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On the congruence $f(x) + g(y) + c \equiv 0 \pmod{xy}$ (completion of Mordell's proof)

by

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L. J. Mordell [4] stated the following theorem, and outlined its proof:

The congruence

$$ax^3 + by^3 + c \equiv 0 \pmod{xy},$$

where a, b, c *are given integers, has an infinite number of solutions in which* $(cx, y) = 1$, *and we can give* x, y *as polynomials in* a, b, c.

He also stated:

The same method proves the existence of an infinity of solutions of

$$ax^m + by^n + c \equiv 0 \pmod{xy},$$

where a, b, c *are given integers, and also of*

$$f(x) + g(y) + c \equiv 0 \pmod{xy}, \tag{1}$$

where

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x$$

and

$$g(y) = b_0y^n + b_1y^{n-1} + \cdots + b_{n-1}y,$$

and the a's and b's are integers.

(See also [5, pp. 293–295]).

Mordell was to a certain extent anticipated by Jacobsthal [2], who assumed $g = f$ and required only $f(x) + c \equiv 0 \pmod{y}$, $f(y) + c \equiv 0 \pmod{x}$.

We shall first assume $m \le 3$, $n = 1$ and prove

Theorem 1. *The congruence*

$$aX^3 + a_1X^2 + a_2X + bY + c \equiv 0 \pmod{XY}, \tag{2}$$

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where $a, a_1, a_2, b, c \in \mathbb{Z}$, has infinitely many solutions in integers if and only if the equation

$$aX^3 + a_1X^2 + a_2X + bY + c = 0 \tag{3}$$

is soluble in integers.

The conditions of Theorem 1 are satisfied for

$$\langle a, a_1, a_2\rangle \in \{\langle 2,0,0\rangle, \langle 0,2,0\rangle, \langle 0,0,2\rangle\}, \qquad b = 2, \qquad c = 1,$$

thus not only Mordell's last assertion above, but also his middle assertion is false for $m \le 3$, $n = 1$. For $m = n = 2$ the falsity of the middle assertion was shown by Jacobsthal [2, §2, Theorem 5] for $a = b = \pm 1$, $c = \mp 2, \mp 3$ (see also Barnes [1], Mills [3]). Moreover the middle assertion is false for $a = b = 0$, $c \ne 0$; $a = 0$, $b \ne 0$, $\sqrt[n]{-c/b} \notin \mathbb{Z}$; $a \ne 0$, $b = 0$, $\sqrt[m]{-c/a} \notin \mathbb{Z}$. Already I. Niven, the reviewer of [4] in *Math. Reviews*, pointed out [6] that *the author seems to assume in the proofs that certain coefficients are not zero without formal hypothesis in the statement of the theorem.* In the case $m = n = 3$, $a > 0$, $b > 0$, $c > 0$ Mordell's argument is valid only for $a > 1$.

Ramasamy and Mohanty [7] found all solutions in positive integers x, y, z of the equation $ax^3 + by + c - xyz = 0$, but even in this special case this does not prove Theorem 1.

We shall prove

THEOREM 2. *If $f(x) = ax^2 + a_1x \in \mathbb{Z}[x]$, $g(y) = by^2 + b_1y \in \mathbb{Z}[y]$, $c \in \mathbb{Z} \setminus \{0\}$, $\operatorname{Rad} c \mid (a_1, b_1a)$ and $|ab| \ge 9$, then the congruence* (1) *has infinitely many solutions in integers x, y such that $(y, c) = 1$. If $0 < |ab| < 9$ and the remaining assumptions of the theorem are satisfied, there are only finitely many exceptions.*

$\operatorname{Rad} c$ means here $\prod_{p\mid c,\, p\ \mathrm{prime}} p$.

Jacobsthal [2, §2, Theorem 4] has shown that if $a = b = 1$, $a_1 = b_1$, $c = \pm 1$, the only exceptions are $a_1 = b_1 = \pm 1$, $c = -1$.

COROLLARY 1. *The congruence*

$$ax^2 + by^2 + c \equiv 0 \pmod{xy},$$

where $a, b, c \in \mathbb{Z} \setminus \{0\}$, has infinitely many solutions in integers x, y such that $(y, c) = 1$ except for $a = b = \pm 1$, $c = \mp 2, \mp 3$.

THEOREM 3. *If $m \ge 4$, $n = 1$, $a_0 \in \mathbb{Z} \setminus \{0\}$, $a_1 = a_{m-1} = 0$ and $b_0, c \in \mathbb{Z} \setminus \{0\}$, then there exist infinitely many solutions of the congruence* (1) *in integers x, y such that $(y, c) = 1$.*

THEOREM 4. *Let $m, n \in \mathbb{Z}$ with $(m-1)(n-1) > 1$,*

$$f(x) = ax^m + \sum_{i=1}^{m-1} a_i x^{m-i} \in \mathbb{Z}[x], \quad g(y) = by^n + \sum_{i=1}^{n-1} b_i y^{n-i} \in \mathbb{Z}[y], \quad c \in \mathbb{Z},$$

$\operatorname{Rad} c \mid a_{m-1}$ *and* $\operatorname{Rad} c \mid b_{n-1}a$ *if* $m=2$, *and either* $|abc|>1$, *or* $a,b,c>0$, $a_i, b_j \geq 0$ $(1\leq i\leq m-1, 1\leq j\leq n-1)$. *Then the congruence* (1) *has infinitely many solutions in integers* x,y *such that* $(y,c)=1$.

Corollary 2. *The congruence*

$$ax^m+by^n+c\equiv 0 \pmod{xy},$$

where $a,b,c,m,n\in\mathbb{Z}\setminus\{0\}$, $(m-1)(n-1)>1$, *has infinitely many solutions in integers* x,y *such that* $(y,c)=1$.

The proofs of Theorems 2–4 use Mordell's method (Lemma 14); some repetitions are due to similarity of the theorems.

Lemma 1. *If* $r^2+s=w^2$, *where* $r,w\in\mathbb{Z}$ *and* $s\neq 0$, *then* $|r|\leq|s|$.

Proof. For $r\neq 0$ we have $|s|\geq r^2-(|r|-1)^2=2|r|-1$, thus

$$|r|\leq \tfrac{1}{2}(|s|+1)\leq |s|,$$

which is also true for $r=0$. ∎

Lemma 2. *If*

$$ax^3+a_1x^2+a_2x+c\equiv 0 \pmod{p},\quad c\equiv 0 \pmod{p},\quad x\not\equiv 0 \pmod{p} \tag{4}$$

and

$$\langle a,a_1,a_2\rangle \not\equiv \langle 0,0,0\rangle \pmod{p}, \tag{5}$$

where a,a_1,a_2,c,x *are integers, and* p *is a prime, then for every positive integer* α *the congruence*

$$aX^3+a_1X^2+a_2X+c\equiv 0 \pmod{p^\alpha} \tag{6}$$

is soluble.

Proof. By Hensel's lemma, if

$$F\in\mathbb{Z}[X],\quad F(x_0)\equiv 0 \pmod{p},\quad F'(x_0)\not\equiv 0 \pmod{p},$$

then for every positive integer α the congruence $F(X)\equiv 0 \pmod{p^\alpha}$ is soluble. Taking in this assertion $F(X)=aX^3+a_1X^2+a_2X+c$ and $x_0=0$, we infer that the congruence (6) is soluble provided $a_2\not\equiv 0 \pmod{p}$. If $a_2\equiv 0 \pmod{p}$, we infer from (4) that the congruence (6) is soluble provided $3ax+2a_1\equiv -a_1\not\equiv 0 \pmod{p}$. If $a_1\equiv a_2\equiv 0 \pmod{p}$, then, by (4), $ax\equiv 0 \pmod{p}$, contrary to (5). ∎

Proof of Theorem 1. Necessity. If the congruence (2) has infinitely many solutions, but the equation (3) is insolvable, then for some integers x,y,z,

$$ax^3+a_1x^2+a_2x+by+c=xyz\neq 0. \tag{7}$$

Now we distinguish four cases: 1. $b=0$; 2. $a=a_1=0$; 3. $a=0$, $a_1b\neq 0$; 4. $ab\neq 0$.

1. If $b=0$, then the existence of infinitely many solutions of the congruence (2) implies that either $ax_0^3+a_1x_0^2+a_2x_0+c=0$ for some $x_0\neq 0$, or $c=0$. Thus (3) has the solution $\langle x_0,0\rangle$ or $\langle 0,0\rangle$.

2. If $a=a_1=0$ then (7) yields

$$|a_2|\,|x|+|b|\,|y|+|c|\geq |a_2x+by+c|=|xyz|\geq |xy|,$$
$$|a_2b|+|c|\geq (|x|-|b|)(|y|-|a_2|),$$

thus either

$$|x|\leq |b|, \tag{8}$$

or

$$|y|\leq |a_2|, \tag{9}$$

or

$$|x|\leq |b|+|a_2b|+|c|,\quad |y|\leq |a_2|+|a_2b|+|c|.$$

(8) implies by (7) either $|y|\leq |a_2x+c|\leq |a_2b|+|c|$ or $a_2x+c=0$; (9) implies by (7) either $|x|\leq |by+c|\leq |a_2b|+|c|$ or $by+c=0$. Therefore, either the number of solutions of (2) is finite, or (3) is soluble.

3. If $a=0$ and $a_1b\neq 0$, then (7) gives

$$(yz^2-a_2z-2a_1b)^2-4a_1(cz^2+a_2bz+a_1b^2)=(2a_1xz+a_2z-yz^2)^2$$

(this identity was first given by J. Browkin), and by Lemma 1 either

$$|yz^2-a_2z-2a_1b|\leq 4|a_1(cz^2+a_2bz+a_1b^2)|, \tag{10}$$

or

$$cz^2+a_2bz+a_1b^2=0. \tag{11}$$

Now (10) gives

$$|yz^2|\leq |a_2z|+2|a_1b|+4|a_1|\,|cz^2+a_2bz+a_1b^2|,$$
$$|y|\leq |a_2|+2|a_1b|+4|a_1|(|c|+|a_2b|+|a_1b^2|)=B,$$

and by (7) either

$$|x|\leq |by+c|\leq |bB|+|c|,$$

or $by+c=0$, which gives an integer solution to (3).

If (11) holds, we put $b=b_1b_2$, where b_1 is the maximal unitary divisor of b dividing z. Then we take

$$x_0\equiv\begin{cases} x \pmod{b_1},\\ \dfrac{b/(b,z)}{z/(b,z)} \pmod{b_2}.\end{cases} \tag{12}$$

(Note that $z/(b,z)$ is prime to b_2.) By (7) and (12) we have

$$a_1x_0^2+a_2x_0+c\equiv a_1x^2+a_2x+c\equiv 0 \pmod{b_1},$$

while by (11) and (12),

$$a_1 x_0^2 + a_2 x_0 + c \equiv a_1 \frac{b^2}{z^2} + a_2 \frac{b}{z} + c \equiv 0 \pmod{b_2},$$

thus

$$a_1 x_0^2 + a_2 x_0 + c \equiv 0 \pmod{b}$$

and (3) is soluble in integers.

4. If $c = 0$, then (3) has the solution $\langle 0, 0\rangle$. If $c \neq 0$, let $\Omega(bc) = n$, where $\Omega(bc)$ is the total number of prime factors of bc. We assume the following (trivially true for $n = 0$):

(13) If $\Omega(bc) < n$, then either (2) has only finitely many solutions X, Y, or (3) is soluble in integers X, Y.

