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Representations by quaternary quadratic forms whose coefficients are 1, 3 and 9

by

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1. Introduction. For $n \in \mathbb{N}$ we set

$$\sigma(n) = \sum_{d \mid n} d, \tag{1.1}$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$.

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we define

$$N(a, b, c, d; n) = \operatorname{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}. \tag{1.2}$$

Clearly

$$N(a, b, c, d; 0) = 1. \tag{1.3}$$

For $q \in \mathbb{C}$ with $|q| < 1$ we have

$$\sum_{n=0}^{\infty} N(a, b, c, d; n) q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d), \tag{1.4}$$

where $\varphi(q)$ denotes *Ramanujan's theta function*,

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.5}$$

There are ten quaternary quadratic forms $ax^2+by^2+cz^2+dt^2$ with $a, b, c, d \in \{1, 3, 9\}$ and $1 = a \le b \le c \le d$. Formulae for $N(a, b, c, d; n)$ $(n \in \mathbb{N})$ with $(a, b, c, d) = (1, 1, 1, 1)$, $(1, 1, 1, 3)$, $(1, 1, 3, 3)$, $(1, 3, 3, 3)$ appear in the literature (see [1], [2]). In this paper we treat the remaining six forms $(1, 1, 1, 9)$, $(1, 1, 3, 9)$, $(1, 1, 9, 9)$, $(1, 3, 3, 9)$, $(1, 3, 9, 9)$ and $(1, 9, 9, 9)$.

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Definition 1.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty} (1 - q^{kn}).$$

The infinite product representation of $\varphi(q)$ is due to Jacobi [7]:

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2}. \tag{1.6}$$

Definition 1.2. For $n \in \mathbb{N}$ we set

(a) $$A(n) := \sum_{d \mid n} d \left(\frac{12}{n/d} \right) = \sum_{d \mid n} \frac{n}{d} \left(\frac{12}{d} \right),$$

(b) $$B(n) := \sum_{d \mid n} d \left(\frac{-3}{d} \right) \left(\frac{-4}{n/d} \right) = \sum_{d \mid n} \frac{n}{d} \left(\frac{-3}{n/d} \right) \left(\frac{-4}{d} \right),$$

(c) $$C(n) := \sum_{d \mid n} d \left(\frac{-3}{n/d} \right) \left(\frac{-4}{d} \right) = \sum_{d \mid n} \frac{n}{d} \left(\frac{-3}{d} \right) \left(\frac{-4}{n/d} \right),$$

(d) $$D(n) := \sum_{d \mid n} d \left(\frac{12}{d} \right) = \sum_{d \mid n} \frac{n}{d} \left(\frac{12}{n/d} \right).$$

It is shown in [2, Sections 4 and 8] for $n \in \mathbb{N}$ that

$$N(1,1,1,3;n) = 6A(n) - 2B(n) + 3C(n) - D(n), \tag{1.7}$$

$$N(1,3,3,3;n) = 2A(n) + 2B(n) - C(n) - D(n). \tag{1.8}$$

We determine an explicit formula for $N(1,3,3,9;n)$ in terms of $\sigma(n/k)$ ($k \in \{1,2,3,4,6,12\}$) by using the formula for $N(1,1,3,3;n)$ proved in [1, p. 297] (see Theorem 1.1). We use the formulae for $N(1,1,1,3;n)$ and $N(1,3,3,3;n)$ proved in [2, pp. 228, 231] to determine explicit formulae for $N(1,1,3,9;n)$ and $N(1,3,9,9;n)$ in terms of $A(n)$, $B(n)$, $C(n)$ and $D(n)$ (see Theorems 1.2 and 1.3). Finally, we use the (p,k)-parametrization of $\varphi(q)$ given in [3] and [5] to determine explicit formulae for $N(1,1,1,9;n)$, $N(1,1,9,9;n)$ and $N(1,9,9,9;n)$ in terms of $\sigma(n/k)$ ($k \in \{1,2,3,6,9,36\}$) and the integers $c(n)$ ($n \in \mathbb{N}$) defined by

$$\sum_{n=1}^{\infty} c(n) q^n = q E_6^4 \tag{1.9}$$

(see Theorems 1.4, 1.5 and 1.6). Clearly $c(n) = 0$ if $n \not\equiv 1 \pmod 6$. It follows from [8, Vol. II, p. 374] (see also [11, p. 121]) that

$$c(n) = \frac{1}{3} \sum_{\substack{x,y=-\infty \\ n=x^2+3xy+3y^2 \\ x\equiv 2 \pmod 3 \\ y\equiv 1 \pmod 2}}^{\infty} (-1)^x x.$$

A numerical study showed that $c(n)$ cannot be expressed linearly in terms of $\sigma(n)$, $A(n)$, $B(n)$, $C(n)$ and $D(n)$ when $n \equiv 1 \pmod 6$.

