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Double integrals on a weighted projective plane and Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$

by

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1. Introduction. The aim of this paper is to give a canonical extension of classical elliptic integrals to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$.

The arrangement of four points on the projective line $\mathbb{P}^1(\mathbb{C})$ is deeply related to elliptic modular functions for the principal congruence subgroup $\Gamma(2)$. The double covering of $\mathbb{P}^1(\mathbb{C})$ branched at four points gives an elliptic curve. The coordinate of the configuration space of four branch points on $\mathbb{P}^1(\mathbb{C})$ gives a modular function for $\Gamma(2)$ via the period mapping of the family of the corresponding elliptic curves.

One of the most successful extensions of the above classical situation to several variables is given by K. Matsumoto, T. Sasaki and M. Yoshida [8]. They showed an interesting relation between the arrangement of six lines on the projective plane $\mathbb{P}^2(\mathbb{C})$ and modular functions on a 4-dimensional bounded symmetric space of type I via the period mapping of the family of $K3$ surfaces coming from the arrangement of six lines.

We shall give another natural extension of classical elliptic integrals to the case of several variables. Hilbert modular functions for real quadratic fields are very popular among modular functions of several variables. However, to the best of the author's knowledge, to obtain simple and geometric extensions of classical elliptic integrals to Hilbert modular cases is a highly non-trivial problem. Although Hilbert modular functions with level 2 structure can be obtained from the moduli of hyperelliptic curves of genus 2, they are characterized by complicated modular equations (see Remark 2.9).

In this paper, we focus on Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Since the real quadratic field $\mathbb{Q}(\sqrt{5})$ gives the smallest discriminant, several researchers (for example, K. B. Gundlach [2], F. Hirzebruch [4], R. Müller [10]) studied this case in detail. We shall give a simple and geometric interpretation of Hilbert modular functions in this case. We consider the double

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integrals of the algebraic function F of (3.2) in two variables on chambers surrounded by the parabola P of (2.2) and the quintic curve Q of (2.3) with the $(2,5)$-cusp. These double integrals are equal to the period integrals of the Kummer surface $K(X,Y)$ of (2.1). The equation (2.1) gives a double covering of the weighted projective plane $\mathbb{P}(1:1:2)$ branched along P and Q, and the complex parameters (X,Y) determine the arrangement of the branch loci. The parameters (X,Y) are regarded as a pair of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ via explicit double integrals (see Remark 2.16 and Theorem 3.9). Our results are coherent with the theory of classical elliptic integrals (see Table 1). The results of this paper are used in [12].

Table 1. Classical elliptic integrals and the result of this paper

	Classical story	Result of this paper
Base space	$\mathbb{P}^1(\mathbb{C})$	$\mathbb{P}(1:1:2)$
Branch loci	4 points	P and Q
Variety	Elliptic curve	Kummer surface $K(X,Y)$
Arrangement	Elliptic modular for $\Gamma(2)$	Hilbert modular for $\mathbb{Q}(\sqrt{5})$

The author conjectures that we can similarly obtain simple and geometric interpretations of other Hilbert modular functions, using suitable weighted projective planes. Our results might give a first step in such an approach to Hilbert modular functions.

2. The Kummer surface $K(X,Y)$ and Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. We consider the period mapping for the family $\mathcal{K} = \{K(X,Y)\}$ of surfaces where

$$K(X,Y) : v^2 = (u^2 - 2y^5)(u - (5y^2 - 10Xy + Y)) \tag{2.1}$$

for $(X,Y) \neq (0,0)$. The equation (2.1) gives a double covering of the (y,u)-space branched along the parabola

$$u = 5y^2 - 10Xy + Y \tag{2.2}$$

and the quintic curve

$$u^2 = 2y^5 \tag{2.3}$$

with the $(2,5)$-cusp $(y,u) = (0,0)$. The parameters (X,Y) define the arrangement of the divisors P and Q. In this section, we study the properties of the family $\mathcal{K}$.

2.1. Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ and the $K3$ surface $S(X,Y)$. In this subsection, we survey the results of [11].

Let $\mathcal{O}$ be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{5})$. Set $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The Hilbert modular group $\mathrm{PSL}(2,\mathcal{O})$ acts on

$\mathbb{H} \times \mathbb{H}$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1, z_2) \mapsto \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right),$$

for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathcal{O})$, where $'$ is the conjugate in $\mathbb{Q}(\sqrt{5})$. We also consider the involution $\tau : (z_1, z_2) \mapsto (z_2, z_1)$.

DEFINITION 2.1. If a holomorphic function g on $\mathbb{H} \times \mathbb{H}$ satisfies the transformation law

$$g\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right) = (\gamma z_1 + \delta)^k (\gamma' z_2 + \delta')^k g(z_1, z_2)$$

for any $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathcal{O})$, we call g a *Hilbert modular form of weight k* for $\mathbb{Q}(\sqrt{5})$. If $g(z_2, z_1) = g(z_1, z_2)$, then g is called a *symmetric modular form.*

If a meromorphic function f on $\mathbb{H} \times \mathbb{H}$ satisfies

$$f\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right) = f(z_1, z_2)$$

for any $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathcal{O})$, we call f a *Hilbert modular function* for $\mathbb{Q}(\sqrt{5})$.

REMARK 2.2. Hirzebruch [4] showed that the symmetric Hilbert modular surface $\overline{(\mathbb{H} \times \mathbb{H})/\langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle}$ is isomorphic to the weighted projective plane $\mathbb{P}(1 : 3 : 5) = \{(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$. The point $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0)$ gives the cusp $(\sqrt{-1}\,\infty, \sqrt{-1}\,\infty)$ of the modular surface. Let

$$X = \frac{\mathfrak{B}}{\mathfrak{A}^3}, \qquad Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}. \tag{2.4}$$

The pair (X, Y) defines a system of affine coordinates of $\{\mathfrak{A} \neq 0\}$ of $\mathbb{P}(1 : 3 : 5)$.

REMARK 2.3. Müller [10] introduced certain Hilbert modular forms g_2 (s_6, s_{10}, s_{15}, resp.) of weight 2 $(6, 10, 15$, resp.). They generate the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$.

A $K3$ surface X is a simply connected compact complex surface with $K_X = 0$. The homology group $H_2(X, \mathbb{Z})$ has a unimodular lattice structure. Let $\mathrm{NS}(X)$, the *Néron–Severi lattice* of X, be the sublattice in $H_2(X, \mathbb{Z})$ generated by the divisors on X. The orthogonal complement $\mathrm{Tr}(X)$ of $\mathrm{NS}(X)$ in $H_2(X, \mathbb{Z})$ is called the *transcendental lattice* of X.

