On the height of algebraic numbers with real conjugates

by

JOHN GARZA (Austin, TX)

1. Introduction. Mahler’s measure of a polynomial $f$, denoted by $M(f)$, is defined as the product of the absolute values of those roots of $f$ that lie outside the unit disk, multiplied by the absolute value of the leading coefficient. If $f(x) = b \prod_{i=1}^{d} (x - \alpha_i)$, then $M(f) = |b| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$. For an algebraic number $\alpha$, let $M(\alpha) \equiv M(f)$ where $f$ is the minimal polynomial of $\alpha$ over $\mathbb{Z}$. If $f \in \mathbb{Z}[x]$, then $M(f) \geq 1$, and it is a theorem of Kronecker that for $f \in \mathbb{Z}[x]$, $M(f) = 1$ if and only if $\pm f$ is a product of a power of $x$ and cyclotomic polynomials. It follows from a result of Schinzel ([2, Corollary 1′]) that if $\alpha \neq 0, \pm 1$ is a totally real algebraic number of degree $d$ then

$$M(\alpha) \geq \left(\frac{1 + \sqrt{5}}{2}\right)^{d/2}.$$  

This article establishes the following generalization of the last inequality.

**Theorem 1.** Let $\alpha$ be an algebraic number, different from $0$ and $\pm 1$. Let $\Lambda$ be the set of Galois conjugates of $\alpha$ that are real and suppose that $|\Lambda| \neq 0$. Let $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ and let $R_{\alpha} \equiv |\Lambda|/d$. Let $\beta = 1 - 1/R_{\alpha}$. Then

$$M(\alpha) \geq \log\left(\frac{2\beta + \sqrt{4\beta^2 + 4}}{2}\right)^{dR_{\alpha}/2}.$$  

It is a natural question to ask whether the full Corollary 1′ of [2] can be generalized in the same way. We mention that in the case $0 < R_{\alpha} < (\log 2)/(3 \log d)$, $\alpha$ an integer, and $d > d_0$, Theorem 2 of Blanksby and Montgomery [1] gives a stronger result.

Amongst the absolute values in a place $v$ of an algebraic number field, $\mathbb{K}$, two play a role in this article. If $v$ is archimedean, let $|| \cdot ||_v$ denote the unique absolute value in $v$ which restricts to the usual absolute value on $\mathbb{Q}$. If $v$ is non-archimedean and $v | p$, let $|| \cdot ||_v$ denote the unique absolute value

2000 Mathematics Subject Classification: Primary 11R04.

Key words and phrases: Mahler measure, Weil height, real roots.
in \( v \) restricting to the usual \( p \)-adic absolute value on \( \mathbb{Q} \). For each place \( v \) of \( \mathbb{K} \), let \( \mathbb{K}_v \) and \( \mathbb{Q}_v \) denote the completions of \( \mathbb{K} \) and \( \mathbb{Q} \) with respect to \( v \) and define the local degree as \( d_v \equiv [\mathbb{K}_v : \mathbb{Q}_v] \). Let \( | \cdot |_v = | \cdot |_{\mathbb{K}_v}^{d_v/d} \).

The absolute values \( | \cdot |_v \) satisfy the product rule: if \( \alpha \in \mathbb{K} \times \), then \( \prod_v |\alpha|_v = 1 \). The absolute (logarithmic) Weil height of \( \alpha \) is defined as \( h(\alpha) = \sum_v \log^+ |\alpha|_v \) where the sum is over all places \( v \) of \( \mathbb{K} \). Because of the way in which the absolute values \( | \cdot |_v \) are normalized, the absolute Weil height of \( \alpha \) does not depend on the field \( \mathbb{K} \) in which \( \alpha \) is contained. If \( \alpha_i \) and \( \alpha_j \) are algebraic numbers, then \( h(\alpha_i \cdot \alpha_j) \leq h(\alpha_i) + h(\alpha_j) \); if \( \alpha_i \) and \( \alpha_j \) are Galois conjugates, then \( h(\alpha_i) = h(\alpha_j) \); and for an algebraic number \( \alpha \), \( h(\alpha) = h(\alpha/1) \).

We define \( d \cdot h(\alpha) = \log M(\alpha) \). We provide the following additional result concerning the Weil height of algebraic numbers.

**Theorem 2.** Let \( \mathbb{K}/\mathbb{Q} \) be a Galois extension of finite degree. Let \( G \equiv \text{Aut}(\mathbb{K}/\mathbb{Q}) \). Let \( \alpha \in \mathbb{K} \times \) have a Galois conjugate not on the archimedean unit circle. Let \( \sigma : \mathbb{K} \rightarrow \mathbb{C} \) be an embedding. Let \( \xi \in G \) correspond to complex conjugation with respect to \( \sigma \). Let \( C_G(\xi) = \{ x \in G : x\xi = \xi x \} \).

Let \( n = [G : C_G(\xi)] \). Let \( \theta(\alpha) = 1 \) if \( \alpha \) has a real Galois conjugate and let \( \theta(\alpha) = 2 \) if \( \alpha \) does not have a real Galois conjugate. Then

\[
h(\alpha) \geq \log \left( \frac{2^{1-n} + \sqrt{4^{1-n} + 4}}{2} \right)^{1/(2\theta(\alpha)n)}.
\]

**2. Proof of Theorem 1.** Let \( | \cdot |_\infty \) be the usual archimedean absolute value on \( \mathbb{R} \). Let \( \delta \equiv 1 - \alpha^2 \). For each place \( v \) of \( \mathbb{K} \) let

\[
b_v \max\{1, \|\alpha^2\|_v\} = \|\delta\|_v.
\]

By the ultrametric inequality, for each \( v \nmid \infty \) we have \( b_v \leq 1 \).