THEOREM 1.1. *For $n \in \mathbb{N}$,*

$$N(1,3,3,9;n) = \begin{cases} 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) & \\ +16\sigma(n/4) + 24\sigma(n/6) - 48\sigma(n/12) & \text{if } n \equiv 0 \pmod 3, \\ 2\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

THEOREM 1.2. *For $n \in \mathbb{N}$,*

$$N(1,3,9,9;n) = \begin{cases} 2A(n/3) + 2B(n/3) - C(n/3) - D(n/3) & \text{if } n \equiv 0 \pmod 3, \\ 2A(n) - \frac{2}{3}B(n) + C(n) - \frac{1}{3}D(n) & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

THEOREM 1.3. *For $n \in \mathbb{N}$,*

$$N(1,1,3,9;n) = \begin{cases} 2A(n/3) + 2B(n/3) - C(n/3) - D(n/3) & \text{if } n \equiv 0 \pmod 3, \\ 4A(n) - \frac{4}{3}B(n) + 2C(n) - \frac{2}{3}D(n) & \text{if } n \equiv 1 \pmod 3, \\ 2A(n) - \frac{2}{3}B(n) + C(n) - \frac{1}{3}D(n) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

THEOREM 1.4. *For $n \in \mathbb{N}$,*

$$N(1,1,1,9;n) = \begin{cases} 2\sigma(n) + 4c(n) & \text{if } n \equiv 1 \pmod 6, \\ 12\sigma(n) - 24\sigma(n/2) & \text{if } n \equiv 2 \pmod 6, \\ 8\sigma(n/3) & \text{if } n \equiv 3 \pmod 6, \\ 6\sigma(n) - 12\sigma(n/2) & \text{if } n \equiv 4 \pmod 6, \\ 4\sigma(n) & \text{if } n \equiv 5 \pmod 6, \\ 24\sigma(n/3) - 48\sigma(n/6) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

THEOREM 1.5. *For* $n \in \mathbb{N}$,

$$N(1,1,9,9;n) = \begin{cases} \frac{4}{3}\sigma(n) + \frac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 6, \\ 4\sigma(n) - 8\sigma(n/2) & \text{if } n \equiv 2,\ 4 \pmod 6, \\ 8\sigma(n/9) & \text{if } n \equiv 3 \pmod 6, \\ \frac{4}{3}\sigma(n) & \text{if } n \equiv 5 \pmod 6, \\ 8\sigma(n/9) - 32\sigma(n/36) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

THEOREM 1.6. *For* $n \in \mathbb{N}$,

$$N(1,9,9,9;n) = \begin{cases} \frac{2}{3}\sigma(n) + \frac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 6, \\ 0 & \text{if } n \equiv 2,\ 5 \pmod 6, \\ 8\sigma(n/9) & \text{if } n \equiv 3 \pmod 6, \\ 2\sigma(n) - 4\sigma(n/2) & \text{if } n \equiv 4 \pmod 6, \\ 8\sigma(n/9) - 32\sigma(n/36) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Theorems 1.1–1.3 appear to be new. In [12] Petr considered representations by the forms $x^2+y^2+z^2+9t^2$, $x^2+y^2+9z^2+9t^2$, $x^2+9y^2+9z^2+9t^2$. A different formulation of Theorem 1.4 has been given by Lomadze [10, Theorem 1a, p. 120]. Theorem 1.5 is due to Lomadze [9, p. 161], who obtained it using the theory of modular functions. Our approach is more elementary.

We close this introduction by recalling the (p,k)-parametrization of theta functions introduced in [3]. We then use it to obtain the (p,k)-parametrization of $\sum_{n=1}^{\infty} c(n)q^n$. As in [3, p. 178] we set

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \tag{1.10}$$

$$k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{1.11}$$

When there is no risk of confusion we write p for $p(q)$ and k for $k(q)$. The following results were proved in [3, Theorems 9, 10].

DUPLICATION PRINCIPLE.

$$p(q^2) = \frac{1+p-p^2-((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{\big(1+p-p^2+((1-p)(1+p)(1+2p))^{1/2}\big)k}{2}.$$

TRIPLICATION PRINCIPLE.

$$p(q^3) = 3^{-1}\big((-4-3p+6p^2+4p^3) + 2^{2/3}(1-2p-2p^2)((1-p)(1+2p)(2+p))^{1/3} + 2^{1/3}(1+2p)((1-p)(1+2p)(2+p))^{2/3}\big),$$

$$k(q^3) = 3^{-2}\big(3+2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} + 2^{4/3}((1-p)(1+2p)(2+p))^{2/3}\big)k.$$

Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q) := qE_1^{24} \tag{1.12}$$

(see [13, eq. (92)], [14, p. 151]). It was shown in [4, eq. (3.32)] that

$$\Delta(q^6) = \frac{1}{256}p^6(1+p)^6(1-p)^2(1+2p)^2(2+p)^2k^{12}. \tag{1.13}$$

By (1.9), (1.12) and (1.13), we obtain

$$\sum_{n=1}^{\infty} c(n)q^n = qE_6^4 = (q^6E_6^{24})^{1/6} = \Delta(q^6)^{1/6} = 2^{-4/3}p(1+p)(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{1/3}k^2.$$

Using $p(1+p) = 2^{-2}((1+2p)^2-1)$, we obtain the following result.

LEMMA 1.1.

$$\sum_{n=1}^{\infty} c(n)q^n = 2^{-10/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2 - 2^{-10/3}(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{1/3}k^2.$$

2. Identities involving φ. In this section we obtain the (p,k)-parametrization of $\varphi(q^9)$ (Theorem 2.2) and deduce some identities involving $\varphi(q)$, $\varphi(q^3)$ and $\varphi(q^9)$, which are used in Sections 6–8 (see Theorems 2.3–2.5).

It was shown in [2, eq. (2.3)] that

$$\varphi(q) = (1+2p)^{3/4}k^{1/2}, \tag{2.1}$$

$$\varphi(q^3) = (1+2p)^{1/4}k^{1/2}. \tag{2.2}$$

It is shown in [1, p. 297] that

$$N(1,1,3,3;n) = 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) + 24\sigma(n/6) - 48\sigma(n/12). \tag{2.3}$$

Our first theorem gives an alternative formulation of the second part of the triplication principle.

