ACTA ARITHMETICA
147.4 (2011)

# On the zeros of linear recurrence sequences

by

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**1. Introduction.** A *linear recurrence sequence* of order $t \geq 1$ is a sequence $\{u_m\}_{m\in\mathbb{Z}}$ of elements in an algebraically closed field $K$ of characteristic zero which satisfies a minimal relation

$$u_{m+t} = c_1 u_{m+t-1} + \cdots + c_t u_m \quad (m \in \mathbb{Z})$$

with $c_1, \ldots, c_t \in K$. We say that $\{u_m\}$ is *simple* if its companion polynomial $\mathcal{P}(z) = z^t - c_1 z^{t-1} - \cdots - c_t$ has only simple roots. Let

$$\mathcal{S}(\{u_m\}) = \{k : u_k = 0\}.$$

The Skolem–Mahler–Lech theorem asserts that for an arbitrary linear recurrence sequence $\{u_m\}$ of order $t \geq 1$ the set $\mathcal{S}(\{u_m\})$ is a finite union of arithmetic progressions, where we adopt the convention that single elements of $\mathbb{Z}$ are trivial arithmetic progressions. In [4], J.-H. Evertse, H.-P. Schlickewei and W. Schmidt prove that for a *simple* linear recurrence sequence of order $t$ the set $\mathcal{S}(\{u_m\})$ is the union of at most

$$\exp\{(6t)^{3t}\}$$

arithmetic progressions.

Let $K$ be a field of characteristic 0. Let $(a_1, \ldots, a_n) \in (K^*)^n$ and $\Gamma$ be a subgroup of $(K^*)^n$ of finite rank $r$. Let us consider the equation

$$a_1\alpha_1 + \cdots + a_n\alpha_n = 1 \quad \text{with } \boldsymbol{\alpha} \in \Gamma. \tag{1.1}$$

We say that a solution of (1.1) is *non-degenerate* if no subsum of the left hand side of (1.1) vanishes. The result on linear recurrence sequences of [4] is a quite straightforward corollary of the bound for the number of non-degenerate solutions of the equation (1.1). In turn, this last estimate depends on two different tools. An application of the Subspace Theorem gives an estimate on the "large" solutions of equation (1.1). To handle the "small" solutions one usually applies a gap principle. For this purpose one needs

2010 *Mathematics Subject Classification*: 11B37, 11D61.
*Key words and phrases*: linear recurrence sequences, exponential equations.

DOI: 10.4064/aa147-4-4

a lower bound for the height of a small solution. In the quoted paper the authors use a result of Schmidt [5]. In [3, Corollary 1.4], we considerably improved such a bound. Thus we can save one exponential in the bound of [4] for the number of non-degenerate solutions of the equation (1.1) (see ([3, Theorem 6.2]). As a further application, we found ([3, Corollary 6.3]) that, for a *simple* linear recurrence sequence of order $t$, the set $\mathcal{S}(\{u_m\})$ is the union of at most

$$(8t)^{4t^5}$$

arithmetic progressions.

In [7], Schmidt generalized the zero estimate of [4] to arbitrary linear recurrence sequences of order $t$. He proved that $\mathcal{S}(\{u_m\})$ is the union of at most

$$\exp\exp\exp(20t) \tag{1.2}$$

arithmetic progressions. This was recently improved to

$$\exp\exp\exp(\sqrt{11t}\log t)$$

by Allen (see [2]). The key change in his proof is an improvement on [7, Lemma 2 on linear independence].

One of the fundamental tools in Schmidt's proof is the estimate of [4] for the number of non-degenerate solutions of the equation (1.1). The aim of the present paper is to briefly show how our results in [3] allow us to substantially improve (1.2), saving an exponential.