If $(x, b) = d > 1$, then $x = dx_1$, $b = db_1$, $c = dc_1$, $c_1 \in \mathbb{Z}$ and, by (7),

$$ad^2 x_1^3 + a_1 d x_1^2 + a_2 x_1 + b_1 y + c_1 = x_1 y z \neq 0. \tag{14}$$

However, $\Omega(b_1 c_1) = n - 2\Omega(d)$ and by the assumption (13) either the congruence

$$ad^2 X^3 + a_1 d X^2 + a_2 X + b_1 Y + c_1 \equiv 0 \pmod{XY}$$

has only finitely many solutions X, Y, or the equation

$$ad^2 X^3 + a_1 d X^2 + a_2 X + b_1 Y + c_1 = 0$$

has an integer solution $\langle x_0, y_0\rangle$. In the former case x_1, y in (14) are bounded and so are x, y; in the latter, (3) has the solution $\langle dx_0, dy_0\rangle$. It remains to consider the case

$$(x, b) = 1. \tag{15}$$

We set

$$b = b_0 b_3 b_4, \tag{16}$$

where b_0 is the maximal unitary divisor of b prime to c, and b_3 is the maximal unitary divisor of b dividing c. For any reduced residue $r \bmod b$, let $\bar{r}$ be the unique reduced residue mod b satisfying $r\bar{r} \equiv 1 \pmod{b}$ and $r\bar{r} = 1 + bs$ with $s \in \mathbb{Z}$. Then $x \equiv r \pmod{b}$ implies

$$b\left(\bar{r}\,\frac{x - r}{b} + s\right) \equiv -1 \pmod{x}. \tag{17}$$

Now (7) gives

$$ax^3 + a_1 x^2 + a_2 x + c = y(xz - b),$$

and in view of (17),

$$y \equiv c\left(\bar{r}\,\frac{x - r}{b} + s\right) \pmod{x},$$

thus

$$y = c\left(\bar{r}\,\frac{x-r}{b} + s\right) + xt, \quad t \in \mathbb{Z}.$$

Substituting in (7) we obtain

$$ax^3 + a_1x^2 + a_2x + c = (xz - b)\left(xt + c\bar{r}\,\frac{x-r}{b}\,x + cs\right),$$

hence on dividing by x and multiplying by b,

$$abx^2 + a_1bx + a_2b = bxzt + c\bar{r}xz - cz - b^2t - bc\bar{r},$$

which gives

$$abx^2 + x(a_1b - bzt - c\bar{r}z) + (a_2b + cz + b^2t + bc\bar{r}) = 0.$$

It follows that

$$(18)\quad (a_1b-bzt-c\bar{r}z)^2 - 4ab(a_2b+cz+b^2t+bc\bar{r}) = (2abx+a_1b-bzt-c\bar{r}z)^2,$$

so by Lemma 1 either

$$(19)\qquad a_2b + cz + b^2t + bc\bar{r} = 0,$$

or

$$(20)\qquad |a_1b - bzt - c\bar{r}z| \le 4|ab|\,|a_2b + cz + b^2t + bc\bar{r}|.$$

In the case (19), $b \mid cz$, hence, by (16),

$$(21)\qquad b_0\,\frac{b_4}{(b_4, c)} \mid z.$$

If for at least one prime $p \mid b_4$ we have

$$(22)\qquad \langle a, a_1, a_2\rangle \equiv \langle 0, 0, 0\rangle \pmod{p},$$

then, by (7),

$$\frac{a}{p}x^3 + \frac{a_1}{p}x^2 + \frac{a_2}{p}x + \frac{b}{p}y + \frac{c}{p} = xy\frac{z}{p},$$

and since $\Omega(bc/p^2) = n - 2$, by the assumption (13) either the congruence

$$\frac{a}{p}X^3 + \frac{a_1}{p}X^2 + \frac{a_2}{p}X + \frac{b}{p}Y + \frac{c}{p} \equiv 0 \pmod{XY}$$

has only finitely many solutions X, Y, or the equation

$$\frac{a}{p}\,X^3 + \frac{a_1}{p}\,X^2 + \frac{a_2}{p}\,X + \frac{b}{p}\,Y + \frac{c}{p} = 0$$

has an integer solution $\langle x_0, y_0\rangle$. In the former case x, y are bounded; in the latter, (3) has the solution $\langle x_0, y_0\rangle$. If (22) holds for no prime $p \mid b_4$, then by Lemma 2 the congruence

$$(23)\qquad aX^3 + a_1X^2 + a_2X + c \equiv 0 \pmod{p^{\operatorname{ord}_p b_4}}$$

has a solution x_p. Taking

$$x_0 \equiv \begin{cases} x \pmod{b_0}, \\ 0 \pmod{b_3}, \\ x_p \pmod{p^{\operatorname{ord}_p b_4}} \quad \text{for all primes } p \mid b_4, \end{cases}$$

we obtain, by (7), (16), (21) and (23),

$$ax_0^3 + a_1x_0^2 + a_2x_0 + c \equiv 0 \pmod{b}, \tag{24}$$

thus (3) is soluble in integers.

In the case (20) we obtain

$$|b|\,|z|\,|t| - |a_1b| - |c\bar{r}|\,|z| \le 4|aa_2|b^2 + 4|abc|\,|z| + 4|ab|b^2|t| + 4|ac\bar{r}|b^2,$$
$$(|z| - 4b^2|a|)(|b|\,|t| - 4|abc| - |c\bar{r}b|) \le |a_1b| + 4|aa_2|b^2| + 4|ac|b^2 + 4b^2|c|(4|abc| + |c\bar{r}|).$$

It follows that either

$$|z| \le 4b^2|a|, \tag{25}$$

or

$$|b|\,|t| \le 4|abc| + |c\bar{r}| \le 4|abc| + |bc|, \quad |t| \le 4|ac| + |c|, \tag{26}$$

or

$$|z| \le 4b^2|a| + |a_1b| + 4|aa_2|b^2 + 4|ac\bar{r}|b^2 + 4b^2|a|(4|abc| + |c|),$$
$$|t| \le 4|ac| + |c| + |a_1| + 4|aa_2b| + 4|ac|b^2 + 4|a|b^2(4|ac| + |c|).$$

In the last case, by (18), there are finitely many possibilities for x and either, by (7), there are finitely many possibilities for y, or $ax^3+a_1x^2+a_2x+c=0$, so (3) is soluble in integers. Thus it remains to consider the cases (25) and (26). In the case (25) we transform (18) to the form

$$(bz^2t + c\bar{r}z^2 - a_1bz - 2ab^2)^2 - 4ab(ab^3 + a_1b^2z + a_2bz^2 + cz^3) = (2abxz + a_1bz - bz^2t - c\bar{r}z^2)^2,$$

and thus, by Lemma 1, either

$$B := ab^3 + a_1b^2z + a_2bz^2 + cz^3 = 0 \tag{27}$$

or

$$|bz^2t + c\bar{r}z^2 - a_1bz - 2ab^2| \le 4|abB|. \tag{28}$$

In the case (27), defining

$$x_0 \equiv \begin{cases} x \pmod{b_1}, \\ \dfrac{b/(b,z)}{z/(b,z)} \pmod{b_2}, \end{cases}$$

we have (24), so (3) is soluble in integers. In the case (28), $|t|$ is bounded. Thus, again by (18) and (7), either there are finitely many possibilities for x and y, or (3) has an integer solution.

In the case (26) we transform (18) to the form

$$\big(z(bt+c\bar r)^2 - a_1 b(bt+c\bar r) - 2abc\big)^2 - 4aa_1b^2c(bt+c\bar r) - 4a^2b^2c^2 \\ - 4ab(bt+c\bar r)^2(a_2b+b^2t+bc\bar r) = (bt+c\bar r)^2(2abx+a_1b-bzt-c\bar r z)^2,$$

and, by Lemma 1, we have the following possibilities:

$$bt+c\bar r = 0,$$
$$4aa_1b^2c(bt+c\bar r)+4a^2b^2c^2+4ab(bt+c\bar r)^2(a_2b+b^2t+bc\bar r) =: 4ab^2C = 0,$$
$$|z(bt+c\bar r)^2 - a_1b(bt+c\bar r)-2abc| \le 4|aC|b^2 \text{ and } (bt+c\bar r)C \ne 0.$$

In the first case, $b\,|\,c$ and (3) is soluble in integers. In the third case, z is bounded and, by (18) and (7), either x and y are bounded, or (3) is soluble in integers. The second case gives

$$c^2(a+a_1\bar r+a_2\bar r^2+c\bar r^3) \equiv 0 \pmod{b(b,c)},$$

hence by (16) and the definition of $\bar r$,

$$ax^3+a_1x^2+a_2x+c \equiv ar^3+a_1r^2+a_2r+c \equiv 0 \ \left(\operatorname{mod} b_0\,\frac{b_4}{(b_4,c)}\right).$$

If for at least one prime $p\,|\,b_4$ we have (22), then either, by (7), $p\,|\,z$ and the argument used after (22) applies, or $p\,|\,y$ and

$$\frac{a}{p}\,x^3+\frac{a_1}{p}\,x^2+\frac{a_2}{p}\,x+b\,\frac{y}{p}+\frac{c}{p} = x\,\frac{y}{p}\,z.$$

Since $\Omega(bc/p)=n-1$, by the assumption (13) either the congruence

$$\frac{a}{p}X^3+\frac{a_1}{p}X^2+\frac{a_2}{p}X+bY+\frac{c}{p} \equiv 0 \pmod{XY}$$

has only finitely many solutions, or the equation

$$\frac{a}{p}X^3+\frac{a_1}{p}X^2+\frac{a_2}{p}X+bY+\frac{c}{p} = 0$$

has an integer solution $\langle x_0, y_0\rangle$. In the former case x and y are bounded; in the latter, (3) has the solution $\langle x_0, py_0\rangle$.

If (22) holds for no prime $p\,|\,b_4$, then, by Lemma 2, the congruence (23) has a solution x_p. Defining suitably x_0 we obtain (24), so (3) is soluble in integers.