THEOREM 2.1.

$$k(q^3) = \left(\frac{(1+2p)^{2/3} + 2^{2/3}(1-p)^{1/3}(2+p)^{1/3}}{3}\right)^3 k. \tag{2.4}$$

Proof. We have

$$\begin{aligned}
&\left(\frac{(1+2p)^{2/3} + 2^{2/3}(1-p)^{1/3}(2+p)^{1/3}}{3}\right)^3 k \\
&\quad = \tfrac{1}{27}\big((1+2p)^2 + 3\cdot 2^{2/3}(1-p)^{1/3}(1+2p)^{4/3}(2+p)^{1/3} \\
&\qquad\quad + 3\cdot 2^{4/3}(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3} + 2^2(1-p)(2+p)\big)k \\
&\quad = \tfrac{1}{27}\big(9 + 3\cdot 2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\
&\qquad\quad + 3\cdot 2^{4/3}((1-p)(1+2p)(2+p))^{2/3}\big)k \\
&\quad = \tfrac{1}{9}\big(3 + 2^{2/3}(1+2p)((1-p)(1+2p)(2+p))^{1/3} \\
&\qquad\quad + 2^{4/3}((1-p)(1+2p)(2+p))^{2/3}\big)k \\
&\quad = k(q^3),
\end{aligned}$$

by the second part of the triplication principle. ■

Theorem 2.1 enables us to give the (p,k)-parametrization of $\varphi(q^9)$.

THEOREM 2.2.

$$\varphi(q^9) = \tfrac{1}{3}(1+2p)^{1/12}\big((1+2p)^{2/3} + 2^{2/3}(1-p)^{1/3}(2+p)^{1/3}\big)k^{1/2}.$$

Proof. Replacing q by q^3 in (1.11) we obtain $\varphi^3(q^9) = \varphi(q^3)k(q^3)$. Then, by Theorem 2.1 and (2.2), we obtain

$$\varphi^3(q^9) = (1+2p)^{1/4}k^{1/2}\left(\frac{(1+2p)^{2/3} + 2^{2/3}(1-p)^{1/3}(2+p)^{1/3}}{3}\right)^3 k.$$

Taking the cube root of each side, we obtain the asserted formula. ■

THEOREM 2.3.

$$\varphi(q)\varphi^3(q^9) = \varphi^2(q)\varphi^2(q^9) - \tfrac{1}{3}\varphi^3(q)\varphi(q^9) + \tfrac{1}{3}\varphi^4(q^3).$$

Proof. This identity follows from (2.1), (2.2) and Theorem 2.2. ■

From Theorem 2.3 we obtain

$$3\,\frac{\varphi(q^9)}{\varphi(q)} = 1 + \left(\frac{9\varphi^4(q^3)}{\varphi^4(q)} - 1\right)^{1/3}, \tag{2.5}$$

which was proved in [6, p. 345].

THEOREM 2.4.

$$\varphi^2(q)\varphi^2(q^9) = \tfrac{1}{6}\varphi^4(q) - \tfrac{2}{3}\varphi^4(q^3) + \tfrac{3}{2}\varphi^4(q^9) + \tfrac{8}{3}\sum_{n=1}^{\infty} c(n)q^n.$$

Proof. By (2.1) and Theorem 2.2, we obtain

$$\begin{aligned}\varphi^2(q)\varphi^2(q^9) &= \tfrac{1}{9}(1+2p)^{5/3}\big((1+2p)^{4/3}+2^{5/3}(1-p)^{1/3}(1+2p)^{2/3}(2+p)^{1/3}\\ &\qquad + 2^{4/3}(1-p)^{2/3}(2+p)^{2/3}\big)k^2\\ &= \tfrac{1}{9}(1+2p)^3k^2 + \tfrac{1}{9}\cdot 2^{5/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2\\ &\quad + \tfrac{1}{9}\cdot 2^{4/3}(1-p)^{2/3}(1+2p)^{5/3}(2+p)^{2/3}k^2.\end{aligned}$$

On the other hand, by Lemma 1.1, (2.1), (2.2), and Theorem 2.2, we obtain

$$\begin{aligned}&\tfrac{1}{6}\varphi^4(q) - \tfrac{2}{3}\varphi^4(q^3) + \tfrac{3}{2}\varphi^4(q^9) + \tfrac{8}{3}\sum_{n=1}^{\infty} c(n)q^n\\ &= \tfrac{1}{6}(1+2p)^3k^2 - \tfrac{2}{3}(1+2p)k^2\\ &\quad + \tfrac{3}{2}\big(\tfrac{1}{27}(-8p^3-12p^2+18p+11)k^2\\ &\qquad + \tfrac{1}{81}\cdot 2^{8/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2\\ &\qquad + \tfrac{1}{27}\cdot 2^{7/3}(1-p)^{2/3}(1+2p)^{5/3}(2+p)^{2/3}k^2\\ &\qquad + \tfrac{1}{81}\cdot 2^{8/3}(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{4/3}k^2\big)\\ &\quad + \tfrac{8}{3}\big(2^{-10/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2\\ &\qquad - 2^{-10/3}(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{1/3}k^2\big)\\ &= \tfrac{1}{9}(1+6p+12p^2+8p^3)k^2\\ &\quad + \tfrac{1}{3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2\\ &\quad \times\left(\frac{2^{5/3}}{9} + \frac{2^{5/3}(1-p)(2+p)}{9(1+2p)^2} + \frac{1}{2^{1/3}} - \frac{1}{2^{1/3}(1+2p)^2}\right)\\ &\quad + \tfrac{1}{9}\cdot 2^{4/3}(1-p)^{2/3}(1+2p)^{5/3}(2+p)^{2/3}k^2\\ &= \tfrac{1}{9}(1+2p)^3k^2 + \tfrac{1}{9}\cdot 2^{5/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2\\ &\quad + \tfrac{1}{9}\cdot 2^{4/3}(1-p)^{2/3}(1+2p)^{5/3}(2+p)^{2/3}k^2.\end{aligned}$$