We consider the family $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mid (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\}\}$ of $K3$ surfaces with an elliptic fibration given by the affine equation

$$S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : z_0^2 = x_0^3 - 4y_0^2(4y_0 - 5\mathfrak{A})x_0^2 + 20\mathfrak{B}y_0^3 x_0 + \mathfrak{C}y_0^4. \tag{2.5}$$

For a generic point $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5)$, the intersection matrix of the Néron–Severi lattice $\mathrm{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ is given by $E_8(-1) \oplus E_8(-1) \oplus$

$\left(\begin{smallmatrix} 2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$ (see [11]). Set $\mathcal{D} = \{\xi \in \mathbb{P}^3(\mathbb{C}) \mid \xi A^t\xi = 0,\ \xi A^t\overline{\xi} > 0\}$, where $A = U \oplus \left(\begin{smallmatrix} 2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$ gives the transcendental lattice of $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$. Here, U is a parabolic lattice of rank 2. Note that $\mathcal{D}$ is composed of two connected components $\mathcal{D}_+$ and $\mathcal{D}_-$. We let $(1 : 1 : -\sqrt{-1} : 0) \in \mathcal{D}_+$. In [11], we considered the multivalued period mapping $\mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\} \to \mathcal{D}_+$ for $\mathcal{F}$ given by

$$\Phi : (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto \Big(\int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega : \int_{\Gamma_4} \omega\Big), \tag{2.6}$$

where ω is a holomorphic 2-form up to a constant factor and $\Gamma_1, \ldots, \Gamma_4$ are 2-cycles on $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$.

Remark 2.4. Let $\{\check{\Gamma}_1, \ldots, \check{\Gamma}_4\}$ be a basis of the transcendental lattice A. We can take 2-cycles $\Gamma_1, \ldots, \Gamma_4$ such that $(\Gamma_j \cdot \check{\Gamma}_k) = \delta_{j,k}$ $(j, k = 1, \ldots, 4)$. These 2-cycles $\Gamma_1, \ldots, \Gamma_4$ give the period mapping (2.6).

Note that we have a biholomorphic mapping $j : \mathbb{H} \times \mathbb{H} \to \mathcal{D}_+$. The multivalued mapping $j^{-1} \circ \Phi$ on $\{\mathfrak{A} \neq 0\}$ is given by

$$(X, Y) \mapsto (z_1, z_2) = \left(-\frac{\int_{\Gamma_3} \omega + \frac{1-\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega}, -\frac{\int_{\Gamma_3} \omega + \frac{1+\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega}\right). \tag{2.7}$$

Theorem 2.5 ([11]). *The multivalued period mapping* (2.7) *gives a developing map of the Hilbert modular orbifold* $\overline{(\mathbb{H} \times \mathbb{H})/\langle \mathrm{PSL}(2, \mathcal{O}), \tau\rangle}$ *with the branch divisor*

$$Y(-1728X^5 + 64(5X^2 - Y)^2 + 720X^3Y - 80XY^2 + Y^3) = 0.$$

The inverse of (2.7) *gives a pair* $(X(z_1, z_2), Y(z_1, z_2))$ *of symmetric Hilbert modular functions for* $\mathbb{Q}(\sqrt{5})$.

Remark 2.6. The icosahedral group is deeply related to Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ (see [4] or [6]). Since the divisor

$$-1728X^5 + 64(5X^2 - Y)^2 + 720X^3Y - 80XY^2 + Y^3 = 0 \tag{2.8}$$

is derived from Klein's icosahedral invariants, this relation is called Klein's icosahedral relation.

Remark 2.7. The inverse $(X(z_1, z_2), Y(z_1, z_2))$ of (2.7) has an explicit expression in terms of Müller's modular forms g_2, s_6, s_{10} (see [11]).

2.2. The Kummer surface for the Humbert surface of invariant 5. In this subsection, we recall the properties of the Humbert surface of invariant 5.

Let $\mathfrak{S}_2$ be the Siegel upper half-plane of degree 2. The symplectic group $\mathrm{Sp}(4, \mathbb{Z})$ acts on $\mathfrak{S}_2$. The quotient space $\mathfrak{S}_2/\mathrm{Sp}(4, \mathbb{Z})$ gives the moduli space of principally polarized Abelian surfaces. Take $\Omega = \left(\begin{smallmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{smallmatrix}\right) \in \mathfrak{S}_2$. Let L_Ω be the lattice generated by the columns of the matrix (Ω, I_2). The complex torus $Z_\Omega = \mathbb{C}/L_\Omega$ of dimension 2 gives a principally polarized Abelian surface. We note that Z_Ω corresponds to the Jacobian variety of a hyperelliptic curve of genus 2.

Let T be the involution of a 2-dimensional complex torus Z induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$ on the universal covering $\mathbb{C}^2$. The minimal resolution $\mathrm{Kum}(Z) = \overline{Z/\langle \mathrm{id}, T\rangle}$ is called the *Kummer surface*. $\mathrm{Kum}(Z)$ is a $K3$ surface. Note that Z is an Abelian surface if and only if $\mathrm{Kum}(Z)$ is an algebraic $K3$ surface.

REMARK 2.8. Let $\Omega \in \mathfrak{S}_2$ and Z_Ω be the corresponding principally polarized Abelian surface. The Kummer surface $\mathrm{Kum}(Z_\Omega)$ can be given by the double covering of $\mathbb{P}^2(\mathbb{C}) = \{(\zeta_0 : \zeta_1 : \zeta_2)\}$ whose branch divisor is given by the six lines $\zeta_2 = 0, \zeta_2 + 2\zeta_1 + \zeta_0 = 0$, $\zeta_0 = 0$ and $\zeta_2 + 2\lambda_j\zeta_1 + \lambda_j^2\zeta_0 = 0$, ($j \in \{1, 2, 3\}$) with three complex parameters λ_1, λ_2 and λ_3. In this paper, this Kummer surface is denoted by $K_H(\lambda_1, \lambda_2, \lambda_3)$.

An element $\Omega = \left(\begin{smallmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{smallmatrix}\right) \in \mathfrak{S}_2$ is said to have a *singular relation* with *invariant* Δ if there exist relatively prime integers a, b, c, d, e such that $a\sigma_1 + b\sigma_2 + c\sigma_3 + d(\sigma_2^2 - \sigma_1\sigma_3) + e = 0$ and $\Delta = b^2 - 4ac - 4de$. Set $\mathcal{N}_5 = \{\Omega \in \mathfrak{S}_2 \mid \sigma \text{ has a singular relation with invariant } \Delta\}$. Let p be the canonical projection $\mathfrak{S}_2 \to \mathfrak{S}_2/\mathrm{Sp}(4,\mathbb{Z})$. Then $\mathcal{H}_5 = p(\mathcal{N}_5)$, called the *Humbert surface of invariant* 5, is the moduli space of principally polarized Abelian surfaces A such that $\mathcal{O} \subset \mathrm{End}(A)$.

REMARK 2.9. Humbert [5] showed that Ω has s singular relation with $\Delta = 5$ if and only if

$$(2.9)\qquad \begin{aligned} &4(\lambda_1^2\lambda_3 - \lambda_2^2 + \lambda_3^2(1-\lambda_1) + \lambda_2^2\lambda_3)(\lambda_1^2\lambda_2\lambda_3 - \lambda_1\lambda_2^2\lambda_3) \\ &\quad = (\lambda_1^2(\lambda_2+1)\lambda_3 - \lambda_2^2(\lambda_1+\lambda_3) + (1-\lambda_1)\lambda_2\lambda_3^2 + \lambda_1(\lambda_2-\lambda_3))^2 \end{aligned}$$

(see also [3, Theorem 2.9]). This relation is called *Humbert's modular equation* for $\Delta = 5$. Let $\mathcal{Q} : \mathcal{M}_{2,2} \to \mathfrak{S}_2/\mathrm{Sp}(4,\mathbb{Z})$ be the natural projection, where $\mathcal{M}_{2,2}$ is the moduli space of genus two curves with level 2 structure. The equation (2.9) defines a component of the inverse image $\mathcal{Q}^{-1}(\mathcal{H}_5)$.