THEOREM 1.1. *Let $\{u_m\}$ be a linear recurrence of order $t$. Then the set $\mathcal{S}(\{u_m\}) = \{k : u_k = 0\}$ is the union of at most*

$$\exp\exp(70t)$$

*arithmetic progressions.*

We then improve some other bounds. Let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of the companion polynomial $\mathcal{P}$ and let $a$ be the maximum of their multiplicities in $\mathcal{P}$. In [7], Schmidt also proves that $\mathcal{S}(\{u_m\})$ is a union of at most

$$\exp\exp(30ak^a\log k)$$

arithmetic progressions. Theorem 1.1 suggests that one could possibly also improve this last estimate by one exponential. Unfortunately, we have not been able to do that. This is due to the double exponential growth in $t$ of the function $Z(t,T)$ which bounds the number of such arithmetic progressions in (3.1) below. However, we can successfully treat the case of non-degenerate sequences. We recall that a sequence is *non-degenerate* if no quotient $\alpha_i/\alpha_j$ $(1 \leq i < j \leq k)$ is a root of unity. For a non-degenerate sequence the Skolem–Mahler–Lech theorem simply asserts that $\mathcal{S}(\{u_m\})$ is a finite set.

Its cardinality is called the *zero multiplicity* of the sequence $\{u_m\}$. The following result improves by one exponential the main theorem of [6].

THEOREM 1.2. *Let $\{u_m\}$ be a non-degenerate linear sequence whose companion polynomial has $k$ distinct roots with multiplicity $\leq a$. Then the zero multiplicity of $\{u_m\}$ is bounded by*

$$(8k^a)^{8k^{6a}} \leq \exp(32ak^{6a}\log k).$$

We finally remark that in Theorem 1.1, one could naturally try to combine our improvement with Allen's refined version of [7, Lemma 2] to obtain a lower bound of the shape

$$\exp\exp(c\sqrt{t}\log t).$$

We cannot do this, because of the double exponential growth in $t$ of the function $Z(t,T)$ and because of the double exponential growth in $n$ for the number of systems of 3-element sets (see the discussion at the end of this article for some more details).

## 2. Ingredients of the proof

**2.1. Auxiliary results.** In what follows we sum up an improved version of some lemmas of [6] which we need in order to obtain the final result. Essentially, we replace the main results of [4] by those of [3, Section 6]. Let us sketch the necessary computations.

For integers $q \geq 1$ and $r \geq 0$ we define

$$C(q,r) = (8q)^{4(q-1)^4(q+r)}.$$

LEMMA 2.1 (Counterpart to [6, Lemma 4]). *Let $\Gamma$ be a finitely generated subgroup of $(\mathbb{C}^*)^q$ of rank $r$, and let $a_1,\ldots,a_q \in \mathbb{C}^*$. Then, up to a factor of proportionality, the equation*

$$a_1x_1 + \cdots + a_qx_q = 0$$

*has less than $C(q,r)$ non-degenerate solutions $\mathbf{x} \in \Gamma$.*

*Proof.* This is an inhomogeneous version of Theorem 6.2 of [3]. Indeed, set $n = q-1$, $b_i = -a_i/a_q$ and $y_i = x_i/x_q$ ($i = 1,\ldots,n$). Then the equation becomes

$$b_1y_1 + \cdots + b_ny_n = 1$$

with $\mathbf{y} = (y_1,\ldots,y_n)$ in a subgroup of rank $\leq r$. By Theorem 6.2 of [3] this last equation has at most

$$(8n)^{4n^4(n+r+1)} < C(q,r)$$

non-degenerate solutions. ∎

For $\boldsymbol{\alpha} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ we denote by $h(\boldsymbol{\alpha})$ the absolute, logarithmic Weil height of $\boldsymbol{\alpha}$. For $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n) \in (\overline{\mathbb{Q}}^*)^n$ we let $\hat{h}(\boldsymbol{\alpha}) = h((1:\alpha_1:\cdots:\alpha_n))$.

Lemma 2.2 (Counterpart to [6, Lemma 5]). *Let $q > 1$ and let $\Gamma$ be a finitely generated subgroup of $(\overline{\mathbb{Q}}^*)^q$ of rank $r$. Consider the set $S$ of solutions of the equation*

$$z_1 + \cdots + z_q = 0 \tag{2.1}$$

*with $\mathbf{z} = \mathbf{x}\mathbf{y}$, $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \mathbb{Q}^q$ and*

$$h(\mathbf{y}) \leq \frac{1}{4q^2} h(\mathbf{x}).$$

*Then $S$ is contained in the union of less than $C(q, r)$ proper linear subspaces of the $(q-1)$-dimensional linear space defined by* (2.1).