This completes the proof of Theorem 2.4. ∎

The following identity is a classical identity due to Jacobi [7]:

(2.6) $$\varphi^4(q) = 1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))q^n.$$

Replacing q by q^3 and q^9 in (2.6), we obtain

(2.7) $$\varphi^4(q^3) = 1 + \sum_{n=1}^{\infty}(8\sigma(n/3) - 32\sigma(n/12))q^n,$$

(2.8) $$\varphi^4(q^9) = 1 + \sum_{n=1}^{\infty}(8\sigma(n/9) - 32\sigma(n/36))q^n.$$

The Eisenstein series $L(q)$ is defined by

$$L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{2.9}$$

It was shown in [4, eq. (3.11), p. 33] that

$$L(q^6) - 2L(q^{12}) = -\left(1 + 2p - p^3 - \tfrac{1}{2}p^4\right)k^2. \tag{2.10}$$

For $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, k-1\}$ we define

$$L_{i,k}(q) := \sum_{\substack{n=1 \\ n \equiv i \,(\mathrm{mod}\, k)}}^{\infty} \sigma(n) q^n. \tag{2.11}$$

It was shown in [3, eq. (14.6), p. 189] that

$$L_{1,2}(q^3) = \left(\tfrac{1}{8}p^3 + \tfrac{1}{16}p^4\right)k^2, \tag{2.12}$$

and in [3, p. 190] that

$$\begin{aligned} L_{1,3}(q) + 2L_{2,3}(q) &= \tfrac{1}{9}(1 + 8p + 18p^2 + 8p^3 + p^4)k^2 \\ &\quad - \tfrac{1}{9} \cdot 2^{-1/3}(1 + 3p - 3p^2 - p^3)((1-p)(1+2p)(2+p))^{1/3}k^2. \end{aligned} \tag{2.13}$$

Applying the duplication principle twice to $L_{1,3}(q) + 2L_{2,3}(q)$, we obtain

$$\begin{aligned} L_{1,3}(q^4) + 2L_{2,3}(q^4) &= \tfrac{1}{36}(-2 - 2p + p^2)^2 k^2 \\ &\quad + \tfrac{1}{72} \cdot 2^{2/3}(-4 - 6p + p^3)((1-p)(1+2p)(2+p))^{1/3}k^2. \end{aligned} \tag{2.14}$$

THEOREM 2.5.

$$\begin{aligned} \varphi^3(q)\varphi(q^9) &= 2L_{1,3}(q) + 4L_{2,3}(q) - 8L_{1,3}(q^4) - 16L_{2,3}(q^4) \\ &\quad - L(q^6) + 2L(q^{12}) + 8L_{1,2}(q^3) + 4\sum_{n=1}^{\infty} c(n)q^n. \end{aligned}$$

Proof. By (2.1) and Theorem 2.2, we have

$$\varphi^3(q)\varphi(q^9) = \tfrac{1}{3}(1+2p)^3k^2 + \tfrac{1}{3} \cdot 2^{2/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2.$$

On the other hand, by (2.10), (2.12), (2.13), (2.14), and Lemma 1.1,

$$\begin{aligned} &2L_{1,3}(q) + 4L_{2,3}(q) - 8L_{1,3}(q^4) - 16L_{2,3}(q^4) \\ &\qquad - L(q^6) + 2L(q^{12}) + 8L_{1,2}(q^3) + 4\sum_{n=1}^{\infty} c(n)q^n \\ &= 2\Big(\tfrac{1}{9}(1 + 8p + 18p^2 + 8p^3 + p^4)k^2 \\ &\qquad - \tfrac{1}{9} \cdot 2^{-1/3}(1 + 3p - 3p^2 - p^3)((1-p)(1+2p)(2+p))^{1/3}k^2\Big) \end{aligned}$$

$$
\begin{aligned}
&- 8\Big(\tfrac{1}{36}(-2-2p+p^2)^2k^2 \\
&\qquad + \tfrac{1}{72}\cdot 2^{2/3}(-4-6p+p^3)((1-p)(1+2p)(2+p))^{1/3}k^2\Big) \\
&\qquad + \big(1+2p-p^3-\tfrac{1}{2}p^4\big)k^2 + 8\big(\tfrac{1}{8}p^3+\tfrac{1}{16}p^4\big)k^2 \\
&\qquad + 4\big(2^{-10/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2 \\
&\qquad\qquad - 2^{-10/3}(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{1/3}k^2\big) \\
={}& \tfrac{2}{9}(1+8p+18p^2+8p^3+p^4)k^2 \\
&- \tfrac{1}{9}\cdot 2^{2/3}(1+3p-3p^2-p^3)((1-p)(1+2p)(2+p))^{1/3}k^2 \\
&- \tfrac{2}{9}(4+8p-4p^3+p^4)k^2 \\
&- \tfrac{1}{9}\cdot 2^{2/3}(-4-6p+p^3)((1-p)(1+2p)(2+p))^{1/3}k^2 + (1+2p)k^2 \\
&+ 2^{-4/3}(1+2p)^2((1-p)(1+2p)(2+p))^{1/3}k^2 \\
&- 2^{-4/3}((1-p)(1+2p)(2+p))^{1/3}k^2 \\
={}& \tfrac{1}{3}(1+2p)^3k^2 \\
&+ ((1-p)(1+2p)(2+p))^{1/3}k^2\left(\frac{2^{2/3}(1+p+p^2)}{3} + \frac{(1+2p)^2}{2^{4/3}} - \frac{1}{2^{4/3}}\right) \\
={}& \tfrac{1}{3}(1+2p)^3k^2 \\
&+ ((1-p)(1+2p)(2+p))^{1/3}k^2\,\frac{4(1+2p)^2}{3\cdot 2^{4/3}} \\
={}& \tfrac{1}{3}(1+2p)^3k^2 + \tfrac{1}{3}\cdot 2^{2/3}(1-p)^{1/3}(1+2p)^{7/3}(2+p)^{1/3}k^2.
\end{aligned}
$$