This modular equation is studied in detail by several researchers (for example, Hashimoto and Murabayashi [3]). However, since (2.9) is complicated, studying the moduli properties of the family $\{K_H(\lambda_1, \lambda_2, \lambda_3)\}$ corresponding to $\mathcal{H}_5$ does not seem to be easy.

2.3. The Shioda–Inose structure. Let X be an algebraic $K3$ surface. Let ω be the unique holomorphic 2-form on X up to a constant factor. If an involution $\iota : X \to X$ satisfies $\iota^*\omega = \omega$, we call ι a *symplectic involution*. Set $G = \langle \iota, \mathrm{id}\rangle \subset \mathrm{Aut}(X)$ and $\tilde{Y} = X/G$. If $Y \to \tilde{Y}$ is the minimal resolution, then Y is a $K3$ surface. We have the rational quotient mapping $\chi : X \dashrightarrow Y$.

Definition 2.10. We say that a $K3$ surface X admits a *Shioda–Inose structure* if there exists a symplectic involution $\iota \in \mathrm{Aut}(X)$ with rational quotient mapping $\chi : X \dashrightarrow Y$ such that Y is a Kummer surface and χ_* induces a Hodge isometry $\mathrm{Tr}(X)(2) \simeq \mathrm{Tr}(Y)$.

Theorem 2.11 (Morrison [9]). *The $K3$ surface X admits a Shioda–Inose structure if and only if there is an embedding $E_8(-1) \oplus E_8(-1) \hookrightarrow \mathrm{NS}(X)$. A symplectic involution ι exchanging the two copies of $E_8(-1)$ induces a Shioda–Inose structure.*

2.4. Kummer surface $K(X,Y)$. By Theorem 2.11, the $K3$ surface $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ for $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \neq (1:0:0)$ admits a Shioda–Inose structure. Therefore, there exists a Kummer surface $K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ and a symplectic involution ι of $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ such that the corresponding rational quotient mapping $\chi : S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \dashrightarrow K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ induces a Hodge isometry $\mathrm{Tr}(S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))(2) \simeq \mathrm{Tr}(K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))$.

We shall obtain an explicit defining equation of $K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ by realizing the above symplectic involution ι. To find such an involution, we need a special elliptic fibration on $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ given by the following lemma.

Lemma 2.12. *The defining equation of $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ in* (2.5) *is birationally equivalent to*

$$z_1^2 = x_1(x_1^2 + (20\mathfrak{A}y_1^2 - 20\mathfrak{B}y_1 + \mathfrak{C})x_1 + 16y_1^5). \tag{2.10}$$

Proof. Apply the birational transformation

$$x_0 = \frac{x_1}{16y_1}, \qquad y_0 = -\frac{x_1}{16y_1^2}, \qquad z_0 = \frac{x_1 z_1}{256 y_1^4}$$

to (2.5). ∎

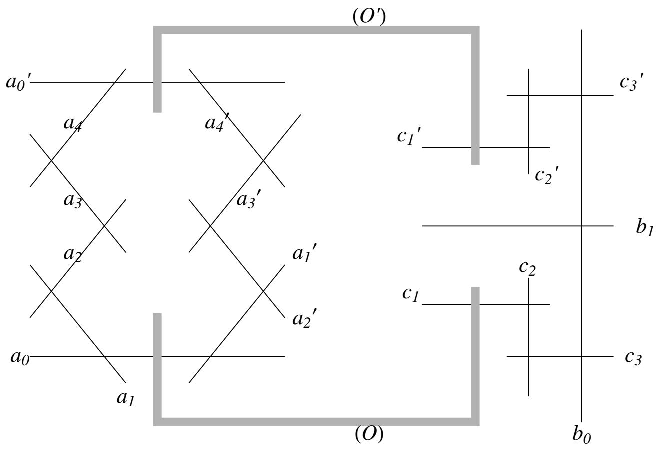


Fig. 1. The singular fibres given by (2.10)

The mapping $\pi_1 : S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \to \mathbb{P}^1(\mathbb{C})$ given by $(x_1, y_1, z_1) \mapsto y_1$ defines an elliptic fibration. The fibre $\pi_1^{-1}(0)$ ($\pi_1^{-1}(\infty)$, resp.) is a singular fibre of π_1 of type I_{10} (III^*, resp.). We set $\pi_1^{-1}(0) = a_0 + a_1 + \cdots + a_4 + a_0' + a_1' + \cdots + a_4'$ and $\pi_1^{-1}(\infty) = b_0 + b_1 + c_1 + c_2 + c_3 + c_1' + c_2' + c_3'$. Let O be the zero of the Mordell–Weil group. Let O' be the section of π_1 given by $(x_1, y_1, z_1) = (0, y_1, 0)$. Note that $2O' = O$ (see Figure 1).

We have an involution ι of $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ given by

$$(x_1, y_1, z_1) \mapsto \left(\frac{16y_1^5}{x_1}, y_1, \frac{-16y_1^5 z_1}{x_1^2} \right).$$

This is a symplectic involution. Note that ι is a van Geemen–Sarti involution for elliptic surfaces (see [1]). Let $G = \langle \mathrm{id}, \iota \rangle$. Set

$$u_1 = x_1 + \frac{16y_1^5}{x_1}, \qquad v_1 = \frac{x_1^2 - 16y_1^5}{z_1}. \tag{2.11}$$

They are G-invariants. We can see that $(x_1, y_1, z_1) \mapsto (u_1, y_1, v_1)$ defines a 2-to-1 mapping.

THEOREM 2.13. *The defining equation of the Kummer surface* $K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ *is given by*

$$v^2 = (u^2 - 2y^5)(u - (5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C})). \tag{2.12}$$

For generic $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5)$, *the intersection matrix of the transcendental lattice* $\mathrm{Tr}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ *is given by*

$$A(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

Proof. We can check directly that ι interchanges the two copies of $E_8(-1)$ in $\mathrm{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ (see Figure 2). Therefore, by Theorem 2.11, the involution ι gives a Shioda–Inose structure on $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$.

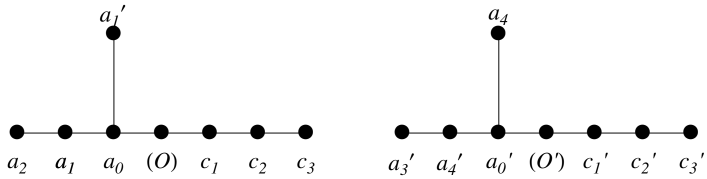


Fig. 2. $E_8(-1)$ lattices in $\mathrm{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$

From (2.10), (2.11) and the birational transformation

$$u_1 = -u, \qquad v_1 = \frac{\sqrt{-1}\, v}{u - (5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C})}, \qquad y_1 = \frac{y}{2},$$

we can check that the defining equation of $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})/G$ is (2.12).