Sufficiency. We shall prove more generally that the solvability of

$$f(x)+by+c=0 \tag{29}$$

implies the existence of infinitely many solutions of (1) with $g(y)=by$. We distinguish two cases: $b=0$ and $b\ne 0$. If $b=0$ and (29) has an integer solution x_0, then either $x_0=0$ or $x_0\neq 0$. If $x_0=0$, then $c=0$ and (1) has infinitely many solutions $(0,t)$ (t an arbitrary non-zero integer). If $x_0\neq 0$, then (1) has infinitely many solutions (x_0,t) (t an arbitrary non-zero integer). If $b\neq 0$ and (29) has a solution (x_0,y_0), then (1) has infinitely many solutions

$$x=x_0+bt\neq 0,\quad y=y_0+b^{-1}(f(x_0)-f(x_0+bt))\neq 0,$$

where t is a suitable integer. ■

NOTATION. Let $abc\neq 0$ and

$$\begin{aligned}
&d_k=\begin{cases} m & \text{for } k \text{ even},\\ n & \text{for } k \text{ odd},\end{cases}\\
&\lambda_1=0,\quad \lambda_2=1,\quad \lambda_k=d_k\lambda_{k-1}-\lambda_{k-2},\\
&\mu_1=-1,\quad \mu_2=0,\quad \mu_k=d_k\mu_{k-1}-\mu_{k-2},\\
&\nu_1=1,\quad \nu_2=m-1,\quad \nu_k=d_k\nu_{k-1}-\nu_{k-2},\\
&\Pi_0=\Pi_1=c,\quad \Pi_k=a^{\lambda_k}b^{\mu_k}c^{\nu_k}\quad (k=2,3,\ldots),\\
&f(x)=ax^m+a_1x^{m-1}+\cdots+a_{m-1}x,\\
&g(y)=by^n+b_1y^{n-1}+\cdots+b_{n-1}y,\\
&g_1(x)=g(x)+c,\quad f_2(x)=x^m+\frac{1}{c}f\left(\frac{c}{x}\right)x^m,\\
&g_{\sigma+1}=\frac{1}{\Pi_{2\sigma-1}}g_\sigma\left(\frac{\Pi_{2\sigma}}{x}\right)x^n,\quad f_{\sigma+1}=\frac{1}{\Pi_{2\sigma-2}}f_\sigma\left(\frac{\Pi_{2\sigma-1}}{x}\right)x^m.
\end{aligned}\tag{30}$$

COROLLARY 3. *$\Pi_2=a\Pi_1^m/\Pi_0$, $\Pi_3=b\Pi_2^n/\Pi_1$, $\Pi_k=\Pi_{k-1}^{d_k}/\Pi_{k-2}$ for $k\geq 4$.*

LEMMA 3. *Let*

$$\alpha=\frac{mn-2+\sqrt{mn(mn-4)}}{2},\quad \beta=\frac{mn-2-\sqrt{mn(mn-4)}}{2}.$$

If $mn\neq 4$, then

$$\lambda_{2k+1}=n\,\frac{\alpha^k-\beta^k}{\alpha-\beta},\quad \lambda_{2k}=\frac{\alpha^k(\beta+1)-\beta^k(\alpha+1)}{\alpha-\beta},\tag{31}$$

$$\mu_{2k+1}=\frac{\alpha^{k-1}(\alpha+1)-\beta^{k-1}(\beta+1)}{\alpha-\beta},\quad \mu_{2k}=m\,\frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta},\tag{32}$$

$$\nu_{2k+1}=\frac{\alpha^{k-1}(\nu_3\alpha-1)-\beta^{k-1}(\nu_3\beta-1)}{\alpha-\beta},\tag{33}$$

$$\nu_{2k}=\frac{\alpha^{k-1}(\nu_2\alpha-1)-\beta^{k-1}(\nu_2\beta-1)}{\alpha-\beta}.\tag{34}$$

If $mn = 4$, *then*

(35) $$\lambda_{2k+1} = nk, \quad \lambda_{2k} = 2k-1;$$

(36) $$\mu_{2k+1} = 2k-1, \quad \mu_{2k} = m(k-1);$$

(37) $$\nu_{2k+1} = (2-n)k+1, \quad \nu_{2k} = (m-2)k+1.$$

Proof. By induction. ■

LEMMA 4. *If* $(m-1)(n-1) > 1$ *and* $|ab| \ge 2$, *then*

(38) $$\lim_{\rho\to\infty}\Big(\sum_{i=1}^{\rho}\log|\Pi_{2i}| - (m-1)\sum_{i=1}^{\rho}\log|\Pi_{2i-1}| - 3\rho\log(mn)\Big) = \infty,$$

(39) $$\lim_{\rho\to\infty}\Big(\sum_{i=1}^{\rho}\log|\Pi_{2i-1}| - (n-1)\sum_{i=1}^{\rho}\log|\Pi_{2i-2}| - 3\rho\log(mn)\Big) = \infty.$$

If $m \ge 5$, $n = 1$ *and* $a_1 = a_{m-1} = 0$, *then*

(40) $$\lim_{\rho\to\infty}\Big(\sum_{i=1}^{\rho}\log|\Pi_{2i}| - (m-2)\sum_{i=1}^{\rho}\log|\Pi_{2i-1}| - 3\rho\log m\Big) = \infty.$$

Proof. By (30) we have

(41) $$\sum_{i=1}^{\rho}\log|\Pi_{2i}| = \sum_{i=1}^{\rho}\lambda_{2i}\log|a| + \sum_{i=1}^{\rho}\mu_{2i}\log|b| + \sum_{i=1}^{\rho}\nu_{2i}\log|c|,$$

(42) $$\sum_{i=1}^{\rho}\log|\Pi_{2i-1}| = \sum_{i=1}^{\rho}\lambda_{2i-1}\log|a| + \sum_{i=1}^{\rho}\mu_{2i-1}\log|b| + \sum_{i=1}^{\rho}\nu_{2i-1}\log|c|.$$

On the other hand, by Lemma 3, if $mn > 4$,

$$\sum_{i=1}^{\rho}\lambda_{2i} = \frac{(\alpha^{\rho+1}-\alpha)(\beta+1)}{(\alpha-1)(\alpha-\beta)} - \frac{(\beta^{\rho+1}-\beta)(\alpha+1)}{(\beta-1)(\alpha-\beta)},$$

$$\sum_{i=1}^{\rho}\lambda_{2i-1} = n\,\frac{\alpha^{\rho}-1}{(\alpha-1)(\alpha-\beta)} - n\,\frac{\beta^{\rho}-1}{(\beta-1)(\alpha-\beta)};$$

$$\sum_{i=1}^{\rho}\mu_{2i} = m\,\frac{\alpha^{\rho}-1}{(\alpha-1)(\alpha-\beta)} - m\,\frac{\beta^{\rho}-1}{(\beta-1)(\alpha-\beta)},$$

$$\sum_{i=1}^{\rho}\mu_{2i-1} = \frac{(\alpha^{\rho-1}-\beta)(\alpha+1)}{(\alpha-1)(\alpha-\beta)} - \frac{(\beta^{\rho-1}-\alpha)(\beta+1)}{(\beta-1)(\alpha-\beta)};$$

$$\sum_{i=1}^{\rho}\nu_{2i} = \frac{(\alpha^{\rho}-1)(\nu_2\alpha-1)}{(\alpha-1)(\alpha-\beta)} - \frac{(\beta^{\rho}-1)(\nu_2\beta-1)}{(\beta-1)(\alpha-\beta)},$$

$$\sum_{i=1}^{\rho}\nu_{2i-1} = \frac{(\alpha^{\rho-1}-\beta)(\nu_3\alpha-1)}{(\alpha-1)(\alpha-\beta)} - \frac{(\beta^{\rho-1}-\alpha)(\nu_3\beta-1)}{(\beta-1)(\alpha-\beta)}.$$

The first difference occurring in (38), by (41) and (42), is asymptotic to

$$\frac{\alpha^\rho}{(\alpha-1)(\alpha-\beta)}(\alpha(\beta+1)-(m-1)n)\log|a| + \frac{\alpha^{\rho-1}}{(\alpha-1)(\alpha-\beta)}(m\alpha-(m-1)(\alpha+1))\log|b| + \frac{\alpha^{\rho-1}}{(\alpha-1)(\alpha-\beta)}(\nu_2\alpha^2-\alpha-(m-1)(\nu_3\alpha-1))\log|c|.$$

Now, (38) follows from the inequalities

$$\begin{aligned}&\alpha(\beta+1)-(m-1)n=\alpha-\nu_3>0,\\ &m\alpha-(m-1)(\alpha+1)=\alpha-\nu_2>0,\\ &\nu_2\alpha^2-\alpha-(m-1)(\nu_3\alpha-1)\\ &\qquad=(m-1)((mn-1)\alpha-1)-\alpha-(m-1)\nu_3\alpha+m-1\\ &\qquad=\alpha((m-1)(n-1)-1)>0.\end{aligned}$$

The differences occurring in (39) in front of $\log|a|,\log|b|,\log|c|$ (after expanding $\log|\Pi_{2i-1}|$ and $\log|\Pi_{2i-2}|$) are, by (41) and (42), asymptotic to

$$\frac{\alpha^\rho}{(\alpha-1)(\alpha-\beta)}(1-(n-1)\beta),\quad \frac{\alpha^{\rho-1}}{(\alpha-1)(\alpha-\beta)}(\alpha+1-m(n-1)),\quad \frac{\alpha^\rho}{(\alpha-1)(\alpha-\beta)}((m-2)\alpha+n-2)$$

for $(m-1)(n-1)>1$ and (39) follows. The proof of (40) is similar. ∎

Lemma 5. *If either* $(m-1)(n-1)>0$ *or* $m\ge 4$, $n=1$, $a_1=a_{m-1}=0$, *and if* $\sigma\ge 2$, *then* $f_\sigma, g_\sigma\in\mathbb{Z}[x]$ *are monic of degree* m,n, *respectively, and* $f_\sigma(0)=\Pi_{2\sigma-2}$ *and* $g_\sigma(0)=\Pi_{2\sigma-1}$. *Moreover, if* $(m-1)(n-1)>1$ *then*

$$L(f_\sigma-x^m-\Pi_{2\sigma-2})\le\frac{|\Pi_{2\sigma-3}\cdots\Pi_1|^{m-1}}{|\Pi_{2\sigma-4}\cdots\Pi_0|}L(f),$$
$$L(g_\sigma-x^n-\Pi_{2\sigma-1})\le\frac{|\Pi_{2\sigma-2}\cdots\Pi_2|^{n-1}}{|\Pi_{2\sigma-3}\cdots\Pi_1|}L(g),$$

where $L(h)$ *denotes the sum of the absolute values of the coefficients of the polynomial* h.

If $m\ge 5$, $n=1$, $a_1=a_{m-1}=0$, *then*

$$L(f_\sigma-x^m-\Pi_{2\sigma-2})\le\frac{|\Pi_{2\sigma-1}\cdots\Pi_1|^{m-2}}{|\Pi_{2\sigma-2}\cdots\Pi_0|}L(f).$$

If $m=4$, $n=1$, $a_1=a_3=0$, *then for* $\sigma\ge 3$,

$$f_\sigma(x)=x^4+a_2b\Pi_{2\sigma-3}x^2+\Pi_{2\sigma-2}. \tag{43}$$

Proof. By induction on σ. For $\sigma = 2$ the assertions are true, except (43), in view of the definition of f_2, since

$$g_2 = x^n + \frac{1}{c}\, g\left(\frac{ac^{m-1}}{x}\right)x^n.$$

The formula (43) is true for $\sigma = 3$, since for $m = 4$, $n = 1$,

$$f_3(x) = \frac{1}{ac^3}\, f_2\left(\frac{abc^2}{x}\right)x^4.$$

Assume that the assertions are true for $\sigma \ge 2$ and (43) for $\sigma \ge 3$. Since $f_\sigma(0) = \Pi_{2\sigma-2}$, $f_{\sigma+1}$ is monic of degree m. Since, by (31) and Lemma 2, $\Pi_{2\sigma-2} \mid \Pi_{2\sigma-1}$ for $(m-1)(n-1) > 1$ and $\Pi_{2\sigma-2} \mid \Pi_{2\sigma-1}^2$ for $m \ge 4$, $n = 1$, we have $f_{\sigma+1} - x^m \in \mathbb{Z}[x]$, thus $f_{\sigma+1} \in \mathbb{Z}[x]$. Also, since $f_{\sigma+1}$ is monic, we obtain $f_{\sigma+1}(0) = \Pi_{2\sigma-1}^m/\Pi_{2\sigma-2} = \Pi_{2\sigma}$ by Corollary 3. If $(m-1)(n-1) > 0$, then

$$\begin{aligned} L(f_{\sigma+1} - x^m - \Pi_{2\sigma}) &\le \frac{|\Pi_{2\sigma-1}|^{m-1}}{|\Pi_{2\sigma-2}|}\, L(f_\sigma - x^m - \Pi_{2\sigma-1}) \\ &\le \frac{|\Pi_{2\sigma-1}\cdots\Pi_1|^{m-1}}{|\Pi_{2\sigma-2}\cdots\Pi_0|}\, L(f). \end{aligned}$$