This completes the proof of Theorem 2.5. ∎

3. Evaluation of $N(1,3,3,9;n)$: Proof of Theorem 1.1. First, we prove the following lemma.

LEMMA 3.1. *Let $n\in\mathbb{N}$. Then*

$$
N(1,3,3,9;n) = \begin{cases} N(1,1,3,3;n/3) & \text{if } n\equiv 0 \pmod 3,\\ \tfrac{1}{2}N(1,1,3,3;n) & \text{if } n\equiv 1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3.\end{cases}
$$

Proof. Let $n\equiv 0 \pmod 3$. If $n = x^2+3y^2+3z^2+9t^2$, then $x\equiv 0 \pmod 3$. Hence

$$
\begin{aligned}
N(1,3,3,9;n) &= \operatorname{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+9t^2\} \\
&= \operatorname{card}\{(x,y,z,t)\in\mathbb{Z}^4 \mid n = x^2+3y^2+3z^2+9t^2,\ x\equiv 0 \pmod 3\} \\
&= \operatorname{card}\{(x_1,y,z,t)\in\mathbb{Z}^4 \mid n = 9x_1^2+3y^2+3z^2+9t^2\} \\
&= \operatorname{card}\{(y,z,x_1,t)\in\mathbb{Z}^4 \mid n = 3y^2+3z^2+9x_1^2+9t^2\} \\
&= \operatorname{card}\{(y,z,x_1,t)\in\mathbb{Z}^4 \mid n/3 = y^2+z^2+3x_1^2+3t^2\} \\
&= N(1,1,3,3;n/3).
\end{aligned}
$$

Let $n \equiv 1 \pmod 3$. If $n = x^2 + y^2 + 3z^2 + 3t^2$, then $x^2 + y^2 \equiv 1 \pmod 3$, and so either $x \equiv 0$, $y \not\equiv 0$ or $x \not\equiv 0$, $y \equiv 0$ modulo 3. Hence

$$\begin{aligned}
&N(1,1,3,3;n)\\
&\quad = \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 3t^2\}\\
&\quad = \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 3t^2,\ x \equiv 0, y \not\equiv 0 \pmod 3\}\\
&\qquad + \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 3t^2,\ x \not\equiv 0, y \equiv 0 \pmod 3\}\\
&\quad = 2\operatorname{card}\{(x_1,y,z,t) \in \mathbb{Z}^4 \mid n = 9x_1^2 + y^2 + 3z^2 + 3t^2,\ y \not\equiv 0 \pmod 3\}\\
&\quad = 2\operatorname{card}\{(y,z,t,x_1) \in \mathbb{Z}^4 \mid n = y^2 + 3z^2 + 3t^2 + 9x_1^2\}\\
&\quad = 2N(1,3,3,9;n).
\end{aligned}$$

If $n \equiv 2 \pmod 3$, then clearly $N(1,3,3,9;n) = 0$, which completes the proof of Lemma 3.1. ∎

Proof of Theorem 1.1. Let $n \equiv 0 \pmod 3$. By Lemma 3.1 and (2.3), we have

$$\begin{aligned}
N(1,3,3,9;n) &= 4\sigma(n/3) - 8\sigma(n/6) - 12\sigma(n/9) + 16\sigma(n/12)\\
&\quad + 24\sigma(n/18) - 48\sigma(n/36).
\end{aligned} \tag{3.1}$$

On the other hand, for $n \equiv 0 \pmod 3$ we have the elementary identities

$$\sigma(n) - 4\sigma(n/3) + 3\sigma(n/9) = 0, \tag{3.2}$$

$$\sigma(n/2) - 4\sigma(n/6) + 3\sigma(n/18) = 0, \tag{3.3}$$

$$\sigma(n/4) - 4\sigma(n/12) + 3\sigma(n/36) = 0. \tag{3.4}$$

By substituting (3.2)–(3.4) into (3.1), we obtain

$$\begin{aligned}
N(1,3,3,9;n) &= 4\sigma(n/3) - 8\sigma(n/6) - 4(4\sigma(n/3) - \sigma(n)) + 16\sigma(n/12)\\
&\quad + 8(4\sigma(n/6) - \sigma(n/2)) - 16(4\sigma(n/12) - \sigma(n/4))\\
&= 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4)\\
&\quad + 24\sigma(n/6) - 48\sigma(n/12),
\end{aligned}$$

which completes the proof of Theorem 1.1 for $n \equiv 0 \pmod 3$. When $n \equiv 1 \pmod 3$, the assertion follows from Lemma 3.1 and (2.3). When $n \equiv 2 \pmod 3$, the assertion follows directly from Lemma 3.1.