*Proof.* Set $n = q - 1$. As in the proof of [6], it is enough to prove the following inhomogeneous version of this lemma. Let $\Gamma$ be a finitely generated subgroup of $(\overline{\mathbb{Q}}^*)^n$ of rank $r$. Let $S'$ be the set of solutions of the equation $z_1 + \cdots + z_n = 1$ with $\mathbf{z} = \mathbf{x}\mathbf{y}$, $\mathbf{x} \in \Gamma$, $\mathbf{y} \in (\mathbb{Q}^*)^n$ and $\hat{h}(\mathbf{y}) \leq (4n^2)^{-1} h(\mathbf{x})$. Then $S'$ is contained in the union of not more than $C(q, r)$ proper linear subspaces of $(\overline{\mathbb{Q}}^*)^n$. We follow the proof of [6] replacing Theorem 2.1 of [4] by Theorem 6.1 of [3]. Then $S'$ is contained in the union of not more than

$$n + 2^{30n^2}(21n^2)^r + (8n)^{(6n^3)(n+r)}$$

proper linear subspaces of $(\overline{\mathbb{Q}}^*)^n$. We clearly assume $n \geq 2$. Using $2 = 16^{1/4} \leq (8n)^{1/4}$ and $1 + 7.5x^2 + 6x^4 \leq 4x^4(x+1)$ for $x \geq 2$, we see that

$$\begin{aligned} n + 2^{30n^2}(21n^2)^r + (8n)^{(6n^3)(n+r)} &\leq (8n)^{1+7.5n^2+2r+(6n^3)(n+r)} \\ &\leq (8n)^{4n^4(n+r+1)} < C(q, r). \ \blacksquare \end{aligned}$$

For a non-zero polynomial $P \in \mathbb{C}[X]$ we put $t(P) = 1 + \deg(P)$ and we agree that $t(0) = 0$. For a vector $\mathbf{P} = (P_1, \ldots, P_k) \in (\mathbb{C}[X])^k$, define $t(\mathbf{P}) = t(P_1) + \cdots + t(P_k)$ and $a(\mathbf{P}) = \max_i t(P_i)$.

Let $\alpha_1, \ldots, \alpha_k \in (\overline{\mathbb{Q}}^*)^k$ be algebraic numbers and let $P_1, \ldots, P_k \in \overline{\mathbb{Q}}[X]$ be non-zero polynomials. We consider the polynomial-exponential equation

$$P_1(x)\alpha_1^x + \cdots + P_k(x)\alpha_k^x = 0. \tag{2.2}$$

Put for simplicity $t = t(\mathbf{P})$ and $t^* = 1 + a(\mathbf{P})$. Assume $t \geq 3$ and $\hbar \in (0, 1]$. We suppose that

$$\max_{i,j} h((\alpha_i : \alpha_j)) \geq \hbar.$$

Let us define

$$E = 16t^2 \cdot t^*/\hbar, \qquad F = (8t)^{4(t-1)^4(t+2)} + 5E \log E.$$

Lemma 2.3 (Counterpart to [6, Lemma 7]). *There exist $k$-tuples* $\mathbf{P}^{(w)} = (P_1^{(w)}, \dots, P_k^{(w)}) \neq (0, \dots, 0)$ $(1 \le w < F)$ *of polynomials with*

$$\deg P_i^{(w)} \le \deg P_i \quad (1 \le w < F,\ 1 \le i < k),$$
$$\deg P_k^{(w)} < \deg P_k \quad (1 \le w < F)$$

*such that every solution* $x \in \mathbb{Z}$ *of* (2.2) *satisfies*

$$P_1^{(w)}(x)\alpha_1^x + \dots + P_k^{(w)}(x)\alpha_k^x = 0$$

*for some* $w \in [1, F)$.