The form of the intersection matrix of $\mathrm{Tr}(K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))$ follows from the fact that ι gives the Shioda–Inose structure. ∎

We thus have the family $\tilde{\mathcal{K}} = \{K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\}$ of Kummer surfaces. The projection $(y,u,v) \mapsto (y,u)$ defines the double covering $\mathcal{P}: K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \to \mathbb{P}(1:1:2) = \{(\zeta_0:\zeta_1:\zeta_2)\}$, where $y = \zeta_1/\zeta_0$ and $u = \zeta_2/\zeta_0^2$ on $\{\zeta_0 \neq 0\}$. Its branch divisor is given by $\tilde{P} \cup \tilde{Q}$, where

$$\begin{aligned} \tilde{P} \cap \{\zeta_0 \neq 0\} &= \{(y,u) \mid u = 5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C}\}, \\ \tilde{Q} \cap \{\zeta_0 \neq 0\} &= \{(y,u) \mid u^2 = 2y^5\}. \end{aligned} \tag{2.13}$$

REMARK 2.14. The equation (2.12) gives an expression of the Kummer surface $\mathrm{Kum}(Z_\Omega)$ for $\Omega \in \mathcal{H}_5$. It is different from the expression of $K_H(\lambda_1,\lambda_2,\lambda_3)$ in Remark 2.9. Our expression has some advantages. For example, our parameter space has a simple compactification by adding the point $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) = (1:0:0)$. This point is equal to the cusp of the Hilbert modular surface $\overline{(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$ (see Remark 2.2).

Let ω_K be the unique holomorphic 2-form on $K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ up to a constant factor. Set $\chi_*(\Gamma_j) = \Delta_j$ for $j \in \{1,2,3,4\}$. The period mapping for $\mathcal{K}$ is given by

$$\Phi_K : (\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \mapsto \Big(\int_{\Delta_1}\omega_K : \int_{\Delta_2}\omega_K : \int_{\Delta_3}\omega_K : \int_{\Delta_4}\omega_K\Big) \in \mathcal{D}. \tag{2.14}$$

Since $\chi^*(\omega_K) = \omega$ and $\chi_*(\Gamma_j) = \Delta_j$, we clearly have the following proposition.

PROPOSITION 2.15.

$$\Big(\int_{\Gamma_1}\omega : \cdots : \int_{\Gamma_4}\omega\Big) = \Big(\int_{\Delta_1}\omega_K : \cdots : \int_{\Delta_4}\omega_K\Big).$$

REMARK 2.16. According to Theorem 2.5 and the above proposition, the inverse of $j^{-1}\circ\Phi_K$ gives the pair (X,Y) of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ via the period mapping Φ_K.

Consider the projection $\pi: K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \to \mathbb{P}^1(\mathbb{C})$ given by $(u,y,v)\mapsto y$. The elliptic surface $K((\mathfrak{A}:\mathfrak{B}:\mathfrak{C}),\pi,\mathbb{P}^1(\mathbb{C}))$ has the singular fibre $\pi^{-1}(0)$ ($\pi^{-1}(\infty)$, resp.) of type I_5 (III^*, resp.) and five other singular fibres $\pi^{-1}(s_1),\ldots,\pi^{-1}(s_5)$ of type I_2.

PROPOSITION 2.17. *The vector space* $\mathrm{NS}(K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))\otimes_{\mathbb{Z}}\mathbb{Q}$ *is generated by the components of the singular fibres, the section O given by the zero of the Mordell–Weil group and a general fibre F of π.*

Proof. $\mathrm{NS}(K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))\otimes_{\mathbb{Z}}\mathbb{Q}$ is an 18-dimensional vector space over $\mathbb{Q}$. Set $\pi^{-1}(y) = \bigcup_{j=0}^{r(y)}\Theta_{y,j}$, where $\Theta_{y,j}$ is a connected component and $\Theta_{y,0}\cap O \neq \emptyset$. By calculating the intersection numbers, we can check that the 18 divisors $\Theta_{0,1},\dots,\Theta_{0,4}$, $\Theta_{s_1,1},\dots,\Theta_{s_5,1}$, $\Theta_{\infty,1},\dots,\Theta_{\infty,7}$, O and F generate a sublattice of $\mathrm{NS}(K(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}))$ of rank 18. Hence the claim follows. ∎

By (2.4) and (2.12), we have $K(X,Y)$ in (2.1).

3. Double integrals of an algebraic function on chambers surrounded by a parabola and a quintic curve. In this section, we obtain an extension of classical elliptic integrals. We shall study a single-valued branch $U_0 \to \mathcal{D}_+$ of the multivalued period mapping Φ_K explicitly where U_0 is the open set in $\mathbb{R}^2$ given by Figure 3. By the analytic continuation of this single-valued branch, we obtain the multivalued period mapping Φ_K of (2.14). The arrangement of P of (2.2) and Q of (2.3) determines the chambers R_1, R_2, R_3 and R_4 in Figure 9. Theorem 3.9 gives an extension of the classical elliptic integrals to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$.

3.1. The elliptic curve $E(y)$. For $y>0$, set $\alpha(y)=y^2\sqrt{2y}$, $\beta(y)=-y^2\sqrt{2y}$ and $p(y)=5y^2-10Xy+Y$ where $\sqrt{y}>0$. Note that $\alpha(y),\beta(y)$ and $p(y)$ are real valued analytic functions for $y\in\mathbb{R}_+$. Set

$$E(y): v^2=(u-\alpha(y))(u-\beta(y))(u-p(y)) \tag{3.1}$$

for $y\in\mathbb{R}_+$. Of course, $E(y)$ gives the fibre for $y\in\mathbb{R}_+$ of the elliptic surface $(K(X,Y),\pi,\mathbb{P}^1(\mathbb{C}))$. The discriminant of the right hand side of (3.1) for u has five roots in the y-plane.

Let U_0 be the domain in $\mathbb{R}^2=\{(X,Y)\}$ described in Figure 3. The curve in Figure 3 is Klein's icosahedral relation in (2.8). If $(X,Y)\in U_0$,

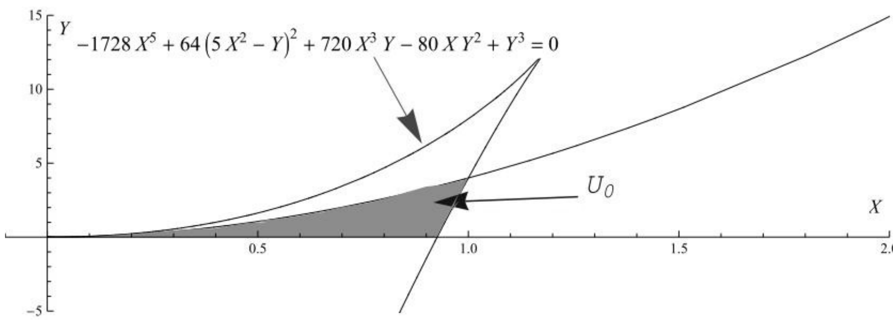


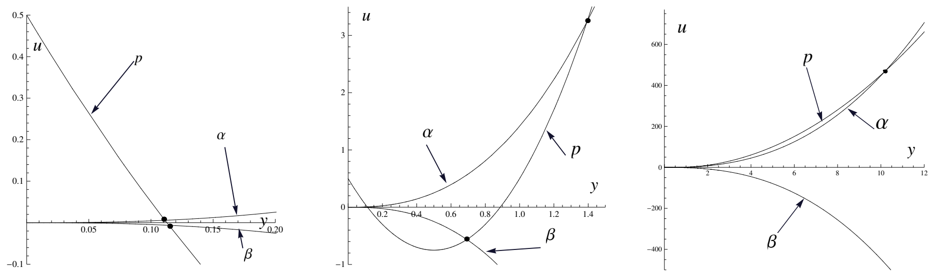
Fig. 3. The domain U_0 in (X,Y)-space $\mathbb{R}^2$.

the five roots of the discriminant of the right hand side of (3.1) for u are in $\mathbb{R}_+$ ($\subset$ y-space). So, we let $s_1=s_1(X,Y)$, $s_2=s_2(X,Y)$, $s_3=s_3(X,Y)$, $s_4=s_4(X,Y)$ and $s_5=s_5(X,Y)$ be these five roots such that $0<s_1<s_2<s_3<s_4<s_5$.