4. Evaluation of $N(1,3,9,9;n)$: Proof of Theorem 1.2.

First, we prove the following lemma.

LEMMA 4.1. *Let $n \in \mathbb{N}$. Then*

$$N(1,3,9,9;n) = \begin{cases} N(1,3,3,3;n/3) & \text{if } n \equiv 0 \pmod 3,\\ \frac{1}{3}N(1,1,1,3;n) & \text{if } n \equiv 1 \pmod 3,\\ 0 & \text{if } n \equiv 2 \pmod 3.\end{cases}$$

Proof. The case $n \equiv 2 \pmod 3$ is trivial. The case $n \equiv 0 \pmod 3$ can be treated in a manner similar to the proof of Lemma 3.1. Thus, we prove the lemma only when $n \equiv 1 \pmod 3$. Clearly $n = x^2 + y^2 + z^2 + 3t^2 \equiv 1 \pmod 3$ if and only if $x^2 + y^2 + z^2 \equiv 1 \pmod 3$. The only possible cases are $x \equiv 0,\ y \equiv 0,\ z \not\equiv 0$ or $x \equiv 0,\ y \not\equiv 0,\ z \equiv 0$ or $x \not\equiv 0,\ y \equiv 0,\ z \equiv 0 \pmod 3$. Hence

$$\begin{aligned}
N(1,1,1,3;n) &= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2\} \\
&= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ x \equiv 0,\ y \equiv 0,\ z \not\equiv 0 \pmod 3\} \\
&\quad + \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ x \equiv 0,\ y \not\equiv 0,\ z \equiv 0 \pmod 3\} \\
&\quad + \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2,\ x \not\equiv 0,\ y \equiv 0,\ z \equiv 0 \pmod 3\} \\
&= 3 \operatorname{card}\{(x_1,y_1,z,t) \in \mathbb{Z}^4 \mid n = 9x_1^2 + 9y_1^2 + z^2 + 3t^2,\ z \not\equiv 0 \pmod 3\} \\
&= 3 \operatorname{card}\{(z,t,x_1,y_1) \in \mathbb{Z}^4 \mid n = z^2 + 3t^2 + 9x_1^2 + 9y_1^2\} \\
&= 3N(1,3,9,9;n),
\end{aligned}$$

so

$$N(1,3,9,9;n) = \tfrac{1}{3} N(1,1,1,3;n), \qquad n \equiv 1 \pmod 3. \tag{4.1}$$ ■

The proof of Theorem 1.2 now follows from Lemma 4.1, (1.7) and (1.8).

5. Evaluation of $N(1,1,3,9;n)$: Proof of Theorem 1.3. First, we prove the following lemma.

LEMMA 5.1. *Let $n \in \mathbb{N}$. Then*

$$N(1,1,3,9;n) = \begin{cases} N(1,3,3,3;n/3) & \text{if } n \equiv 0 \pmod 3, \\ \frac{2}{3} N(1,1,1,3;n) & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{3} N(1,1,1,3;n) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 3$. Then $n = x^2 + y^2 + 3z^2 + 9t^2 \equiv 0 \pmod 3$ if and only if $x \equiv y \equiv 0 \pmod 3$. Hence

$$\begin{aligned}
N(1,1,3,9;n) &= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 9t^2\} \\
&= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 9t^2,\ x \equiv y \equiv 0 \pmod 3\} \\
&= \operatorname{card}\{(x_1,y_1,z,t) \in \mathbb{Z}^4 \mid n = 9x_1^2 + 9y_1^2 + 3z^2 + 9t^2\} \\
&= \operatorname{card}\{(z,x_1,y_1,t) \in \mathbb{Z}^4 \mid n/3 = z^2 + 3x_1^2 + 3y_1^2 + 3t^2\} \\
&= N(1,3,3,3;n/3).
\end{aligned}$$

Let $n \equiv 1 \pmod 3$. Then $n = x^2 + y^2 + 3z^2 + 9t^2 \equiv 1 \pmod 3$ if and only if $x^2 + y^2 \equiv 1 \pmod 3$. Thus, either $x \equiv 0$, $y \not\equiv 0 \pmod 3$, or $x \not\equiv 0$, $y \equiv 0 \pmod 3$. Hence

$$\begin{aligned} N(1,1,3,9;n) &= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 9t^2\} \\ &= 2\operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + 3z^2 + 9t^2,\ x \equiv 0, y \not\equiv 0 \pmod 3\} \\ &= 2\operatorname{card}\{(x_1,y,z,t) \in \mathbb{Z}^4 \mid n = 9x_1^2 + y^2 + 3z^2 + 9t^2\} \\ &= 2\operatorname{card}\{(y,z,x_1,t) \in \mathbb{Z}^4 \mid n = y^2 + 3z^2 + 9x_1^2 + 9t^2\} \\ &= 2N(1,3,9,9;n). \end{aligned}$$