If $m \ge 5$, $n = 1$, then

$$\begin{aligned} L(f_{\sigma+1} - x^m - \Pi_{2\sigma}) &\le \frac{|\Pi_{2\sigma-1}|^{m-2}}{|\Pi_{2\sigma-2}|}\, L(f_\sigma - x^m - \Pi_{2\sigma-2}) \\ &\le \frac{|\Pi_{2\sigma-1}\cdots\Pi_1|^{m-2}}{|\Pi_{2\sigma-2}\cdots\Pi_0|}\, L(f). \end{aligned}$$

Finally, by Corollary 3, for $m = 4$, $n = 1$,

$$\begin{aligned} f_{\sigma+1}(x) &= \frac{1}{\Pi_{2\sigma-2}}\, f_\sigma\left(\frac{\Pi_{2\sigma-1}}{x}\right)x^4 \\ &= \frac{1}{\Pi_{2\sigma-2}}(\Pi_{2\sigma-2}x^4 + a_2 b\Pi_{2\sigma-3}\Pi_{2\sigma-1}^2 x^2 + \Pi_{2\sigma-1}^4) = x^4 + a_2 b\Pi_{2\sigma-1}x^2 + \Pi_{2\sigma}. \end{aligned}$$

Similarly, since $g_\sigma(0) = \Pi_{2\sigma-1}$, it follows that $g_{\sigma+1}$ is monic of degree n. Since $\Pi_{2\sigma-1} \mid \Pi_{2\sigma}$, we have $g_{\sigma+1} - x^n \in \mathbb{Z}[x]$, so $g_{\sigma+1} \in \mathbb{Z}[x]$. Also, since g_σ is monic, $g_{\sigma+1}(0) = \Pi_{2\sigma}^n/\Pi_{2\sigma-1} = \Pi_{2\sigma+1}$ by Corollary 3. Finally, if $(m-1)(n-1) > 0$, then

$$\begin{aligned} L(g_{\sigma+1} - x^n - \Pi_{2\sigma-1}) &\le \frac{|\Pi_{2\sigma}|^{n-1}}{|\Pi_{2\sigma-1}|}\, L(g_\sigma - x^n - \Pi_{2\sigma-1}) \\ &\le \frac{|\Pi_{2\sigma}\cdots\Pi_2|^{n-1}}{|\Pi_{2\sigma-1}\cdots\Pi_1|}\, L(g). \ \blacksquare \end{aligned}$$

LEMMA 6. *If* $m=n=2$, $\sigma\ge 2$, *then*

$$|f_\sigma(x)-x^2-\Pi_{2\sigma-2}|\le |ab|^{\sigma-2}L(f)\max(1,|x|),$$
$$|g_\sigma(x)-x^2-\Pi_{2\sigma-1}|\le |ab|^{\sigma-2}|a|L(g)\max(1,|x|),$$

Proof. For $m=n=2$ by Lemma 5 we have, for $\sigma\ge 2$,

$$L(f_\sigma-x^2-\Pi_{2\sigma-2})\le \frac{|\Pi_{2\sigma-3}\cdots\Pi_1|}{|\Pi_{2\sigma-4}\cdots\Pi_0|}L(f),$$
$$L(g_\sigma-x^2-\Pi_{2\sigma-1})\le \frac{|\Pi_{2\sigma-2}\cdots\Pi_2|}{|\Pi_{2\sigma-3}\cdots\Pi_1|}L(g).$$

However, by (30) and Lemma 3,

$$\Pi_0=\Pi_1=c,\quad \Pi_2=ac,\quad \Pi_k=a^{k-1}b^{k-2}c,$$

hence for $\sigma\ge 2$,

$$|f_\sigma(x)-x^2-\Pi_{2\sigma-2}|\le |ab|^{\sigma-2}L(f)\max\{1,|x|\},$$
$$|g_\sigma(x)-x^2-\Pi_{2\sigma-1}|\le |ab|^{\sigma-2}|a|L(g)\max\{1,|x|\}. \ \blacksquare$$

LEMMA 7. *For* $(m-1)(n-1)>1$, $|abc|>1$ *and for* ρ *sufficiently large in terms of* m,n, *if* $2\le\sigma\le\rho$ *then*

$$|f_\sigma(x)-x^m-\Pi_{2\sigma-2}|\le \max\{1,|x|\}^{m-1}|\Pi_{2\rho-3}|^{m-1}, \tag{44}$$
$$|g_\sigma(x)-x^n-\Pi_{2\sigma-1}|\le \max\{1,|x|\}^{n-1}|\Pi_{2\rho-2}|^{m-1}. \tag{45}$$

For $m\ge 5$, $n=1$, $a_1=a_{m-1}=0$, $|abc|>1$, *and* ρ *sufficiently large in terms of* m, *if* $2\le\sigma\le\rho$ *then*

$$|f_\sigma(x)-x^m-\Pi_{2\sigma-2}|\le \max\{1,|x|\}^{m-2}|\Pi_{2\rho-3}|^{m-2}. \tag{46}$$

Proof. By Lemma 5 we have

$$L(f_\sigma-x^m-\Pi_{2\sigma-2})\le \max\{1,|x|\}^{m-1}L(f_\sigma-x^m-\Pi_{2\sigma-2})$$
$$\le \max\{1,|x|\}^{m-1}\frac{|\Pi_{2\sigma-3}\cdots\Pi_1|^{m-1}}{|\Pi_{2\sigma-4}\cdots\Pi_0|}L(f).$$

In order to show (44) it is enough to show that

$$\lim_{\rho\to\infty}\max_{2\le\sigma\le\rho}\frac{|\Pi_{2\sigma-3}\cdots\Pi_1|^{m-1}}{|\Pi_{2\sigma-4}\cdots\Pi_0|\,|\Pi_{2\sigma-3}|^{m-1}}=0,$$

but for $\sigma\le\rho$ we have $|\Pi_{2\rho-3}|\ge|\Pi_{2\sigma-3}|$ and by Lemma 4 for every $\varepsilon>0$ and $\sigma>\sigma_0(\varepsilon)$,

$$\frac{|\Pi_{2\sigma-3}\cdots\Pi_1|^{m-1}}{|\Pi_{2\sigma-4}\cdots\Pi_0\Pi_{2\rho-3}^{m-1}|}<\varepsilon.$$

For $\sigma\le\sigma_0(\varepsilon)$ and $\rho>\rho_0(\varepsilon)$ the same inequality holds. This proves (44). The proofs of (45) and of (46) are similar. $\blacksquare$

LEMMA 8. *For every real t we have*

$$e^t \geq 1+t, \tag{47}$$

and for every $t \in [0,1]$,

$$e^{-t} \leq 1-t/2. \tag{48}$$

Proof. The inequality (47) is well known, while (48) is equivalent to

$$\frac{t}{2}-\frac{t^2}{2}+\sum_{i=3}^{\infty}(-1)^{1-i}\frac{t^i}{i!} \geq 0,$$

which clearly holds for $t \in [0,1]$. ∎

LEMMA 9. *The numbers*

$$c_1=\frac{\log 3}{\log 2},\quad c_2=\frac{\log 7}{\log 2},\quad c_3=0,\quad c_4=\frac{\log(23/12)}{\log 2}$$

for every $d \geq 2$ satisfy the inequalities

$$\begin{aligned} &d^{c_1} > d^{c_2-3}+d^{c_3+1}, \quad d^{c_2} > d^{c_1-3}+d^{c_4+1},\\ &d^{c_3+3} > d^{c_4}+d^{c_1+1}, \quad d^{c_4+3} > d^{c_2+1}+d^{c_3}. \end{aligned} \tag{49}$$

Proof. For $d=2$ the inequalities in question take the form

$$3 > \frac{7}{8}+2,\quad 7 > \frac{3}{8}+\frac{23}{6},\quad 8 > \frac{23}{12}+6,\quad \frac{46}{3} > 14+1,$$

and since $c_1 > \max\{c_2-3, c_3+1\}$, $c_2 > \max\{c_1, c_4+1\}$, $c_3+3 > \max\{c_4, c_1+1\}$, $c_4+3 > \max\{c_2+1, c_3\}$, the inequalities (49) hold for all $d \geq 2$. ∎

LEMMA 10. *If $m=n=2$, $|ab| \geq 9$ and ρ is large enough, and for $2 \leq \sigma \leq \rho+1$, x_σ and $y_{\sigma-1}$ are given using backward induction by the formulae*

$$y_{\rho+1}=1, \tag{50}$$

$$x_{\rho+1}=1, \tag{51}$$

$$y_{\sigma-1}=\frac{f_\sigma(x_\sigma)}{y_\sigma} \quad (\sigma \leq \rho+1), \tag{52}$$

$$x_\sigma=\frac{g_\sigma(y_\sigma)}{x_{\sigma+1}} \quad (\sigma \leq \rho), \tag{53}$$

then for every non-negative integer $\tau < \rho$,

(54)

$$\exp(-2^{3(\tau-\rho)+c_2-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}} \leq |x_{\rho-\tau+1}| \leq \exp(2^{3(\tau-\rho)+c_1-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}},$$

$$\exp(-2^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{\lambda_{2\tau+2}} \leq |y_{\rho-\tau}| \leq \exp(2^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}. \tag{55}$$

Proof by induction on τ. For $\tau=0$ the inequality (54) follows from (50). For (55), if $\tau=0$ in view of Lemma 6 we have, for $\rho \geq 2$,

$$|y_\rho-1-\Pi_{2\rho}| < |ab|^{\rho-2}L(f);$$

then in view of Lemmas 3 and 8, (55) follows for ρ large enough from

$$\lim_{n\to\infty} |\Pi_{2\rho}|\,|ab|^{2-\rho}2^{-3\rho}=\infty.$$

Assume now that (54) and (55) are true for $\tau<\rho-1$. Then by Lemmas 6 and 9 and the inductive assumption, for ρ large enough,

$$\begin{aligned}|g_{\rho-\tau}(y_{\rho-\tau})| &\le |y_{\rho-\tau}|^2+\max\{1,|y_{\rho-\tau}|\}|ab|^{\rho-\tau-2}|a|L(g)+|\Pi_{2\rho-2\tau-1}|\\ &\le \exp(2\cdot 2^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{2\lambda_{2\tau+2}}\\ &\quad+\exp(2^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}|ab|^{\rho-\tau}|a|L(g)+|\Pi_{2\rho-1}|\\ &< \exp(2^{3(\tau-\rho)}(2^{c_1}-2^{c_2-3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+2}},\end{aligned}$$

hence, by (53) and the inductive assumption,

$$|x_{\rho-\tau}|\le \frac{\exp(2^{3(\tau-\rho)}(2^{c_1}-2^{c_2-3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+2}}}{\exp(-2^{3(\tau-\rho)+c_2-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}}}=\exp(2^{3(\tau-\rho)+c_1})|\Pi_{2\rho}|^{\lambda_{2\tau+3}}. \tag{56}$$