For $(X, Y) \in U_0$ and $s_{j-1} < y < s_j$ $(j = 0, \dots, 6)$, we denote the right hand side of $E(y)$ by $(u - w_1(y))(u - w_2(y))(u - w_3(y))$, where $w_1(y) < w_2(y) < w_3(y)$ (see Table 2 and Figure 4).

Table 2. The correspondence between $\{w_1, w_2, w_3\}$ and $\{\alpha, \beta, p\}$

	$0 < y < s_1$	$s_1 < y < s_2$	$s_2 < y < s_3$	$s_3 < y < s_4$	$s_4 < y < s_5$	$s_5 < y$
$w_1(y)$	$\beta(y)$	$\beta(y)$	$p(y)$	$\beta(y)$	$\beta(y)$	$\beta(y)$
$w_2(y)$	$\alpha(y)$	$p(y)$	$\beta(y)$	$p(y)$	$\alpha(y)$	$p(y)$
$w_3(y)$	$p(y)$	$\alpha(y)$	$\alpha(y)$	$\alpha(y)$	$p(y)$	$\alpha(y)$



$0 < y < 0.2$ $\qquad$ $0 < y < 1.5$ $\qquad$ $0 < y < 12$

Fig. 4. The graph of $u = \alpha(y), \beta(y), p(y)$

Since $\alpha(y)$, $\beta(y)$ and $p(y)$ are real for $y \in \mathbb{R}_+$, the function

$$F(y, u_+) = \sqrt{(u_+ - \alpha(y))(u_+ - \beta(y))(u_+ - p(y))}$$

is single-valued on $\{(y, u_+) \mid y \in \mathbb{R}_+, \operatorname{Im}(u_+) > 0\}$. Hence,

$$F(y, u) = \lim_{t \to 0} F(y, u + \sqrt{-1}\, t) \in \mathbb{R} \tag{3.2}$$

is single-valued for $s_{j-1} < y < s_j$ and $u \notin \{\alpha(y), \beta(y), p(y), \infty\}$, as is seen in Table 3.

Table 3. The values of $F(u, y)$

	$-\infty < u < w_1$	$w_1 < u < w_2$	$w_2 < u < w_3$	$w_3 < u < \infty$
$F(u, y)$	$-\sqrt{-1}\,\mathbb{R}_+$	$-\mathbb{R}_+$	$\sqrt{-1}\,\mathbb{R}_+$	$\mathbb{R}_+$

Take a base point $b \in (s_2, s_3)$ $(\subset \mathbb{R})$. We can take a basis $\{\gamma_1, \gamma_2\}$ of the homology group $H_1(\pi^{-1}(b), \mathbb{Z})$ such that $(\gamma_1 \cdot \gamma_2) = 1$ and

$$\int_{\gamma_1} \omega = 2 \int_{\beta(b)}^{p(b)} \frac{du}{\sqrt{F(b, u)}}, \qquad \int_{\gamma_2} \omega = 2 \int_{\alpha(b)}^{\beta(b)} \frac{du}{\sqrt{F(b, u)}}.$$

For $j \in \{0, 1, 2\}$ $(\in \{3, 4, 5\}$, resp.), we set $l_j = \{(s_j, -\sqrt{-1}\, t) \mid t \geq 0\}$ $(= \{(s_j, \sqrt{-1}\, t) \mid t \geq 0\}$, resp.). We call l_j the *cut line* for s_j. For y in

$\mathbb{C} - \{l_0, \ldots, l_5\}$, take an arc α_y which does not meet the cut lines l_j ($j \in \{0, \ldots, 5\}$) with the start (end, resp.) point b (y, resp.). Let $u \mapsto a_y(u)$ ($0 \le u \le 1$) be the parametric representation of α_y. Take a 1-cycle γ on $E(b)$. For $\gamma \in H_1(\pi^{-1}(b), \mathbb{Z})$, we choose a 1-cycle $\gamma_{\alpha_y}(u)$ on $\pi^{-1}(a_y(u))$ which depends continuously on u with $\gamma_{\alpha_y}(0) = \gamma$. If α'_y is homotopic to α_y in $\mathbb{C} - \{l_0 \cup \cdots \cup l_5\}$, we have $\gamma_{\alpha_y}(1) = \gamma_{\alpha'_y}(1)$. So, we have a well-defined correspondence $\mathbb{C} - \{l_0 \cup \cdots \cup l_5\} \ni y \mapsto \gamma_{\alpha_y}(1) \in H_1(\pi^{-1}(y), \mathbb{Z})$. Then, we set

$$\gamma = \gamma_{\alpha_y}(1) \in H_1(\pi^{-1}(y), \mathbb{Z}) \qquad (y \in \mathbb{C} - \{y_0, \ldots, y_5\}). \tag{3.3}$$

Next, let r_j ($j = 0, 1, \ldots, 5$) be a closed arc in $\mathbb{C} - \{0, s_1, \ldots, s_5\}$, starting at b, going around s_j with the positive orientation and ending at b. We assume that r_j does not meet the cut line l_k if $j \neq k$. Let $t \mapsto u_j(t)$ ($0 \le t \le 1$) be the parametric representation of r_j. For instance, we can take an arc r_1 as in Figure 5. We choose 1-cycles $\gamma_1(t)$ and $\gamma_2(t)$ on $\pi^{-1}(u_j(t))$ which depend continuously on t such that $\gamma_1(0) = \gamma_1$ and $\gamma_2(0) = \gamma_2$. So, we have

$$\begin{pmatrix} \gamma_1(1) \\ \gamma_2(1) \end{pmatrix} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

where $a_j, b_j, c_j, d_j \in \mathbb{Z}$ and $a_j d_j - b_j c_j = 1$. The correspondence $r_j \mapsto M_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ gives a representation of the fundamental group $\pi_1(\mathbb{C} - \{0, s_1, \ldots, s_5\})$. We call M_j the *monodromy matrix* for r_j.

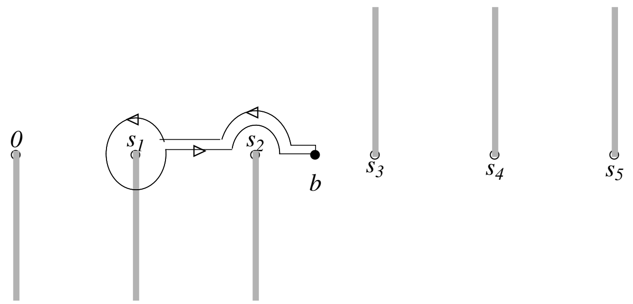


Fig. 5. The points $0, s_1, \ldots, s_5$, the cut lines and an arc r_1 going around s_1

Remark 3.1. If an arc r in the base space of an elliptic fibration goes around a singular fibre with the positive orientation, the monodromy matrix M_r is obtained by K. Kodaira [7, Theorem 9.1]. For example, if the singular fibre is of type I_b ($b > 0$) or III^*, the monodromy matrix M_r is given by $B^{-1} M_r^0 B$, where M_r^0 is given by Table 4 and $B \in \mathrm{GL}(2, \mathbb{Z})$.