*Proof.* We replace, in Schmidt's proof, Lemma 5 of [6] by Lemma 2.2. From (4.10) of [6] and from the discussion following that formula, we see that we can take

$$F = C(t,2) + 5E\log E = (8t)^{4(t-1)^4(t+2)} + 5E\log E. \ \blacksquare$$

Let $\alpha$, $\beta$ be complex numbers. We write $\alpha \approx \beta$ if $\alpha$, $\beta$ are non-zero and $\alpha/\beta$ is a root of unity.

Let $a_1, \dots, a_q, \alpha_1, \dots, \alpha_q \in \mathbb{C}$ and

$$f(x) = a_1\alpha_1^x + \dots + a_q\alpha_q^x.$$

Consider a partition of the summands such that $a_i\alpha_i^x$ and $a_j\alpha_j^x$ are in the same part if and only if $\alpha_i \approx \alpha_j$. After relabeling, one can write

$$f(x) = f_1(x) + \dots + f_g(x)$$

where

$$f_i(x) = a_{i,1}\alpha_{i,1}^x + \dots + a_{i,q_i}\alpha_{i,q_i}^x \qquad (i = 1, \dots, g)$$

with $q_1 + \dots + q_g = q$ and

$$\alpha_{i,j} \approx \alpha_{i,k} \quad \text{when } 1 \le i \le g,\ 1 \le j,k \le q_i;$$
$$\alpha_{i,j} \not\approx \alpha_{i',k} \quad \text{when } 1 \le i \neq i' \le g,\ 1 \le j \le q_i,\ 1 \le k \le q_{i'}.$$

Lemma 2.4 (Counterpart to [6, Lemma 8]). *All but at most*

$$G(q) = (8q)^{4(q-1)^3q^2}$$

*solutions* $x \in \mathbb{Z}$ *of* $f(x) = 0$ *have* $f_1(x) = \dots = f_g(x) = 0$.

*Proof.* In the proof of [6, Lemma 8], we replace [6, Lemma 4] by our Lemma 2.1. From the last formula of [6], p. 258, we see that it is enough to show that $C(q,1) + 2^qG(q-1) \le G(q)$ for $q \ge 2$. This arises from

$$\begin{aligned} C(q,1) + 2^qG(q-1) &\le (8q)^{4(q-1)^4(q+1)} + 2^q(8(q-1))^{4(q-2)^3(q-1)^2} \\ &\le (1+2^q)(8q)^{4(q-1)^2\max((q-1)^2(q+1),(q-2)^3)} \\ &\le (8q)^{q+4(q-1)^2\max((q-1)^2(q+1),(q-2)^3)} \le (8q)^{4(q-1)^3q^2} \end{aligned}$$

(use $x + 4(x-1)^2\max((x-1)^2(x+1),(x-2)^3) \le 4(x-1)^3x^2$ for $x \ge 2$). $\blacksquare$

**2.2. Main proposition.** We improve the main proposition at the beginning of Section 3 of [7] replacing in (3.4) of that proposition the value of $H(T)$ by $(8T)^{4(T-1)T^4}$. For the convenience of the reader we recall the statement of that proposition, which is the core of [7]. Let

$$M_j(\mathbf{X}) = a_{1,j}X_1 + \cdots + a_{k,j}X_k \quad (j = 1, \ldots, n)$$

be linear forms with algebraic coefficients which are linearly independent over $\mathbb{Q}$. We write $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n})$ and assume that each $\mathbf{a}_i \neq (0, \ldots, 0)$ $(i = 1, \ldots, k)$. We define $t_i$ to be the integer such that

$$\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,t_i}, 0, \ldots, 0)$$

with $a_{i,t_i} \neq 0$. Set

$$t = t_1 + \cdots + t_k, \qquad T = \min(k^n, e^{12t}), \qquad \hbar = \hbar(T) = e^{-6T^4}.$$

PROPOSITION 2.5. *Suppose that $\alpha_1, \ldots, \alpha_k$ are non-zero algebraic numbers. Consider $x \in \mathbb{Z}$ for which*

$$M_1(\alpha_1^x, \ldots, \alpha_k^x), \ldots, M_n(\alpha_1^x, \ldots, \alpha_k^x)$$