Since the function $t\mapsto t^2-At$ is increasing for $t\ge A/2$ ($A\ge 0$), and we have, for large ρ, by the inductive assumption,

$$|y_{\rho-\tau}|\ge \frac{1}{e}|\Pi_{2\rho}|^{\lambda_{2\tau+2}}\ge \frac{1}{2}|ab|^{\rho-\tau}|a|L(g),$$

it follows from Lemmas 6 and 9 that

$$\begin{aligned}|g_{\rho-\tau}(y_{\rho-\tau})| &\ge |y_{\rho-\tau}|^2-\max\{1,|y_{\rho-\tau}|\}|ab|^{\rho-\tau}|a|L(g)-|\Pi_{2\rho-2\tau-1}|\\ &\ge \exp(-2\cdot 2^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{2\lambda_{2\tau+2}}\\ &\quad-\exp(-2^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}|ab|^{\rho-\tau}|a|L(g)-|\Pi_{2\rho-2}|\\ &\ge \exp(-2^{3(\tau-\rho)}(2^{c_2}-2^{c_1-3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+2}},\end{aligned}$$

hence, by (53) and the inductive assumption,

$$\begin{aligned}|x_{\rho-\tau}| &\ge \frac{\exp(-2^{3(\tau-\rho)}(2^{c_2}-2^{c_1-3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+2}}}{\exp(2^{3(\tau-\rho)+c_1-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}}}\\ &= \exp(-2^{3(\tau-\rho)+c_2})|\Pi_{2\rho}|^{\lambda_{2\tau+3}}.\end{aligned} \tag{57}$$

Similarly, by Lemmas 6 and 9 and (56), for ρ large enough and $\tau<\rho-1$,

$$\begin{aligned}|f_\rho(x_{\rho-\tau})| &\le |x_{\rho-\tau}|^2+\max\{1,|x_{\rho-\tau}|\}|ab|^{\rho-\tau}L(f)+|\Pi_{2\rho-2\tau-1}|\\ &\le \exp(2\cdot 2^{3(\tau-\rho)+c_1})|\Pi_{2\rho}|^{2\lambda_{2\tau+3}}\\ &\quad+\exp(2^{3(\tau-\rho)+c_1})|\Pi_{2\rho}|^{\lambda_{2\tau+3}}|ab|^{\rho-\tau}L(f)+|\Pi_{2\rho-1}|\\ &\le \exp(2^{3(\tau-\rho)}(2^{c_3+3}-2^{c_4}))|\Pi_{2\rho}|^{2\lambda_{2\tau+3}},\end{aligned}$$

hence, by (52), the inductive assumption and Lemma 9,

$$|y_{\rho-\tau-1}| \le \frac{\exp(2^{3(\tau-\rho)}(2^{c_3+3}-2^{c_4}))|\Pi_{2\rho}|^{2\lambda_{2\tau+3}}}{\exp(-2^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}} = \exp(2^{3(\tau-\rho)+c_3+3})|\Pi_{2\rho}|^{\lambda_{2\tau+4}}.$$

Since the function $t \mapsto t^2 - Bt$ is increasing for $t \ge B/2$ ($B \ge 0$) and we have for large ρ, by (57),

$$|x_{\rho-\tau}| \ge \frac{1}{e}|\Pi_{2\rho}|^{\lambda_{2\tau+3}} \ge \frac{1}{2}|ab|^{\rho-\tau}L(f),$$

it follows from Lemmas 6 and 9 and (57) that, for large ρ,

$$\begin{aligned}|f_{\rho-\tau}(x_{\rho-\tau})| &\ge |x_{\rho-\tau}|^2 - \max\{1,|x_{\rho-\tau}|\}|ab|^{\rho-\tau}L(f) - |\Pi_{2\rho-2}| \\ &\ge \exp(2\cdot 2^{3(\tau-\rho)+c_2})|\Pi_{2\rho}|^{2\lambda_{2\tau+3}} \\ &\quad - \exp(2^{3(\tau-\rho)+c_2})|\Pi_{2\rho}|^{\lambda_{2\tau+3}}|ab|^{\rho-\tau}L(f) - |\Pi_{2\rho-2}| \\ &\ge \exp(-2^{3(\tau-\rho)}(2^{c_4+3}-2^{c_3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+3}},\end{aligned}$$

hence, by (52) and the inductive assumption,

$$|y_{\rho-\tau-1}| \ge \frac{\exp(-2^{3(\tau-\rho)}(2^{c_4+3}-2^{c_3}))|\Pi_{2\rho}|^{2\lambda_{2\tau+3}}}{\exp(2^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}} = \exp(-2^{3(\tau-\rho)+c_4+3})|\Pi_{2\rho}|^{\lambda_{2\tau+4}}. \ \blacksquare$$

LEMMA 11. *If $f(x) = ax^4 + a_2x^2$, $g = by$, a, a_2, b, c integers, $|abc| > 1$, ρ is large enough in terms of a, a_2, b, c, and x_σ, y_σ are given by* (50)–(53), *then $2 \le \sigma \le \rho$ implies*

$$|x_\sigma| > \max\{|\Pi_{2\rho-2}|, \sigma|x_{\sigma+1}|\}. \tag{58}$$

Proof by backward induction on σ. For $\sigma = \rho$ we have by (51), (53) and Lemma 5, for large ρ,

$$\begin{aligned}|x_\rho| &= |f_{\rho+1}(1) + \Pi_{2\rho-1}| = |1 + (a_2b+1)\Pi_{2\rho-1} + \Pi_{2\rho}| \\ &\ge |\Pi_{2\rho}| - |a_2b+1|\,|\Pi_{2\rho-1}| - 1 \\ &= |a|^{2\rho-1}|b|^{4\rho-4}|c|^{2\rho+1} - |a_2b+1|\,|a|^{\rho-1}|b|^{2\rho-3}|c|^{\rho} - 1 \\ &> |a|^{2\rho-3}|b|^{4\rho-8}|c|^{2\rho-1} = \max\{|\Pi_{2\rho-2}|, \rho|x_{\rho+1}|\}.\end{aligned}$$

Assume now that (58) holds for $3 \le \sigma+1 \le \rho$. Then, by (52)–(53), we have

$$x_\sigma = \frac{y_\sigma + \Pi_{2\sigma-1}}{x_{\sigma+1}}, \quad y_\sigma = \frac{x_{\sigma+1}^4 + a_2b\Pi_{2\sigma-1}x_{\sigma+1}^2 + \Pi_{2\sigma}}{y_{\sigma+1}},$$

$$x_{\sigma+1} = \frac{y_{\sigma+1} + \Pi_{2\sigma+1}}{x_{\sigma+2}},$$

hence

$$y_{\sigma+1}=x_{\sigma+1}x_{\sigma+2}-\Pi_{2\sigma+1},$$

$$|y_\sigma|=\frac{|x_{\sigma+1}^4+a_2b\Pi_{2\sigma-1}x_{\sigma+1}^2+\Pi_{2\sigma}|}{|x_{\sigma+1}x_{\sigma+2}-\Pi_{2\sigma+1}|}\ge\frac{x_{\sigma+1}^4-|a_2b|\,|\Pi_{2\sigma-1}|x_{\sigma+1}^2-|\Pi_{2\sigma}|}{\frac{x_{\sigma+1}^2}{\sigma+1}+|\Pi_{2\sigma+1}|},$$

$$|x_\sigma|=\frac{|y_\sigma+\Pi_{2\sigma-1}|}{|x_{\sigma+1}|}\ge\frac{x_{\sigma+1}^4-|\Pi_{2\sigma}|-|\Pi_{2\sigma-1}|\left(|a_2b|x_{\sigma+1}^2+\frac{x_{\sigma+1}^2}{\sigma+1}+|\Pi_{2\sigma+1}|\right)}{|x_{\sigma+1}|\left(\frac{x_{\sigma+1}^2}{\sigma+1}+|\Pi_{2\sigma+1}|\right)},$$

and the inequality (58) follows from

$$\text{(59)}\quad x_{\sigma+1}^4-|\Pi_{2\sigma}|-|\Pi_{2\sigma-1}|\left(|a_2b|x_{\sigma+1}^2+\frac{x_{\sigma+1}^2}{\sigma+1}+|\Pi_{2\sigma+1}|\right)$$
$$\ge\max\left\{|\Pi_{2\rho-2}|\left(\frac{|x_{\sigma+1}^3|}{\sigma+1}+|x_{\sigma+1}|\,|\Pi_{2\sigma+1}|\right),\frac{\sigma x_{\sigma+1}^4}{\sigma+1}+\sigma x_{\sigma+1}^2|\Pi_{2\sigma+1}|\right\}.$$

For $|x_{\sigma+1}|\ge|\Pi_{2\rho-2}|$ the second term of the maximum is greater and the difference between the left-hand side and the right-hand side of (59) for ρ large enough is at least

$$\frac{\Pi_{2\rho-2}^4}{\sigma+1}-|\Pi_{2\sigma}|-|\Pi_{2\sigma-1}\Pi_{2\sigma+1}|-\Pi_{2\rho-2}^2\left(|a_2b|\,|\Pi_{2\sigma-1}|+\frac{|\Pi_{2\sigma-1}|}{\sigma+1}+\sigma|\Pi_{2\sigma+1}|\right),$$

which is positive for ρ large enough. ∎

LEMMA 12. *If either* $(m-1)(n-1)>1$, $|abc|\ge 2$, *or* $m\ge 5$, $n=1$, $a_1=a_{m-1}=0$, $|abc|\ge 2$ *and* ρ *is large enough in terms of* m,n, *and for* $2\le\sigma\le\rho+1$, x_σ *and* $y_{\sigma-1}$ *are given by* (50)–(53), *then for every non-negative integer* $\tau<\rho$,

$$\text{(60)}\quad \exp(-(mn)^{3(\tau-\rho)+c_2-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}}\le|x_{\rho-\tau+1}|\le\exp((mn)^{3(\tau-\rho)+c_1-3})|\Pi_{2\rho}|^{\lambda_{2\tau+1}},$$

$$\text{(61)}\quad \exp(-(mn)^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}\le|y_{\rho-\tau}|\le\exp((mn)^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}.$$

Proof by induction on τ. For $\tau=0$ the inequality (60) follows from (51). For (61), if $\tau=0$ in view of Lemma 7 we have

$$|y_\rho-1-\Pi_{2\rho}|\le|\Pi_{2\rho-1}|^{m-\varepsilon},$$

where $\varepsilon=1$ if $(m-1)(n-1)>1$ and $\varepsilon=2$ if $m\ge 5$, $n=1$, thus in view of Lemma 8, (61) follows for ρ large enough from

$$\lim_{n\to\infty}|\Pi_{2\rho}|\,|\Pi_{2\rho-1}|^{1-m}(mn)^{-3\rho}=\lim_{\rho\to\infty}|\Pi_{2\rho-1}/\Pi_{2\rho-2}|(mn)^{-3\rho}=\infty$$

and

$$\lim_{n\to\infty}|\Pi_{2\rho}|\,|\Pi_{2\rho-1}|^{2-m}(mn)^{-3\rho}=\lim_{\rho\to\infty}|\Pi_{2\rho-1}^2/\Pi_{2\rho-2}|(mn)^{-3\rho}=\infty$$

for $(m-1)(n-1) > 1$ or $m \ge 5$, $n = 1$, respectively, which in turn follows from (30) and Lemma 3.