We have

$$N(1,1,3,9;n) = 2N(1,3,9,9;n), \quad n \equiv 1 \pmod 3. \tag{5.1}$$

Then, by (4.1) and (5.1), we obtain

$$N(1,1,3,9;n) = \tfrac{2}{3}N(1,1,1,3;n), \quad n \equiv 1 \pmod 3.$$

Let $n \equiv 2 \pmod 3$. Then $n = x^2 + y^2 + z^2 + 3t^2 \equiv 2 \pmod 3$ if and only if $x \equiv 0$, $y \not\equiv 0$, $z \not\equiv 0$ or $x \not\equiv 0$, $y \equiv 0$, $z \not\equiv 0$ or $x \not\equiv 0$, $y \not\equiv 0$, $z \equiv 0 \pmod 3$. Hence

$$\begin{aligned} N(1,1,1,3;n) &= \operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2\} \\ &= 3\operatorname{card}\{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + 3t^2, \\ &\qquad\qquad x \equiv 0,\ y \not\equiv 0, z \not\equiv 0 \pmod 3\} \\ &= 3\operatorname{card}\{(x_1,y,z,t) \in \mathbb{Z}^4 \mid n = 9x_1^2 + y^2 + z^2 + 3t^2,\ y \not\equiv 0,\ z \not\equiv 0 \pmod 3\} \\ &= 3\operatorname{card}\{(y,z,t,x_1) \in \mathbb{Z}^4 \mid n = y^2 + z^2 + 3t^2 + 9x_1^2\} \\ &= 3N(1,1,3,9;n), \end{aligned}$$

which completes the proof of Lemma 5.1. ∎

Theorem 1.3 now follows from Lemma 5.1, (1.7) and (1.8).

6. Evaluation of $N(1,1,1,9;n)$: Proof of Theorem 1.4. By Theorem 2.5 and (1.4), (2.10)–(2.14) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(1,1,1,9;n)q^n &= 2L_{1,3}(q) + 4L_{2,3}(q) - 8L_{1,3}(q^4) - 16L_{2,3}(q^4) \\ &\quad - L(q^6) + 2L(q^{12}) + 8L_{1,2}(q^3) + 4\sum_{n=1}^{\infty} c(n)q^n \end{aligned}$$

$$= 2 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n)q^n + 4 \sum_{\substack{n=1 \\ n\equiv 2 \pmod 3}}^{\infty} \sigma(n)q^n - 8 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} \sigma(n)q^{4n}$$

$$- 16 \sum_{\substack{n=1 \\ n\equiv 2 \pmod 3}}^{\infty} \sigma(n)q^{4n} - \Big(1 - 24\sum_{n=1}^{\infty} \sigma(n)q^{6n}\Big)$$

$$+ 2\Big(1 - 24\sum_{n=1}^{\infty} \sigma(n)q^{12n}\Big) + 8 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n)q^{3n} + 4\sum_{n=1}^{\infty} c(n)q^n$$

$$= 1 + 24\sum_{n=1}^{\infty} \sigma(n/6)q^n - 48\sum_{n=1}^{\infty} \sigma(n/12)q^n$$

$$+ \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} (2\sigma(n) - 8\sigma(n/4))q^n + \sum_{\substack{n=1 \\ n\equiv 2 \pmod 3}}^{\infty} (4\sigma(n) - 16\sigma(n/4))q^n$$

$$+ 8 \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \sigma(n/3)q^n + 4\sum_{n=1}^{\infty} c(n)q^n.$$

Equating the coefficients of q^n ($n \in \mathbb{N}$), we obtain

$$\text{(6.1)} \qquad N(1,1,1,9;n) = \begin{cases} 2\sigma(n) + 4c(n) & \text{if } n \equiv 1 \pmod 6, \\ 4\sigma(n) - 16\sigma(n/4) & \text{if } n \equiv 2 \pmod 6, \\ 8\sigma(n/3) & \text{if } n \equiv 3 \pmod 6, \\ 2\sigma(n) - 8\sigma(n/4) & \text{if } n \equiv 4 \pmod 6, \\ 4\sigma(n) & \text{if } n \equiv 5 \pmod 6, \\ 24\sigma(n/6) - 48\sigma(n/12) & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

When $n \equiv 0 \pmod 2$ we have the elementary identity

$$\text{(6.2)} \qquad \sigma(n) - 3\sigma(n/2) + 2\sigma(n/4) = 0.$$

Then

$$N(1,1,1,9;n) = 4\sigma(n) - 16\sigma(n/4) = 12\sigma(n) - 24\sigma(n/2) \quad \text{if } n \equiv 2 \pmod 6,$$

$$N(1,1,1,9;n) = 2\sigma(n) - 8\sigma(n/4) = 6\sigma(n) - 12\sigma(n/2) \quad \text{if } n \equiv 4 \pmod 6,$$

$$N(1,1,1,9;n) = 24\sigma(n/6) - 48\sigma(n/12) = 24\sigma(n/3) - 48\sigma(n/6) \quad \text{if } n \equiv 0 \pmod 6,$$

which completes the proof of Theorem 1.4.

7. Evaluation of $N(1,1,9,9;n)$: Proof of Theorem 1.5. By (1.4), Theorem 2.4, and (2.6)–(2.8), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,9,9;n)q^n &= \varphi^2(q)\varphi^2(q^9) \\
&= \tfrac{1}{6}\varphi^4(q) - \tfrac{2}{3}\varphi^4(q^3) + \tfrac{3}{2}\varphi^4(q^9) + \tfrac{8}{3}\sum_{n=1}^{\infty} c(n)q^n \\
&= \tfrac{1}{6}\Big(1 + \sum_{n=1}^{\infty}(8\sigma(n) - 32\sigma(n/4))q^n\Big) \\
&\quad - \tfrac{2}{3}\Big(1 + \sum_{n=1}^{\infty}(8\sigma(n/3) - 32\sigma(n/12))q^n\Big) \\
&\quad + \tfrac{3}{2}\Big(1 + \sum_{n=1}^{\infty}(8\sigma(n/9) - 32\sigma(n/36))q^n\Big) + \tfrac{8}{3}\sum_{n=1}^{\infty} c(n)q^n \\
&= 1 + \sum_{n=1}^{\infty}\big(\tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/3) - \tfrac{16}{3}\sigma(n/4) \\
&\qquad + 12\sigma(n/9) + \tfrac{64}{3}\sigma(n/12) - 48\sigma(n/36) + \tfrac{8}{3}c(n)\big)q^n.
\end{aligned}$$

