-l)/p \rfloor} e^{-2\pi i\alpha p j},$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. If $l \in \{1, \ldots, |S|\}$ then
$$\left\lfloor \frac{N-1-l}{p} \right\rfloor = \frac{N-1}{p} + \left\lfloor \frac{-l}{p} \right\rfloor = \frac{N-1}{p} - 1;$$

if $\alpha \in S \setminus \{0\}$ then

$$C(\alpha) := \sum_{j=0}^{(N-1)/p-1} e^{-2\pi i\alpha pj} = \frac{1-(e^{-2\pi i\alpha p})^{(N-1)/p}}{1-e^{-2\pi i\alpha p}} = \frac{1-e^{2\pi i\alpha}}{1-e^{-2\pi i\alpha p}} \neq 0.$$

Let $c = c_0 = \cdots = c_{p-1}$. By the above,

$$\sum_{\alpha\in S} e^{-2\pi i\alpha j} f(\alpha) = c$$

for every $j = 0,\ldots,|S|-1$, where

$$f(0) = \frac{N-1}{p}\sum_{s=1}^{k}\frac{\lambda_s}{n_s}$$

and

$$f(\alpha) = e^{-2\pi i\alpha}C(\alpha)\sum_{\substack{s=1\\ \alpha n_s\in\mathbb{Z}}}^{k}\frac{\lambda_s}{n_s}\,e^{2\pi i\alpha a_s} \quad \text{for } \alpha\in S\setminus\{0\}.$$

Let $\alpha_0 = 0, \alpha_1,\ldots,\alpha_{|S|-1}$ be all the distinct elements of S. Now that

$$\sum_{t=0}^{|S|-1} e^{-2\pi i\alpha_t j} f(\alpha_t) = c \quad \text{for each } j = 0,\ldots,|S|-1,$$

by Cramer's rule $D_t = Df(\alpha_t)$ vanishes for every $t = 1,\ldots,|S|-1$, where $D = \det\left((e^{-2\pi i\alpha_t})^j\right)_{0\le j,t<|S|}$ is of Vandermonde's type and hence nonzero. Therefore

$$\sum_{\substack{s=1\\ \alpha n_s\in\mathbb{Z}}}^{k}\frac{\lambda_s}{n_s}\,e^{2\pi i\alpha a_s} = 0 \quad \text{for all } \alpha\in S\setminus\{0\}$$

and hence $w_{\mathcal{A}}(x) = \sum_{s=1}^{k}\lambda_s/n_s$ for all $x\in\mathbb{Z}$ by (2.2). It follows that

$$0 = \sum_{r=0}^{N-1} w_{\mathcal{A}}(r)\zeta_p^r = \sum_{s=1}^{k}\frac{\lambda_s}{n_s}\,(1+\zeta_p+\zeta_p^2+\cdots+\zeta_p^{N-1}) = \sum_{s=1}^{k}\frac{\lambda_s}{n_s}\cdot\frac{1-\zeta_p^N}{1-\zeta_p}.$$

So $\sum_{s=1}^{k}\lambda_s/n_s = 0$ and hence $\mathcal{A}\sim\emptyset$. We are done.

References

[FFKPY] M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu, *Sieving by large integers and covering systems of congruences*, J. Amer. Math. Soc. 20 (2007), 495–517.

[GO] R. L. Graham and K. O'Bryant, *A discrete Fourier kernel and Fraenkel's tiling conjecture*, Acta Arith. 118 (2005), 283–304.

[G04] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, New York, 2004.
[N71] M. Newman, *Roots of unity and covering sets*, Math. Ann. 191 (1971), 279–282.
[S89] Z. W. Sun, *Systems of congruences with multipliers*, Nanjing Univ. J. Math. Biquarterly 6 (1989), no. 1, 124–133.
[S91] —, *On a generalization of a conjecture of Erdős*, Nanjing Univ. J. Natur. Sci. 27 (1991), no. 1, 8–15.
[S95] —, *Covering the integers by arithmetic sequences*, Acta Arith. 72 (1995), 109–129.
[S96] —, *Covering the integers by arithmetic sequences II*, Trans. Amer. Math. Soc. 348 (1996), 4279–4320.
[S01] —, *Algebraic approaches to periodic arithmetical maps*, J. Algebra 240 (2001), 723–743.
[S02] —, *On covering equivalence*, in: Analytic Number Theory (Beijing/Kyoto, 1999), Dev. Math. 6, Kluwer, Dordrecht, 2002, 277–302.
[S03a] —, *On the function* $w(x) = |\{1 \le s \le k : x \equiv a_s \pmod{n_s}\}|$, Combinatorica 23 (2003), 681–691.
[S03b] —, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announc. Amer. Math. Soc. 9 (2003), 51–60.
[S04] —, *Arithmetic properties of periodic maps*, Math. Res. Lett. 11 (2004), 187–196.
[S05a] —, *On the range of a covering function*, J. Number Theory 111 (2005), 190–196.
[S05b] —, *A local-global theorem on periodic maps*, J. Algebra 293 (2005), 506–512.
[Z75a] Š. Znám, *A simple characterization of disjoint covering systems*, Discrete Math. 12 (1975), 89–91.
[Z75b] —, *On properties of systems of arithmetic sequences*, Acta Arith. 26 (1975), 279–283.

Department of Mathematics
Shanghai Jiaotong University
Shanghai 200240
People's Republic of China
E-mail: haopan79@yahoo.com.cn

Department of Mathematics
Nanjing University
Nanjing 210093
People's Republic of China
E-mail: zwsun@nju.edu.cn

Received on 9.4.2007 (5429)