*are linearly independent over $\mathbb{Q}$. These numbers fall into at most*

$$H(T) = (8T)^{4(T-1)T^4} \tag{2.3}$$

*classes with the following properties. For each class $C$ there is a natural number $m$ such that*

(a) *solutions $x$, $x'$ in $C$ have $x \equiv x' \bmod m$,*
(b) *there are $i \neq j$ such that either $\alpha_i \not\approx \alpha_j$ and $h(\alpha_i/\alpha_j) \geq \hbar$, or $\alpha_i \approx \alpha_j$ and $\operatorname{ord}(\alpha_i^m/\alpha_j^m) \leq \hbar^{-1}$.*

*Proof.* We remark that $t \geq k$. Moreover, $t \geq n$. Indeed $\max t_i = n$, otherwise we had $M_n = 0$, contradicting the hypothesis on the linear independence of $M_1, \ldots, M_n$.

The case $k = 1$ is trivial, as remarked at the beginning of [7, Section 6]. Assume $k \geq 2$ and $n = 1$. In the proof, we replace Lemma 8 of [6] by our Lemma 2.4. Then the last equation of [7, Section 6, p. 625] can be replaced by

$$\begin{aligned} G(k) + 2^k \cdot k^{3k^2} &= (8k)^{4(k-1)^3k^2} + 2^k \cdot k^{3k^2} \\ &\leq (8k)^{4(k-1)^3k^2+3k^2} \leq H(k) = H(T) \end{aligned}$$

because $4(x-1)^3x^2 + 3x^2 \leq 4(x-1)x^4$ for $x \geq 2$ and in addition $n = 1$ yields $T = k$.

As in [7, Section 7] we assume $k \geq 2$ and $n \geq 2$. Thus

$$T = \min(k^n, e^{12t}) \geq \min(2^n, e^{12n}) = 2^n.$$

Again, we replace Lemma 8 of [6] by our Lemma 2.4. Then [7, (7.11)] is replaced by (use $(3T)^n \le (8T)^T$)

$$|\mathcal{S}''|G(T) < (3T)^n(8T)^{4(T-1)^3T^2} < (8T)^{T+4(T-1)^3T^2}.$$

The estimate at the end of [7, p. 630] becomes (use $3^n \le 4^n \le T^2$)

$$\begin{aligned}(8T)^{T+4(T-1)^3T^2} + \exp(5T^3 + 3^nT) &\le (8T)^{T+4(T-1)^3T^2} + \exp(6T^3)\\ &\le (8T)^{T+4(T-1)^3T^2+6T^3} \le H(T)\end{aligned}$$

because $x + 4(x-1)^3x^2 + 6x^3 \le 4(x-1)x^4$ for $x \ge 3$ and $T \ge 2^n \ge 4$. ∎

**3. Conclusion.** We closely follow [7, Section 3]. For a non-zero polynomial $P \in \mathbb{C}[X]$ we recall that $t(P) = 1 + \deg(P)$ (with the convention $t(0) = 0$). For a vector $\mathbf{P} = (P_1, \ldots, P_k) \in (\mathbb{C}[X])^k$, as before we put $t(\mathbf{P}) = t(P_1) + \cdots + t(P_k)$ and $a(\mathbf{P}) = \max_i t(P_i)$. Moreover, we let

$$T(\mathbf{P}) = \min(k^{a(\mathbf{P})}, e^{12t(\mathbf{P})}).$$

For a set $\mathcal{Z}$ of integers, let $\nu(\mathcal{Z})$ be the minimum $\nu$ such that $\mathcal{Z}$ can be expressed as the union of $\nu$ arithmetic progressions. We agree that single elements of $\mathbb{Z}$ are trivial arithmetic progressions and that $\nu(\mathcal{Z}) = \infty$ if $\mathcal{Z}$ cannot be expressed as such a union. Notice that for a finite set $\mathcal{Z}$, $\nu(\mathcal{Z})$ is simply the cardinality of $\mathcal{Z}$.