Equating the coefficients of q^n ($n \in \mathbb{N}$), and recalling that $c(n) = 0$ if $n \not\equiv 1 \pmod 6$, we obtain

$$\text{(7.1)}\quad N(1,1,9,9;n) = \begin{cases}
\tfrac{4}{3}\sigma(n) + \tfrac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 6, \\
\tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/4) & \text{if } n \equiv 2,\, 4 \pmod 6, \\
\tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/3) + 12\sigma(n/9) & \text{if } n \equiv 3 \pmod 6, \\
\tfrac{4}{3}\sigma(n) & \text{if } n \equiv 5 \pmod 6, \\
\tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/3) & \\
\quad - \tfrac{16}{3}\sigma(n/4) + 12\sigma(n/9) & \\
\quad + \tfrac{64}{3}\sigma(n/12) - 48\sigma(n/36) & \text{if } n \equiv 0 \pmod 6.
\end{cases}$$

When $n \equiv 0 \pmod 3$ we have the elementary identity

$$\text{(7.2)}\qquad \sigma(n) - 4\sigma(n/3) + 3\sigma(n/9) = 0.$$

Then, by (7.1), (6.2) and (7.2), we obtain

$$\begin{aligned}
N(1,1,1,9;n) &= \tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/4) = 4\sigma(n) - 8\sigma(n/2) \quad \text{if } n \equiv 2,\, 4 \pmod 6, \\
N(1,1,1,9;n) &= \tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/4) + 12\sigma(n/9) \\
&= 8\sigma(n/9) \quad \text{if } n \equiv 3 \pmod 6,
\end{aligned}$$

$$\begin{aligned}N(1,1,1,9;n) &= \tfrac{4}{3}\sigma(n) - \tfrac{16}{3}\sigma(n/3) - \tfrac{16}{3}\sigma(n/4)\\ &\quad + 12\sigma(n/9) + \tfrac{64}{3}\sigma(n/12) - 48\sigma(n/36)\\ &= 8\sigma(n/9) - 32\sigma(n/36) \quad \text{if } n \equiv 0 \pmod 6,\end{aligned}$$

which completes the proof of Theorem 1.5.

8. Evaluation of $N(1,9,9,9;n)$: Proof of Theorem 1.6. By (1.4) and Theorem 2.3, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,9,9,9;n)q^n = \varphi(q)\varphi^3(q^9)\\ &= \varphi^2(q)\varphi^2(q^9) - \tfrac{1}{3}\varphi^3(q)\varphi(q^9) + \tfrac{1}{3}\varphi^4(q^3)\\ &= \sum_{n=0}^{\infty} N(1,1,9,9;n)q^n - \tfrac{1}{3}\sum_{n=0}^{\infty} N(1,1,1,9;n)q^n + \tfrac{1}{3}\sum_{n=0}^{\infty} N(1,1,1,1;n)q^{3n}.\end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$, we obtain

(8.1) $N(1,9,9,9;n) = N(1,1,9,9;n) - \tfrac{1}{3}N(1,1,1,9;n) + \tfrac{1}{3}N(1,1,1,1;n/3).$

From (2.7) we have

(8.2) $$N(1,1,1,1;n/3) = 8\sigma(n/3) - 32\sigma(n/12), \qquad n \in \mathbb{N}.$$

By Theorem 1.4, Theorem 1.5 and (8.2), we obtain

(8.3) $$N(1,9,9,9;n) = \begin{cases} \tfrac{2}{3}\sigma(n) + \tfrac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 6,\\ 0 & \text{if } n \equiv 2,\,5 \pmod 6,\\ 8\sigma(n/9) & \text{if } n \equiv 3 \pmod 6,\\ 2\sigma(n) - 4\sigma(n/2) & \text{if } n \equiv 4 \pmod 6,\\ -\tfrac{16}{3}\sigma(n/3) + 16\sigma(n/6) + 8\sigma(n/9) & \\ \quad - \tfrac{32}{3}\sigma(n/12) - 32\sigma(n/36) & \text{if } n \equiv 0 \pmod 6.\end{cases}$$

If $n \equiv 0 \pmod 6$, then $n \equiv 0 \pmod 2$, and so by (6.2), we have

$$-\tfrac{16}{3}\sigma(n/3) + 16\sigma(n/6) - \tfrac{32}{3}\sigma(n/12) = 0,$$

which completes the proof of Theorem 1.6. ∎

We conclude the paper by noting that for $n \in \mathbb{N}$,

$$N(1,1,9,9;n) = \begin{cases} N(1,1,1,1;n/9) & \text{if } n \equiv 0 \pmod 3,\\ 2N(1,9,9,9;n) & \text{if } n \equiv 1 \pmod 3,\\ \tfrac{1}{3}N(1,1,1,9;n) & \text{if } n \equiv 2 \pmod 3,\end{cases}$$

which follows from Theorem 1.4, Theorem 1.6 and (8.2). This result can also be proved independently of Theorems 1.4 and 1.6 in a similar manner to the proof of Lemma 5.1.

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