Lemma 3.2. *The monodromy matrices* M_j *for* $\{\gamma_1, \gamma_2\}$ *are given by Table* 5.

Table 4. The matrices M_r^0 for the singular fibres of type I_b and III^*

Singular fibre	Matrix M_r^0
I_b	$\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$
III^*	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

Table 5. The monodromy matrices M_j $(j = 0, 1, \ldots, 5, \infty)$

	Type of singular fibre	Monodromy matrix for γ_1, γ_2
y_1	I_2	$M_1 = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$
y_2	I_2	$M_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$
y_3	I_2	$M_3 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$
y_4	I_2	$M_4 = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$
y_5	I_2	$M_5 = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$
0	I_5	$M_0 = \left(\begin{smallmatrix} 1 & -5 \\ 0 & 1 \end{smallmatrix}\right)$
∞	III^*	$M_\infty = \left(\begin{smallmatrix} 3 & 5 \\ -2 & -3 \end{smallmatrix}\right)$

Proof. Let us determine the matrix M_2 around s_2. The fibre $\pi^{-1}(s_2)$ is a singular fibre of type I_2. So, the monodromy matrix M_2 is of the form

$$M_2 = B^{-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}B,$$

where $B \in \mathrm{GL}(2, \mathbb{Z})$. Observe that $p(y) = w_1^{(3)}(y)$ converges to $\beta(y) = w_2^{(3)}(y)$ when $y \to y_2 + 0$. So, the matrix M_2 fixes the 1-cycle $\gamma_1 = \gamma_1^{(3)}$. Hence, $B = I_2$ and $M_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. By the same argument, we obtain Table 5. ∎

3.2. The transcendental lattice $\langle D_1, \ldots, D_4 \rangle$. From Table 5, we have the following relations:

$$(3.4)\qquad \begin{aligned} &M_1M_2M_4M_3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, && M_1M_2M_5M_3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\ &M_2^{-1}M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, && M_0M_1M_2M_0^{-1}M_3^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}. \end{aligned}$$

The transformation given by the matrix $M_1M_2M_4M_3$ fixes the 1-cycle γ_2. Let ρ_1 be a closed curve in the y-plane starting from the base point b and going around s_1, s_2, s_4 and s_3 successively. Let $t \mapsto s(t)$ be a parametric representation of ρ_1. For $0 \le t \le 1$, we define a 1-cycle $\gamma^{(1)}(t)$ on the elliptic curve $\pi^{-1}(s(t))$. The 1-cycle $\gamma^{(1)}(t)$ depends continuously on t and $\gamma^{(1)}(0) = \gamma^{(1)}(1) = \gamma_2$ on $\pi^{-1}(b) = \pi^{-1}(s(0)) = \pi^{-1}(s(1))$. Then the set

$$C_1 = \bigcup_{0 \le t \le 1} \gamma^{(1)}(t)$$

defines a 2-cycle on the surface $K(X,Y)$. Similarly, we have the 2-cycles C_2, C_3 in Figure 6 and C_4 in Figure 7.

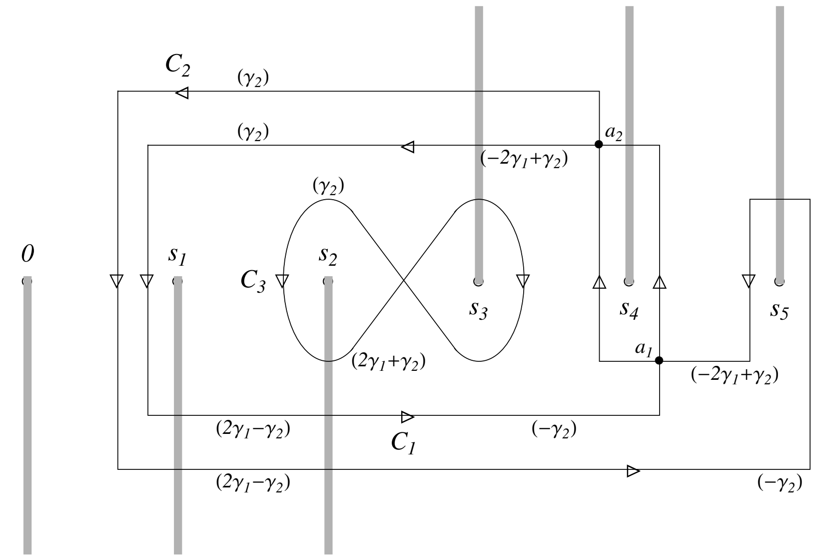


Fig. 6. 2-cycles C_1, C_2, C_3

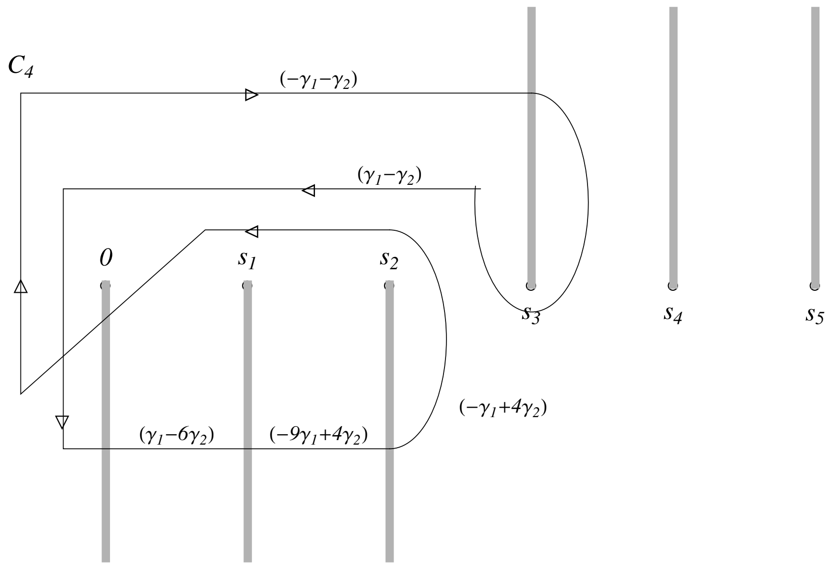


Fig. 7. 2-cycle C_4

Lemma 3.3. *The intersection matrix for* $\{C_1, C_2, C_3, C_4\}$ *is*

$$((C_j \cdot C_k))_{j,k=1,\dots,4} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & -6 & -4 \end{pmatrix}.$$