Let $\{u_m\}$ be a linear recurrence of order $t$ with companion polynomial $\mathcal{P}$. Write

$$\mathcal{P}(z) = c_0 \prod_{i=1}^{k} (z - \alpha_i)^{a_i}$$

with distinct roots $\alpha_1, \ldots, \alpha_k$. Then

$$u_m = P_1(m)\alpha_1^m + \cdots + P_k(m)\alpha_k^m$$

where $P_i$ is a polynomial of degree $< a_i$ ($i = 1, \ldots, k$). Thus, we have to consider the polynomial-exponential equation

$$P_1(x)\alpha_1^x + \cdots + P_k(x)\alpha_k^x = 0.$$

Let $\mathcal{Z} = \mathcal{Z}(\mathbf{P})$ be the set of integers $x$ satisfying this equation. We have $t = t(\mathbf{P})$. Put for simplicity $a = a(\mathbf{P}) = \max a_i$ and $T = T(\mathbf{P})$.

**3.1. Proof of Theorem 1.1.** By induction on $t$, we prove that

$$\nu(\mathcal{Z}) \le Z(t, T) = (8T)^{(2^t-1)8T^5}. \tag{3.1}$$

As in [7], we may suppose $k \ge 2$ and $t \ge 3$. Since $k \ge 2$, we have $k^{a-1} \ge a$. Thus $k^a \ge ka \ge t$ and $T \ge t \ge 3$.

We consider the solutions of [7, (3.9)]

$$\sum_{r=1}^{n}\Big(\sum_{j=1}^{a} c_{j,r}x^{j-1}\Big)M_r(\alpha_1^x,\dots,\alpha_k^x)=0. \tag{3.2}$$

There are fewer than $a$ numbers $x\in\mathbb{Z}$ such that each polynomial $\sum_{j=1}^{a} c_{j,r}x^{j-1}$ ($r=1,\dots,n$) vanishes. For other solutions of (3.2) the numbers $M_r(\alpha_1^x,\dots,\alpha_k^x)$ ($r=1,\dots,n$) are linearly dependent over $\mathbb{Q}$. By Proposition 2.5, these numbers fall into at most

$$H(T)=(8T)^{4(T-1)T^4}$$

classes. Fix one class $C$.

Proposition 2.5 leads to two cases. Let us consider first the case where there are $i\neq j$ such that $\alpha_i\approx\alpha_j$ and $\operatorname{ord}(\alpha_i^m/\alpha_j^m)\le\hbar(T)^{-1}$. In this case, by [7, (3.12) and the inequality just before (3.12)], the set $\mathcal{Z}_C$ of solutions in our class satisfies

$$\nu(\mathcal{Z}_C)\le\exp(6T^4)Z(t-1,T)\le(8T)^{4(T-1)T^4}Z(t-1,T)$$

(use $6x^4\le 4(x-1)^2x^4$ for $x\ge 3$).

We now consider the case where there are $i\neq j$ such that $\alpha_i\not\approx\alpha_j$ and $h(\alpha_i/\alpha_j)\ge\hbar(T)$. We replace [6, Lemma 7] by our Lemma 2.3. Let, as in that lemma,

$$F=(8t)^{4(t-1)^4(t+2)}+5E\log E.$$

In the present situation, thanks to the inequality just before [7, (3.14)], we have $E\log E<\exp(8T^4)$. Thus (3.14) is replaced by

$$\begin{aligned}F&<(8t)^{4(t-1)^4(t+2)}+5\exp(8T^4)\le 6(8T)^{\max(4(T-1)^4(T+2),4T^4)}\\&\le(8T)^{4(T-1)T^4}\end{aligned}$$