Assume now that (60) and (61) are true for $\tau < \rho - 1$. Then by Lemma 7 and the inductive assumption, for ρ large enough,

$$\begin{aligned}|g_{\rho-\tau}(y_{\rho-\tau})| &\le |y_{\rho-\tau}|^n + \max\{1, |y_{\rho-\tau}|\}^{n-\varepsilon}|\mathit{\Pi}_{2\rho-2}|^{n-\varepsilon} + |\mathit{\Pi}_{2\rho-2\tau-1}| \\ &\le \exp(n(mn)^{3(\tau-\rho)+c_3})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}} \\ &\quad + \exp((n-\varepsilon)(mn)^{3(\tau-\rho)+c_3})|\mathit{\Pi}_{2\rho}|^{(n-\varepsilon)\lambda_{2\tau+2}}|\mathit{\Pi}_{2\rho-2}|^{n-\varepsilon} + |\mathit{\Pi}_{2\rho-1}| \\ &\le \exp((mn)^{3(\tau-\rho)+c_3+1})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}}.\end{aligned}$$

Hence by (53), the inductive assumption and Lemma 9,

$$\begin{aligned}|x_{\rho-\tau}| &\le \frac{\exp((mn)^{3(\tau-\rho)+c_3+1})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}}}{\exp(-(mn)^{3(\tau-\rho)+c_1-3})|\mathit{\Pi}_{2\rho}|^{\lambda_{2\tau+2}}} \\ &\le \exp((mn)^{3(\tau-\rho)+c_1})|\mathit{\Pi}_{2\rho}|^{\lambda_{2\tau+3}}.\end{aligned} \tag{62}$$

Since the functions $t \mapsto t^n - At^{n-\varepsilon}$ are increasing for $t \ge A > 0$, and by the inductive assumption we have

$$|y_{\rho-\tau}| \ge \frac{1}{e}|\mathit{\Pi}_{2\rho}|^{\lambda_{2\tau+2}} \ge |\mathit{\Pi}_{2\rho-2}|^{n-\varepsilon},$$

it follows from Lemma 7 that

$$\begin{aligned}|g_{\rho-\tau}(y_{\rho-\tau})| &\ge |y_{\rho-\tau}|^n - \max\{1, |y_{\rho-\tau}|\}^{n-\varepsilon}|\mathit{\Pi}_{2\rho-2}|^{n-\varepsilon} - |\mathit{\Pi}_{2\rho-2\tau-1}| \\ &\ge \exp(-n(mn)^{3(\tau-\rho)+c_4})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}} \\ &\quad - \exp(-(n-\varepsilon)(mn)^{3(\tau-\rho)+c_4})|\mathit{\Pi}_{2\rho}|^{(n-\varepsilon)\lambda_{2\tau+2}}|\mathit{\Pi}_{2\rho-2}|^{n-\varepsilon} - |\mathit{\Pi}_{2\rho-1}| \\ &\ge \exp(-(mn)^{3(\tau-\rho)+c_4+1})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}},\end{aligned}$$

hence by (53), the inductive assumption and Lemma 9,

$$\begin{aligned}|x_{\rho-\tau}| &\ge \frac{\exp(-(mn)^{3(\tau-\rho)+c_4+1})|\mathit{\Pi}_{2\rho}|^{n\lambda_{2\tau+2}}}{\exp((mn)^{3(\tau-\rho)+c_1-3})|\mathit{\Pi}_{2\rho}|^{\lambda_{2\tau+1}}} \\ &\ge \exp(-(mn)^{3(\tau-\rho)+c_2})|\mathit{\Pi}_{2\rho}|^{\lambda_{2\tau+3}}.\end{aligned} \tag{63}$$

Similarly, by Lemmas 7 and 9 and (62), for ρ large enough and $\tau < \rho - 1$,

$$\begin{aligned}|f_{\rho-\tau}(x_{\rho-\tau})| &\le |x_{\rho-\tau}|^m + \max\{1, |x_{\rho-\tau}|\}^{m-\varepsilon}|\mathit{\Pi}_{2\rho-3}|^{m-\varepsilon} + |\mathit{\Pi}_{2\rho-2\tau-2}| \\ &\le \exp(m(mn)^{3(\tau-\rho)+c_1})|\mathit{\Pi}_{2\rho}|^{m\lambda_{2\tau+3}} \\ &\quad + \exp((m-\varepsilon)(mn)^{3(\tau-\rho)+c_1})|\mathit{\Pi}_{2\rho}|^{(m-\varepsilon)\lambda_{2\tau+3}}|\mathit{\Pi}_{2\rho-3}|^{m-\varepsilon} + |\mathit{\Pi}_{2\rho-2}| \\ &\le \exp\big((mn)^{3(\tau-\rho)}((mn)^{c_3+3} - (mn)^{c_4})\big)|\mathit{\Pi}_{2\rho}|^{m\lambda_{2\tau+3}},\end{aligned}$$

hence by (52) and the inductive assumption

$$|y_{\rho-\tau-1}| \le \frac{\exp\big((mn)^{3(\tau-\rho)}((mn)^{c_3+3}-(mn)^{c_4})\big)|\Pi_{2\rho}|^{m\lambda_{2\tau+3}}}{\exp(-(mn)^{3(\tau-\rho)+c_4})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}} = \exp((mn)^{3(\tau-\rho)+c_3+3})|\Pi_{2\rho}|^{\lambda_{2\tau+4}}.$$

Finally, since the functions $t\mapsto t^m - Bt^{m-\varepsilon}$ are increasing for $t\ge B\ge 0$, and by (63) we have

$$|x_{\rho-\tau}| \ge \frac{1}{e}\,|\Pi_{2\rho}|^{\lambda_{2\tau+3}} \ge |\Pi_{2\rho-3}|^{m-\varepsilon},$$

it follows by Lemmas 7 and 9 and (63) that, for large ρ,

$$\begin{aligned}|f_{\rho-\tau}(x_{\rho-\tau})| &\ge |x_{\rho-\tau}|^m - \max\{1,|x_{\rho-\tau}|\}^{m-\varepsilon}|\Pi_{2\rho-3}|^{m-\varepsilon} - |\Pi_{2\rho-2\tau-2}|\\ &\ge \exp(m(mn)^{3(\tau-\rho)+c_2})|\Pi_{2\rho}|^{m\lambda_{2\tau+3}}\\ &\quad - \exp((m-\varepsilon)(mn)^{3(\tau-\rho)+c_2})|\Pi_{2\rho}|^{(m-\varepsilon)\lambda_{2\tau+3}}|\Pi_{2\rho-3}|^{m-2} - |\Pi_{2\rho-2}|\\ &\ge \exp\big(-(mn)^{3(\tau-\rho)}((mn)^{c_4+3}-(mn)^{c_3})\big)|\Pi_{2\rho}|^{\lambda_{2\tau+3}},\end{aligned}$$

hence, by (52) and the inductive assumption,

$$|y_{\rho-\tau-1}| \ge \frac{\exp\big(-(mn)^{3(\tau-\rho)}((mn)^{c_4+3}-(mn)^{c_3})\big)|\Pi_{2\rho}|^{m\lambda_{2\tau+3}}}{\exp((mn)^{3(\tau-\rho)+c_3})|\Pi_{2\rho}|^{\lambda_{2\tau+2}}} = \exp(-(mn)^{3(\tau-\rho)+c_4+3})|\Pi_{2\rho}|^{\lambda_{2\tau+4}}. \blacksquare$$

LEMMA 13. *If* $(m-1)(n-1)>0$, $a,b,c>0$, $a_i,b_j\ge 0$ $(0<i<m,$ $0<j<n)$ *and, for* $2\le\sigma\le\rho+1$, x_σ *and* $y_{\sigma-1}$ *are given by* (50)–(53), *then for* $1\le\sigma\le\rho$,

$$0 < x_{\sigma+1} < y_\sigma < x_\sigma. \tag{64}$$

Proof by backward induction. For $\sigma=\rho$ the first and second inequality are clear, and the third follows from

$$x_\rho = g_\rho(y_\rho) > y_\rho.$$

Assume that the inequality (64) holds for $\sigma+1<\rho$. Then

$$y_\sigma = \frac{f_{\sigma+1}(x_{\sigma+1})}{y_{\sigma+1}} > \frac{f_{\sigma+1}(x_{\sigma+1})}{x_{\sigma+1}} > x_{\sigma+1},$$
$$x_\sigma = \frac{g_\sigma(y_\sigma)}{x_{\sigma+1}} > \frac{g_\sigma(y_\sigma)}{y_\sigma} > y_\sigma. \blacksquare$$

COROLLARY 4. *Under the assumptions of Theorems* 2–4 *the numbers* $x_\sigma, y_{\sigma-1}$ *given for* $2\le\sigma\le\rho+1$ *by* (50)–(53) *are non-zero.*

Proof. Clear from (54), (55), (58), (60), (61) and (64). ■

Lemma 14. *Under the assumptions of Theorems 2–4, let the numbers* $x_\sigma, y_{\sigma-1}$ *for* $2 \le \sigma \le \rho+1$, $\rho \ge 2$, *be given by* (50)–(53) *and moreover set*

$$x_1 = \frac{g_1(y_1)}{x_2}.$$

Then for $\sigma \le \rho + 1$,

$$x_\sigma \in \mathbb{Z} \quad (\sigma \ge 1), \tag{65}$$

$$y_{\sigma-1} \in \mathbb{Z} \quad (\sigma \ge 2), \tag{66}$$

and for $\sigma \ge 2$,

$$(x_\sigma, \mathit{\Pi}_{2\sigma-2}) = 1, \tag{67}$$

$$(y_{\sigma-1}, \mathit{\Pi}_{2\sigma-3}) = 1. \tag{68}$$

Proof by backward induction on σ. For $\sigma = \rho+1$, (65)–(67) are clear. Now,

$$y_\rho = f_{\rho+1}(1) = \frac{1}{\mathit{\Pi}_{2\rho-2}} f_\rho(\mathit{\Pi}_{2\rho-1}),$$

and since by the assumption $\operatorname{Rad} c \mid a_{m-1}$, in any case we have

$$(y_\rho, \mathit{\Pi}_{2\rho-1}) = 1.$$

Assume now that (65)–(68) are true for $\sigma+1 \le \rho+1$ and $\sigma \ge 2$. In the case of (65) the last step of the induction is from $\sigma = 2$ to $\sigma = 1$. Thus, by Corollary 4, $x_{\sigma+1} \ne 0$, and by (53) with $g_\sigma(y) = \sum_{i=0}^n g_{\sigma i} y^{n-i}$,

$$\begin{aligned} y_{\sigma+1}^n x_\sigma &= \frac{y_{\sigma+1}^n g_\sigma(y_s)}{x_{\sigma+1}} \\ &= \frac{(y_\sigma y_{\sigma+1})^n + \sum_{i=1}^{n-1} g_{\sigma i} y_{\sigma+1}^i (y_\sigma y_{\sigma+1})^{n-i} + \mathit{\Pi}_{2\sigma-1} y_{\sigma+1}^n}{x_{\sigma+1}} \\ &\equiv \frac{\mathit{\Pi}_{2\sigma}^n + \sum_{i=1}^{n-1} g_{\sigma i} y_{\sigma+1}^i \mathit{\Pi}_{2\sigma}^{n-i} + \mathit{\Pi}_{2\sigma-1} y_{\sigma+1}^n}{x_{\sigma+1}} \\ &= \mathit{\Pi}_{2\sigma-1} \frac{\frac{1}{\mathit{\Pi}_{2\sigma-1}} g_\sigma(\frac{\mathit{\Pi}_{2\sigma}}{y_{\sigma+1}}) y_{\sigma+1}^n}{x_{\sigma+1}} \equiv \mathit{\Pi}_{2\sigma-1} \frac{g_{\sigma+1}(y_{\sigma+1})}{x_{\sigma+1}} \pmod 1. \end{aligned}$$

For $\sigma = \rho$ the right-hand side is clearly an integer; for $\sigma < \rho$ it is equal to $\mathit{\Pi}_{2\sigma-1} x_{\sigma+2}$, hence it is also an integer. Thus

$$y_{\sigma+1}^n x_\sigma \in \mathbb{Z}. \tag{69}$$

Moreover, by the inductive assumption

$$(x_{\sigma+1}, \mathit{\Pi}_{2\sigma}) = 1,$$

and it follows from (52) and Lemma 5 that

$$(y_\sigma y_{\sigma+1}, \mathit{\Pi}_{2\sigma-1} x_{\sigma+1}) = 1. \tag{70}$$

Since by (53), $x_\sigma x_{\sigma+1} \in \mathbb{Z}$, it follows from (69) and (70) that $x_\sigma \in \mathbb{Z}$.