Proof. Let ρ_j be the base arc of C_j. For $y \in \rho_j$, let $\gamma^{(j)}(y) = C_j \cap \pi^{-1}(y)$. Suppose the base arcs ρ_j and ρ_k intersect in s points $y_1, \dots, y_s$ in the y-plane. Then the intersection number $(C_j \cdot C_k)$ is given by

$$(C_j \cdot C_k) = \sum_{l=1}^{s} (-1)(\rho_j \cdot \rho_k)_{y_l} (\gamma^{(j)}(y_l) \cdot \gamma^{(k)}(y_l)), \tag{3.5}$$

where $(\rho_j \cdot \rho_k)_{y_l}$ is the intersection number of the base arcs ρ_j and ρ_k at the point y_l, and $(\gamma^{(j)}(y_l) \cdot \gamma^{(k)}(y_l))$ is the intersection number of 1-cycles on the elliptic curve $\pi^{-1}(y_j)$. See Figure 6. The base arcs ρ_1 and ρ_2 intersect in two points a_1 and a_2. We have $(\rho_1 \cdot \rho_2)_{a_1} = +1$ and $(\rho_1 \cdot \rho_2)_{a_2} = -1$. Then, from (3.5), we have

$$\begin{aligned}(C_1 \cdot C_2) &= (-1)(+1)(-\gamma_2 \cdot -2\gamma_1 + \gamma_2) + (-1)(-1)(-2\gamma_1 + \gamma_2 \cdot -2\gamma_1 + \gamma_2) \\ &= (-1)(-2) + 0 = 2.\end{aligned}$$

By the same argument, the claim follows. ∎

The following corollary to the above lemma is obvious.

Corollary 3.4. *Set*

$$D_1 = C_1, \quad D_2 = C_2, \quad D_3 = C_4 - C_3, \quad D_4 = C_4. \tag{3.6}$$

Then the intersection matrix for $\{D_1, \dots, D_4\}$ *is*

$$((D_j \cdot D_k))_{j,k=1,\dots,4} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}. \tag{3.7}$$

Proposition 3.5. *The system* $\{D_1, D_2, D_3, D_4\}$ *gives a basis of the transcendental lattice of* $K(X,Y)$ *with intersection matrix* $A(2)$.

Proof. By the above construction, the 2-cycle D_j $(j = 1, \dots, 4)$ does not meet the singular fibres of $(K(X,Y), \pi, \mathbb{P}^1(\mathbb{C}))$. So, from Theorem 2.5 and Proposition 2.17, the system $\{D_1, \dots, D_4\}$ gives a basis of $\mathrm{Tr}(K(X,Y))$. ∎

3.3. The 2-cycles $L_1, \dots, L_6$**.** Next, we define 2-cycles $L_1, \dots, L_6$ on $K(X,Y)$. Let ϱ_j $(j = 1, \dots, 6)$ be an arc in the y-plane with a parametric representation $t \mapsto q_j(t)$ $(0 \le t \le 1)$ whose start point and end point are given by Table 6. We take them so that ϱ_j does not meet the cut lines l_k

Table 6. The arc ϱ_j and 1-cycles for 2-cycles L_j ($j = 1, \ldots, 6$)

	L_1	L_2	L_3	L_4	L_5	L_6
start point of ϱ_j	s_5	s_4	s_3	s_2	s_1	0
end point of ϱ_j	∞	s_5	∞	s_3	s_4	∞
1-cycle $\delta^{(j)}$	$\gamma_1 - \gamma_2$	$\gamma_1 - \gamma_2$	γ_1	γ_1	$\gamma_1 - \gamma_2$	γ_2

($k \in \{0, \ldots, 5\}$) if $0 < t < 1$. Hence, we can define a 1-cycle $\delta^{(j)}(q_j(t))$ on $\pi^{-1}(q_j(t))$ as in Table 6 in the manner of (3.3). Then we can see that $L_j = \bigcup_{0 \le t \le 1} \delta^{(j)}(q_j(t))$ gives a 2-cycle on $K(X, Y)$ (see Figure 8).

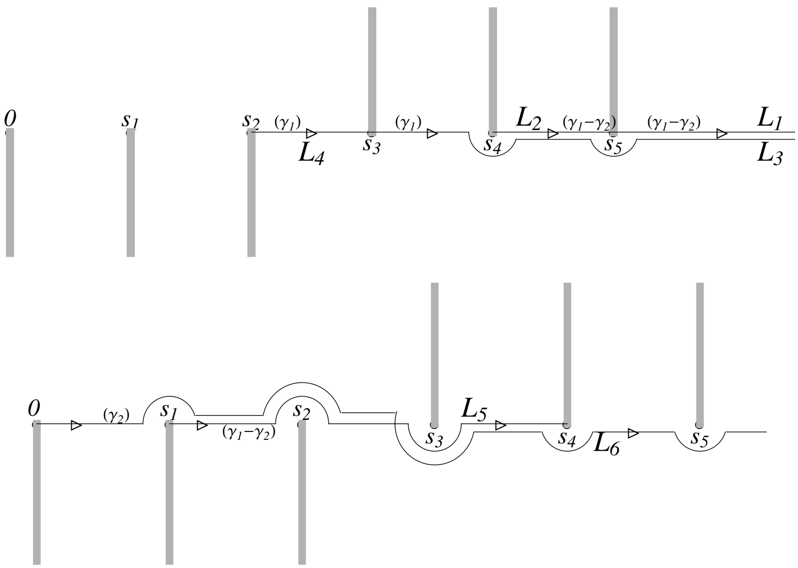


Fig. 8. 2-cycles L_1, L_2, L_3, L_4, L_5 and L_6

Just as we proved Lemma 3.3, we can prove the following lemma and corollary.

LEMMA 3.6.

$$((L_j \cdot C_k))_{1 \le j \le 6,\, 1 \le k \le 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{3.8}$$

Corollary 3.7.

$$((L_j \cdot D_k))_{1\le j\le 6,\,1\le k\le 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & -1 \\ 0 & 0 & -1 & -3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -2 & 0 \end{pmatrix}. \tag{3.9}$$

Proposition 3.8. *A branch of the period mapping Φ_K in* (2.14) *on U_0 has the following expression:*

$$\begin{aligned} &\int_{\Delta_1} \omega_K = \int_{L_1+L_2} \omega_K, \qquad \int_{\Delta_2} \omega_K = \int_{L_1} \omega_K = \int_{L_5-L_4} \omega_K, \\ &\int_{\Delta_3} \omega_K = \int_{-L_4-3(L_6+L_5-L_4-L_3+L_2+L_1)} \omega_K, \\ &\int_{\Delta_4} \omega_K = \int_{L_6+L_5-L_4-L_3+L_2+L_1} \omega_K. \end{aligned} \tag{3.10}$$

Proof. According to Proposition 3.5, $\{D_1,\dots,D_4\}$ gives a basis of $\mathrm{Tr}(K(X,Y))$. Recall the construction of the 2-cycles $\Gamma_1,\dots,\Gamma_4$ on $S(X,Y)$ in Remark 2.4. Together with Proposition 2.15, it is sufficient to take 2-cycles $\Delta_1,\dots,\Delta_4 \in \mathrm{H}_2(K(X,Y),\mathbb{Z})$ such that $(\Delta_j \cdot D_k) = \delta_{jk}$. By Corollary 3.7, we can check that the 2-cycles on the right hand side of (3.10) have these properties. ∎