(recall that $t\le T$ and use $e\le\sqrt{8}$, $1+\max(4(x-1)^4(x+2),4x^4)\le 4(x-1)x^4$ for $x\ge 3$). Inequality (3.17) of [7] now reads

$$\nu(\mathcal{Z}_C)\le FZ(t-1,T)^2<(8T)^{4(T-1)T^4}Z(t-1,T)^2. \tag{3.3}$$

Therefore, in both cases of Proposition 2.5,

$$\nu(\mathcal{Z}_C)\le(8T)^{4(T-1)T^4}Z(t-1,T)^2.$$

Thus, using the new value (2.3) of $H(T)$ in Proposition 2.5 and the inductive hypothesis, the inequality which follows (3.17) in [7] becomes

$$\begin{aligned}\nu(\mathcal{Z})&<a+H(T)(8T)^{4(T-1)T^4}Z(t-1,T)^2\\&\le T+(8T)^{4(T-1)T^4+4(T-1)T^4+2(2^{t-1}-1)8T^5}\\&\le(8T)^{1-8T^4+(2^t-1)8T^5}\le Z(t,T).\end{aligned} \tag{3.4}$$

Hence (3.1) is established. Since $T \le e^{12t}$, we deduce

$$\begin{aligned}\nu(\mathcal{Z}) &\le (8e^{12t})^{2^t\cdot 8e^{60t}} \le \exp\exp(t\log 2 + \log 8 + 60t + \log(\log 8 + 12t))\\ &\le \exp\exp(70t). \ \blacksquare\end{aligned}$$

**3.2. Proof of Theorem 1.2.** We follow the proof above. We show by induction on $t$ that

$$|\mathcal{Z}| \le Z(t,T) = (8T)^{8T^5 t}. \tag{3.5}$$

As in the proof of Theorem 1.1, Proposition 2.5 leads to two cases. However, the case

$$\exists i \ne j, \quad \alpha_i \approx \alpha_j, \quad \operatorname{ord}(\alpha_i^m/\alpha_j^m) \le \hbar(T)^{-1}$$

does not occur, since $\{u_m\}$ is not degenerate. More importantly, the case

$$\exists i \ne j, \quad \alpha_i \not\approx \alpha_j, \quad h(\alpha_i/\alpha_j) \ge \hbar(T)$$

has no additional troubles with non-trivial arithmetic progressions (see the paragraph in [7] between (3.14) and (3.15)). Thus, inequality (3.3) can be replaced by

$$\nu(\mathcal{Z}_C) \le FZ(t-1,T) < (8T)^{4(T-1)T^4} Z(t-1,T)$$

saving a square on $Z(t-1,T)$. In turn, (3.4) becomes

$$\begin{aligned}\nu(\mathcal{Z}) &< a + H(T)(8T)^{4(T-1)T^4} Z(t-1,T)\\ &\le T + (8T)^{4(T-1)T^4 + 4(T-1)T^4 + (t-1)8T^5} \le (8T)^{1-8T^4+8T^5 t} \le Z(t,T).\end{aligned}$$

Hence (3.5) is established. Since $t \le T \le k^n \le k^a$ and $k \ge 2$, we immediately deduce that

$$|\mathcal{Z}| \le (8k^a)^{8k^{6a}} = \exp(8k^{6a}(3\log 2 + a\log k)) \le \exp(32ak^{6a}\log k). \ \blacksquare$$

As mentioned in the introduction, in Theorem 1.1, we could try to combine our improvement with Allen's refined version of [7, Lemma 2], to obtain a lower bound of the shape

$$\exp\exp(c\sqrt{t}\log t). \tag{3.6}$$

We have not been able to do that. In the degenerate case the growth in $t$ of $Z(t,T)$ is double exponential. So, as for Theorem 1.2, we are not able to get further advantage. Neither in the non-degenerate case can we obtain a bound of the kind (3.6). Allen takes

$$H(T) = \exp(4(6T)^{3T})$$

in the main proposition (see [1, Proposition in Section 5.6]) and replaces

$$T = \min(k^n, e^{12t})$$

by $\min(k^n, e^{\sqrt{2}t})$. This does not work with a function $H(T)$ which has a simple exponential growth. Indeed, the number of systems of 3-element sets has a double exponential growth in $n$ ([6, Section 11]).

**Acknowledgments.** We thank Y. Bugeaud for calling our attention to Schmidt's result [7].

Research of E. Viada was supported by the Fonds National Suisse (FNS).

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*Received on 17.10.2009*
*and in revised form on 16.4.2010* (6178)