Similarly, since $y_\sigma\neq 0$, by (52) and (53) with $f_\sigma(x)=\sum_{i=0}^m f_{\sigma i}x^{m-i}$ we have

$$(71)\quad x_{\sigma+1}^m y_{\sigma-1}=\frac{x_{\sigma+1}^m f_\sigma(x_\sigma)}{y_\sigma}$$
$$=\frac{(x_\sigma x_{\sigma+1})^m+\sum_{i=1}^{m-1} f_{\sigma i}x_{\sigma+1}^i(x_\sigma x_{\sigma+1})^{m-i}+\Pi_{2\sigma-2}x_{\sigma+1}^m}{y_s}$$
$$\equiv\frac{\Pi_{2\sigma-1}^m+\sum_{i=1}^{m-1} f_{\sigma i}x_{\sigma+1}^i\Pi_{2\sigma-1}^{m-i}+\Pi_{2\sigma-2}x_{\sigma+1}^m}{y_\sigma}$$
$$=\Pi_{2\sigma-1}\frac{f_{\sigma+1}(x_{\sigma+1})}{y_\sigma}=\Pi_{2\sigma-1}y_{\sigma+1} \pmod 1.$$

Since by (52), $y_{\sigma-1}y_\sigma\in\mathbb{Z}$, it follows from (70) and (71) that $y_{\sigma-1}\in\mathbb{Z}$.

Moreover, under the assumptions of the lemma,

$$\operatorname{Rad}\Pi_{2\sigma-2}\mid g_\sigma(y_\sigma)-y_\sigma^n,$$

hence by (53) and (70),

$$(72)\qquad (x_\sigma,\Pi_{2\sigma-2})=1.$$

Finally, under the assumptions of the lemma,

$$\operatorname{Rad}\Pi_{2\sigma-3}\mid f_\sigma(x_\sigma)-x_\sigma^m,$$

so by (52) and (72),

$$(y_{\sigma-1},\Pi_{2\sigma-3})=1. \blacksquare$$

Lemma 15. *If $a,b,c\neq 0$, a_1,b_1,z are integers and the equation $ax^2-zxy+by^2+a_1x+b_1y+c=0$ has a solution in integers x,y such that $(y,c)=1$, then it has infinitely many such solutions provided*

$$D=z^2-4ab\quad \textit{is positive, but not a perfect square}$$

and

$$\Delta=4abc-za_1b_1-ab_1^2-ba_1^2-cz^2\neq 0.$$

Proof. The proof follows the proof of Theorem 2 in [5, p. 59]. Only the solution of the Pell equation $T^2-Du^2=1$ has to be chosen so that $T\equiv 1 \pmod{Dc}$, $u\equiv 0 \pmod{Dc}$. $\blacksquare$

Notation. For $\varepsilon,\eta\in\{1,-1\}$ set

$$\Delta(\varepsilon,\eta)=4abc-(a+b+\varepsilon a_1+\eta b_1+c)\varepsilon\eta a_1b_1-ab_1^2-ba_1^2-c(a+b+\varepsilon a_1+\eta b_1+c)^2.$$

Lemma 16. *If $abc\Delta(\varepsilon,\eta)\neq 0$, then either the congruence*

$$ax^2+a_1x+by^2+b_1y+c\equiv 0 \pmod{xy}$$

has infinitely many solutions in integers x,y such that $(y,c)=1$, or $|a+\varepsilon a_1+b+\eta b_1+c|\leq 4|ab|$.

Proof. The equation

$$ax^2 + a_1x + by^2 + b_1y + c = (a + a_1\varepsilon + b + b_1\eta + c)xy$$

has a solution $x = \varepsilon$, $y = \eta$, hence by Lemma 15 either it has infinitely many solutions in integers such that $(y, c) = 1$, or

(73) $$(a + \varepsilon a_1 + b + \eta b_1 + c)^2 - 4ab \le 0,$$

or

(74) $$(a + \varepsilon a_1 + b + \eta b_1 + c)^2 - 4ab \quad \text{is a perfect square.}$$

In the case (73) the assertion is clear; in the case (74) we use Lemma 1. ∎

LEMMA 17. *If $a, b \ne 0$, c, a_1, b_1 are integers and*

(75) $$\Delta(\varepsilon, \eta) = \Delta(-\varepsilon, -\eta) = 0,$$

then either $\varepsilon\eta a_1b_1 + 2c(a + b + c) = 0$, $a_1^2 + b_1^2 > 0$, or $b_1 = -\varepsilon\eta a_1$, $c = 0$, or a_1, b_1, c are bounded in terms of a, b.

Proof. The equations (75) give on subtraction

$$-2\varepsilon\eta(\varepsilon a_1 + \eta b_1)a_1b_1 - 4c(\varepsilon a_1 + \eta b_1)(a + b + c) = 0,$$

and if

$$\varepsilon\eta a_1b_1 + 2c(a + b + c) \ne 0 \quad \text{or} \quad a_1^2 + b_1^2 = 0,$$

we obtain

$$\varepsilon a_1 + \eta b_1 = 0, \quad b_1 = -\varepsilon\eta a_1.$$

On substituting in (75) we obtain

$$4abc + (a + b + c)a_1^2 - aa_1^2 - ba_1^2 - c(a + b + c)^2 = 0,$$

thus either $c = 0$, or

$$4ab + a_1^2 - (a + b + c)^2 = 0$$

and, by Lemma 1, a_1, $a+b+c$ are bounded in terms of a, b. Since $b_1 = -\varepsilon\eta a_1$, the same applies to a_1, b_1, c. ∎

LEMMA 18. *If $a, b \ne 0$, c, a_1, b_1 are integers and*

(76) $$\Delta(\varepsilon, \eta) = \Delta(\varepsilon, -\eta) = 0,$$

then either $b_1 = 0$, or a_1, b_1, c are bounded in terms of a, b.

Proof. The equations (76) give on subtraction

$$-2\varepsilon\eta(a + b + c + \varepsilon a_1)a_1b_1 - 4c\eta b_1(a + b + c + \varepsilon a_1) = 0,$$

hence either

(77) $$a + b + c + \varepsilon a_1 = 0,$$

or

(78) $$\varepsilon\eta a_1b_1 + 2c\eta b_1 = 0.$$

In the case (77) substituting in (76) we obtain

$$4abc - \varepsilon a_1 b_1^2 - ab_1^2 - ba_1^2 - cb_1^2 = 0,$$
$$4abc - (\varepsilon a_1 + a + c)b_1^2 - ba_1^2 = 0, \quad 4abc + bb_1^2 - ba_1^2 = 0,$$

and on dividing by b,

$$4ac = a_1^2 - b_1^2 = (a_1 + b_1)(a_1 - b_1).$$

Since the numbers $a_1 + b_1$ and $a_1 - b_1$ are of the same parity, they are even. Thus we obtain, for some integers x, β, γ, δ,

$$a = \alpha\beta, \quad c = \gamma\delta, \quad a_1 + b_1 = 2\alpha\gamma, \quad a_1 - b_1 = 2\beta\delta, \tag{79}$$

hence

$$a_1 = \alpha\gamma + \beta\delta, \quad b_1 = \alpha\gamma - \beta\delta, \tag{80}$$

and the equation (77) gives

$$\alpha\beta + b + \gamma\delta + \varepsilon(\alpha\gamma + \beta\delta) = 0,$$

thus

$$b = -(\alpha + \varepsilon\delta)(\beta + \varepsilon\gamma),$$

which gives finitely many choices for $\alpha + \varepsilon\delta$, $\beta + \varepsilon\gamma$. However, by (79) there are only finitely many choices for α and β, thus there are only finitely many choices for δ and γ, hence by (79) and (80) also for c, a_1, b_1.

Consider now the case (78). If $b_1 \neq 0$, we obtain $\varepsilon a_1 + 2c = 0$, hence by (76),

$$\begin{aligned}
0 &= 4abc - \varepsilon\eta(a + b - c + \eta b_1)a_1 b_1 - ab_1^2 - ba_1^2 - c(a + b - c + \eta b_1)^2 \\
&= 4abc + 2c\eta(a + b - c + \eta b_1)b_1 - ab_1^2 - 4bc^2 - c(a + b - c + \eta b_1)^2 \\
&= 4abc + c(a + b - c + \eta b_1)(2\eta b_1 - a - b + c - \eta b_1) - ab_1^2 - 4bc^2 \\
&= 4abc + c(b_1^2 - (a + b - c)^2) - ab_1^2 - 4bc^2 \\
&= 4abc - c(a + b - c)^2 - 4bc^2 + (c - a)b_1^2 \\
&= 4abc - a^2c - 2abc + 2ac^2 - b^2c + 2bc^2 - c^3 - 4bc^2 + (c - a)b_1^2 \\
&= -c(a - b - c)^2 + (c - a)b_1^2.
\end{aligned}$$

It follows that

$$\left(\frac{a - b - c}{b_1}\right)^2 = \frac{c - a}{c},$$

and for some integers $\alpha, \beta, \gamma, \delta$,

$$a - b - c = \alpha\beta, \quad b_1 = \alpha\gamma, \quad c - a = \delta\beta^2, \quad c = \delta\gamma^2, \quad a = \delta\gamma^2 - \delta\beta^2,$$

hence β, γ, δ are bounded in terms of a, and c is bounded. If $\beta = 0$, then $a - b - c = 0$, $c - a = 0$, $b = 0$. Therefore $\beta \neq 0$ and α is bounded, b_1 is bounded, and so is $a_1 = -2\varepsilon c$. ∎

Lemma 19. *If* a, $b \neq 0$, c, a_1, b_1 *are integers and*

$$\Delta(\varepsilon,\eta) = \Delta(-\varepsilon,\eta) = 0,$$

then either $a_1 = 0$, *or* a_1, b_1, c *are bounded in terms of* a, b.

The proof is analogous to the proof of Lemma 18.