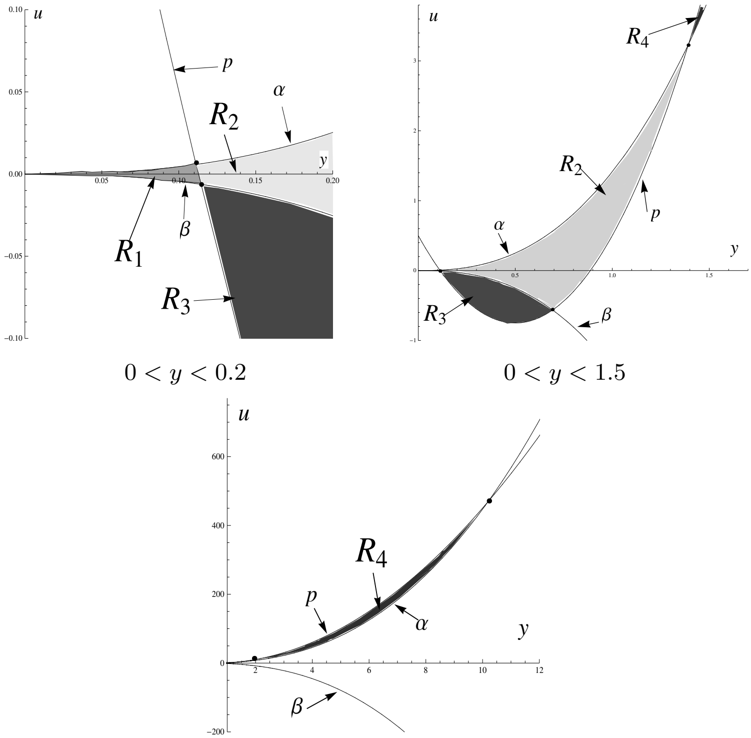
3.4. The chambers R_1, R_2, R_3 and R_4. We define the following chambers in $\mathbb{R}^2$ (see Figure 9):

$$\begin{aligned} R_1 &= \{(u,y) \mid 0 \le y \le s_2,\ w_1(y) \le u \le w_2(y)\}, \\ R_2 &= \{(u,y) \mid s_1 \le y \le s_4,\ w_2(y) \le u \le w_3(y)\}, \\ R_3 &= \{(u,y) \mid s_2 \le y \le s_3,\ w_1(y) \le u \le w_2(y)\}, \\ R_4 &= \{(u,y) \mid s_4 \le y \le s_5,\ w_2(y) \le u \le w_3(y)\}. \end{aligned} \tag{3.11}$$

They are surrounded by the branch divisors P and Q. From Table 2, we obtain Table 7.

Theorem 3.9. *A branch of the period mapping Φ_K in* (2.14) *on U_0 is given by the following double integrals on the chambers R_1, R_2, R_3 and R_4:*

$$\begin{aligned} &\int_{\Delta_1} \omega_K = 2\int_{R_2} \frac{du\,dy}{F(u,y)} + 2\int_{R_4} \frac{du\,dy}{F(u,y)}, \qquad \int_{\Delta_2} \omega_K = 2\int_{R_2} \frac{du\,dy}{F(u,y)}, \\ &\int_{\Delta_3} \omega_K = 6\int_{R_1} \frac{du\,dy}{F(u,y)} + 2\int_{R_3} \frac{du\,dy}{F(u,y)}, \qquad \int_{\Delta_4} \omega_K = -2\int_{R_1} \frac{du\,dy}{F(u,y)}. \end{aligned} \tag{3.12}$$



$0 < y < 0.2$ $\quad$ $0 < y < 1.5$

$0 < y < 12$

Fig. 9. The chambers R_1, R_2, R_3 and R_4

Table 7. Elliptic integrals on $E(y)$ for (s_{j-1}, s_j)

y	$\frac{1}{2}\left(\int_{\gamma_1(y)} \omega_y\right)$	$\frac{1}{2}\left(\int_{\gamma_2(y)} \omega_y\right)$
$0 < y < s_1$	$\int_{\alpha(y)}^{p(y)} \frac{du}{F(u,y)} + \int_{p(y)}^{\infty} \frac{du}{F(u,y)}$	$\int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u,y)}$
$s_1 < y < s_2$	$\int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$	$\int_{\alpha(y)}^{p(y)} \frac{du}{F(u,y)} + \int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$
$s_2 < y < s_3$	$\int_{\beta(y)}^{p(y)} \frac{du}{F(u,y)}$	$\int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u,y)}$
$s_3 < y < s_4$	$\int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$	$\int_{\alpha(y)}^{p(y)} \frac{du}{F(u,y)} + \int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$
$s_4 < y < s_5$	$\int_{\alpha(y)}^{p(y)} \frac{du}{F(u,y)} + \int_{p(y)}^{\infty} \frac{du}{F(u,y)}$	$\int_{p(y)}^{\infty} \frac{du}{F(u,y)}$
$s_5 < y$	$\int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$	$\int_{\alpha(y)}^{p(y)} \frac{du}{F(u,y)} + \int_{p(y)}^{\beta(y)} \frac{du}{F(u,y)}$

Proof. From Proposition 3.8 and Tables 6 and 7, we have

$$\int_{\Delta_2} \omega_K = \int_{L_5} \omega_K - \int_{L_4} \omega_K = 2\int_{s_1}^{s_4} \int_{\gamma_1(y)-\gamma_2(y)} \frac{dy\,du}{F(u,y)} - 2\int_{s_2}^{s_3} \int_{\gamma_1(y)} \omega_K$$

$$= 2\int_{s_1}^{s_4} \int_{p(y)}^{\alpha(y)} \frac{dy\,du}{F(u,y)} = \int_{R_2} \frac{dy\,du}{F(u,y)}.$$

Similarly,

$$\int_{\Delta_1} \omega_K = \int_{L_5} \omega_K - \int_{L_4} \omega_K + \int_{L_2} \omega_K = 2\int_{R_2} \frac{dy\,du}{F(u,y)} + 2\int_{s_4}^{s_5}\int_{\alpha(y)}^{p(y)} \frac{dy\,du}{F(u,y)}$$
$$= 2\int_{R_2} \frac{dy\,du}{F(u,y)} + 2\int_{R_4} \frac{dy\,du}{F(u,y)},$$
$$\int_{\Delta_4} \omega_K = \int_{L_6} \omega_K + \int_{L_5} \omega_K - \int_{L_4} \omega_K - \int_{L_3} \omega_K + \int_{L_2} \omega_K + \int_{L_1} \omega_K$$
$$= 2\int_0^{s_1}\int_{\beta(y)}^{\alpha(y)} \frac{dy\,du}{F(u,y)} + 2\int_{s_1}^{s_2}\int_{p(y)}^{\beta(y)} \frac{dy\,du}{F(u,y)} = -\int_{R_1} \frac{dy\,du}{F(u,y)},$$
$$\int_{\Delta_3} \omega_K = -\int_{L_4} \omega_K - 3\int_{\Delta_4} \omega_K = 2\int_{s_2}^{s_3}\int_{p(y)}^{\beta(y)} \frac{dy\,du}{F(u,y)} + 6\int_{R_1} \frac{dy\,du}{F(u,y)}$$
$$= 2\int_{R_3} \frac{dy\,du}{F(u,y)} + 6\int_{R_1} \frac{dy\,du}{F(u,y)}. \blacksquare$$

By the analytic continuation of the single-valued branch on U_0 given by the integrals in (3.12), we obtain the multivalued period mapping Φ_K for the family $\mathcal{K}$. Hence, the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ are closely connected with the arrangement of the divisors P of (2.2) and Q of (2.3). The above theorem gives a canonical extension of the classical elliptic integrals to the Hilbert modular case with the smallest discriminant.

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