Proof of Theorem 2. If $|ab| \geq 9$ and $\operatorname{Rad} c \mid (a_1, b_1 a)$, by Lemmas 10 and 14, for ρ large enough there exist arbitrarily large (in absolute value) integers x_1, y_1, x_2, y_2 such that

$$\begin{aligned} x_1x_2 &= g_1(y_1) = g(y_1) + c, \\ y_1y_2 &= f_2(x_2) = x_2^2 + \frac{1}{c} f\left(\frac{c}{x_2}\right)x_2^2 \end{aligned} \tag{81}$$

and

$$(y_1, c) = 1. \tag{82}$$

We have

$$\begin{aligned} c(f(x_1) + c) &= acx_1^2 + a_1cx_1 + c^2 \\ &\equiv (x_1x_2)^2 + \frac{1}{c}\left(a\left(\frac{c}{x_2}\right)^2 + a_1\left(\frac{c}{x_2}\right)\right)(x_1x_2)^2 = x_1^2 f_2(x_2) \equiv 0 \pmod{y_1} \end{aligned}$$

and by (82),

$$f(x_1) + c \equiv 0 \pmod{y_1}.$$

Since by (81),

$$g(y_1) + c \equiv 0 \pmod{x_1},$$

and by (81) and (82),

$$(x_1, y_1) = (y_1, c) = 1,$$

it follows that

$$f(x_1) + g(y_1) + c \equiv 0 \pmod{x_1y_1}. \tag{83}$$

It remains to show that for $0 < |ab| < 9$ there exist only finitely many triples of integers a_1, b_1, c such that the congruence

$$ax^2 + a_1x + by^2 + b_1y + c \equiv 0 \pmod{xy} \tag{84}$$

has only finitely many solutions in integers x, y with $(y, c) = 1$. Assuming this is false, we shall use Lemmas 16–19.

If $a_1 = b_1 = 0$ and $\Delta(1, 1) \neq 0$, then by Lemma 16, c is bounded in terms of a, b. If $a_1 = b_1 = 0$ and $\Delta(1, 1) = 0$, then $\Delta(-1, -1) = 0$, thus by Lemma 17, c is bounded in terms of a, b.

If $a_1^2 + b_1^2 > 0$ and $a_1b_1 = 0$, then we may assume without loss of generality that $a_1 = 0$ and $b_1 \neq 0$. If $\Delta(1, 1) \neq 0$ and $\Delta(1, -1) \neq 0$, then we use Lemma 16. If $\Delta(1, 1) \neq 0$ and $\Delta(1, -1) = 0$, then by Lemma 16,

$$|a + b + b_1 + c| \leq 4|ab| \tag{85}$$

and

$$4abc-ab_1^2-c(a+b-b_1+c)^2=0. \tag{86}$$

(86) implies $c\,|\,a(b_1+c)^2$, and since, by (85), b_1+c is bounded in terms of a,b, we conclude that either b_1 and c are bounded in terms of a,b, or $b_1+c=0$, which gives, by (86) and the assumption $c\neq 0$, $c\,|\,(a-b)^2$. Hence either b_1 and c are bounded in terms of a,b, or $a=b$ and, by (86), $9a+4c=0$, and c and b_1 are determined by a,b.

If $\Delta(1,1)=0$ and $\Delta(1,-1)\neq 0$, the argument is analogous. If $\Delta(1,1)=\Delta(1,-1)=0$, then by Lemma 18, b_1,c are bounded in terms of a,b. If $a_1b_1\neq 0$ and, for an $\varepsilon=\pm 1$, $\Delta(\varepsilon,\varepsilon)\neq 0$, $\Delta(-1,1)\neq 0$, $\Delta(1,-1)\neq 0$, then we use Lemma 16. If $\Delta(\varepsilon,\varepsilon)\neq 0$, $\Delta(-1,1)\neq 0$ and $\Delta(1,-1)=0$, then by Lemmas 18 and 19 either $\Delta(-\varepsilon,-\varepsilon)\neq 0$, or a_1,b_1,c are bounded in terms of a,b. In the former case we use Lemma 16 again. If $\Delta(\varepsilon,\varepsilon)\neq 0$, $\Delta(-1,1)=0$ and $\Delta(1,-1)\neq 0$, the argument is analogous. If $\Delta(\varepsilon,\varepsilon)\neq 0$ and $\Delta(1,-1)=\Delta(-1,1)=0$, then by Lemma 16,

$$|a+b+c+\varepsilon a_1+\varepsilon b_1|\le 4|ab|, \tag{87}$$

and by Lemma 17 either

$$-a_1b_1+2c(a+b+c)=0, \tag{88}$$

or

$$c=0,\quad b_1=a_1, \tag{89}$$

or a_1,b_1,c are bounded in terms of a,b.

If $\Delta(-\varepsilon,-\varepsilon)\neq 0$, then by Lemma 16 we have

$$|a+b+c-\varepsilon a_1-\varepsilon b_1|\le 4|ab|,$$

hence by (87),

$$|a+b+c|\le 4|ab|,$$

c is bounded in terms of a,b and, by (88), so are a_1,b_1 different from 0. The case (89) is excluded by the assumption of the theorem.

If $\Delta(-\varepsilon,-\varepsilon)=0$, then, by Lemma 18, a_1,b_1,c are bounded in terms of a,b.

If $\Delta(1,1)=\Delta(-1,-1)=0$, then by Lemma 18 either $\Delta(1,-1)\neq 0$ and $\Delta(-1,1)\neq 0$, or a_1,b_1,c are bounded in terms of a,b. In the former case, by Lemma 16,

$$\begin{aligned}|a+a_1+b-b_1+c|&\le 4|ab|,\\ |a-a_1+b+b_1+c|&\le 4|ab|,\end{aligned} \tag{90}$$

hence

$$|a+b+c|\le 4|ab|,$$

and c is bounded in terms of a, b. On the other hand, by Lemma 17 either

$$a_1b_1 + 2c(a+b+c) = 0, \tag{91}$$

or

$$c = 0, \quad b_1 = -a_1, \tag{92}$$

or a_1, b_1, c are bounded in terms of a, b. In the case (91), a_1, b_1 are bounded in terms of a, b. The case (92) is excluded by the assumptions of the theorem. ∎

Proof of Corollary 1. An analysis of the proof of Theorem 2 shows that for $a_1 = b_1 = 0$ it works for $|ab| > 2$. Therefore, it suffices to consider $|ab| \le 2$ and we may assume without loss of generality that $a = 1$ or $b = 1$. Lemma 15 with $x = y = 1$, $z = a + b + c$ leaves open the cases where $(a+b+c)^2 - 4ab$ is negative or a perfect square, thus

$$a = b = 1, \quad c = -4, -3, -2, -1;$$
$$\{a, b\} = \{1, 2\}, \quad c = -6, -5, -4, -3, -2, -1;$$
$$a = 1,\ b = -2,\ c = 2; \quad a = -2,\ b = 1,\ c = 2.$$

For $a = b = 1$, $c = -4$ we take $x = 2t-1$, $y = 2t+1$ (t an arbitrary integer). For $a = b = 1$, $c = -3, -2$ there are only finitely many solutions (see [1] or [2]). For $a = b = 1$, $c = -1$; $a = 1$, $b = 2$, $c = -4$; $a = 1$, $b = 2$, $c = -2$; $a = 1$, $b = 2$, $c = -1$; and $a = 1$, $b = -2$, $c = 2$, we take respectively $x = 1$, y arbitrary; $x = 2$, y arbitrary odd; x arbitrary, $y = 1$; $x = 1$, y arbitrary; and $y = 1$, x arbitrary.

For $a = 1$, $b = 2$, $c = -6$; $a = 1$, $b = 2$, $c = -5$; $a = 1$, $b = 2$, $c = -3$; and $a = 2$, $b = 1$, $c = -4$, we take in Lemma 15 respectively $x = 1$, $y = 5$, $z = 9$; $x = 1$, $y = 4$, $z = 7$; $x = 5$, $y = 22$, $z = 9$; and $x = 3$, $y = 1$, $z = 5$.

For $a = 2$, $b = 1$, $c = -2$ we take $x = 1$, y arbitrary odd; for $a = 2$, $b = 1$, $c = -1$ we take x arbitrary, $y = 1$; for $a = -2$, $b = 1$, $c = 2$ we take $x = 1$, y arbitrary odd. ∎

Proof of Theorem 3. If $m \ge 4$, $n = 1$ and $|abc| \ge 2$, then by Lemmas 10, 11 and 14, for ρ large enough in terms of m there exist arbitrarily large (in absolute value) integers x_1, y_1, x_2, y_2 such that (81) and (82) hold. We infer, as in the proof of Theorem 2, that (83) holds.

It remains to consider the case $m \ge 4$, $n = 1$ and $|abc| = 1$. Then $a, b, c \in \{1, -1\}$ and the congruence (1) has infinitely many solutions satisfying $(y, c) = 1$ given by $x \ne 0$ arbitrary, $y = -b(f(x) + c) \ne 0$. ∎

Proof of Theorem 4. If $(m-1)(n-1) > 1$ and either $|abc| > 1$, or $a, b, c > 0$, $a_i, b_j \ge 0$ ($0 < i < m$, $0 < j < n$), by Lemmas 12 and 14 or by Lemmas 13 and 14, respectively, for ρ large enough in terms of m, n there exist arbitrarily large (in absolute value) integers x_1, y_1, x_2, y_2 such that (81) and (82) hold. We infer, as in the proof of Theorem 2, that (83) holds, namely

$$c^{m-1}(f(x_1) + c) \equiv (x_1x_2)^m + \frac{1}{c}(x_1x_2)^m f(x_1) \equiv x_1^m f_2(x_2) \equiv 0 \pmod{y_1}. \blacksquare$$

Proof of Corollary 2. It remains to consider the case $|abc| = 1$. If $\Pi_{2\rho} = 1$ and for $2 \le \sigma \le \rho + 1$, x_σ and $y_{\sigma-1}$ are given by (50)–(53), then we shall show by backward induction that for $2 \le \sigma \le \rho$,

$$0 < y_\sigma < x_\sigma < y_{\sigma-1}. \tag{93}$$

For $\sigma = \rho$ we have, by (50)–(53),

$$y_\rho = 1 + \Pi_{2\rho} = 2,$$
$$x_\rho = \frac{g_\rho(y_\rho)}{x_{\rho+1}} = 2^n + \Pi_{2\rho-1} \ge 2^n - 1 > 2,$$
$$y_{\rho-1} = \frac{f_\rho(x_\rho)}{y_\rho} \ge \frac{x_\rho^m + \Pi_{2\rho-2}}{x_\rho - 1} \ge \frac{x_\rho^m - 1}{x_\rho - 1} > x_\rho.$$

Assuming now that (93) holds for $\sigma \ge 3$ we have

$$x_{\sigma-1} = \frac{g_{\sigma-1}(y_{\sigma-1})}{x_\sigma} \ge \frac{y_{\sigma-1}^n + \Pi_{2\sigma-3}}{y_{\sigma-1} - 1} > y_{\sigma-1},$$
$$y_{\sigma-2} = \frac{f_{\sigma-1}(x_{\sigma-1})}{y_{\sigma-1}} \ge \frac{y_{\sigma-1}^m + \Pi_{2\sigma-1}}{x_{\sigma-1} - 1} > x_{\sigma-1}.$$

Thus by Lemma 14, for ρ large enough in terms of m, n, there exist arbitrarily large x_1, x_2, y_1, y_2 such that (81)–(82) hold. We infer as in the proof of Theorem 2 that (83) holds. If $\Pi_{2\rho} = -1$ for all large ρ, since the congruence (1) can be multiplied by -1 we may assume that $c = 1$ and then the condition $\Pi_{2\rho} = -1$ for all large ρ implies $a = b = -1$, $\lambda_{2\rho} + \mu_{2\rho} \equiv 1 \pmod 2$, which in view of symmetry in x and y implies $m \equiv n \equiv 0 \pmod 2$. Taking $x = 1$, $y \ne 0$ arbitrary, we obtain infinitely many solutions of (1) satisfying $(y, c) = 1$. ∎

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