For each \( \gamma \in \Lambda \) define

\[
\|1 - \gamma^2\|_\infty = a_\gamma \max\{1, \|\gamma^2\|_\infty\}.
\]

Then

\[
a_\gamma = \begin{cases} \|1 - 1/\gamma^2\|_\infty & \text{if } \|\gamma\|_\infty > 1, \\ \|1 - \gamma^2\|_\infty & \text{if } \|\gamma\|_\infty < 1. \end{cases}
\]

We define

\[
\gamma' = \begin{cases} 1/\gamma & \text{if } \|\gamma\|_\infty > 1, \\ \gamma & \text{if } \|\gamma\|_\infty < 1. \end{cases}
\]

We thus have

\[
\prod_{\gamma \in \Lambda} (\gamma')^2 \geq \frac{1}{(e^{dh(\alpha)})^4}.
\]
Using the arithmetic-geometric mean inequality twice we have

\[ \prod_{\gamma \in \Lambda} (1 - (\gamma')^2) \leq \left( \frac{1}{|\Lambda|} \left( \sum_{\gamma \in \Lambda} (1 - (\gamma')^2) \right) \right)^{|\Lambda|} = \left( 1 - \frac{1}{|\Lambda|} \sum_{\gamma \in \Lambda} (\gamma')^2 \right)^{|\Lambda|} \]

\[ \leq \left( 1 - \left( \prod_{\gamma \in \Lambda} (\gamma')^2 \right)^{1/|\Lambda|} \right)^{|\Lambda|} \leq \left( 1 - \left( \frac{1}{(e^{dh(\alpha)})^{4}} \right)^{1/dR_\alpha} \right)^{dR_\alpha}. \]

By the triangle inequality, \( b_v \leq 2 \) for all \( v | \infty \). Let

\[ B \equiv \prod_v b_v^{d_v/d}. \]

We recall that \( \sum_{v | \infty} d_v = d \). From the Galois action on places we have

\[ B \leq 2^{1-R_\alpha} \left( 1 - \left( \frac{1}{(e^{dh(\alpha)})^{4}} \right)^{1/dR_\alpha} \right)^{R_\alpha}. \]

If \( dR_\alpha = |\Lambda| \) is sufficiently large in comparison to \( e^{dh(\alpha)} \) it follows that \( B < 1 \).

Fix \( v \). We have \( \|\delta\|_v = \|\delta\|^{d/d_v}_v = b_v \max\{1, \|\alpha\|^2_v\} \). Consequently,

\[ \log |\delta|_v = (d_v/d)(\log b_v + 2 \log^+ \|\alpha\|_v). \]

Summing over all places and using the product rule yields

\[ 0 = \sum_v \log |\delta|_v, \]

\[ 0 = \sum_v \log b_v^{d_v/d} + 2 \sum_v \log^+ |\alpha|_v, \]

\[ 0 = \log B + 2h(\alpha). \]

We thus have

\[ h(\alpha) = \frac{1}{2} \log(1/B), \]

\[ h(\alpha) \geq \frac{1}{2} \log \left( 2^{R_\alpha - 1} \left( 1 - \left( \frac{1}{(e^{dh(\alpha)})^{4}} \right)^{1/dR_\alpha} \right)^{-R_\alpha} \right), \]

\[ dh(\alpha) \geq \frac{d}{2} \log \left( 2^{R_\alpha - 1} \left( 1 - \left( \frac{1}{(e^{dh(\alpha)})^{4}} \right)^{1/dR_\alpha} \right)^{-R_\alpha} \right), \]

\[ dh(\alpha) \geq \log \left( 2^{R_\alpha - 1} \left( 1 - \left( \frac{1}{(e^{dh(\alpha)})^{4}} \right)^{1/dR_\alpha} \right)^{-R_\alpha} \right)^{d/2}. \]

We notice that for fixed \( d \) and \( R_\alpha \), if \( h(\alpha) \) decreases the right hand side of the inequality increases. As a result, the inequality implies a lower bound
on $h(\alpha)$. We now deduce as follows:

$$e^{dh(\alpha)} \geq \left(2^{R_{\alpha}-1} \left(1 - \frac{1}{(e^{dh(\alpha)})^{1/dR_{\alpha}}} \right)^{1/dR_{\alpha}} - R_{\alpha} \right)^{d/2},$$

$$(e^{dh(\alpha)})^{2/d} \geq 2^{R_{\alpha}-1} \left(\frac{(e^{dh(\alpha)})^{1/dR_{\alpha}}}{(e^{dh(\alpha)})^{1/dR_{\alpha}} - 1} \right)^{R_{\alpha}},$$

$$(e^{dh(\alpha)})^{2/dR_{\alpha}} \geq 2^{\beta} \frac{(e^{dh(\alpha)})^{4/dR_{\alpha}}}{(e^{dh(\alpha)})^{4/dR_{\alpha}} - 1},$$

$$1 \geq 2^{\beta} \frac{(e^{dh(\alpha)})^{2/dR_{\alpha}}}{(e^{dh(\alpha)})^{4/dR_{\alpha}} - 1},$$

$$(e^{dh(\alpha)})^{4/dR_{\alpha}} - 1 \geq 2^{\beta} (e^{dh(\alpha)})^{2/dR_{\alpha}},$$

$$((e^{dh(\alpha)})^{2/dR_{\alpha}})^{2} - 2^{\beta} (e^{dh(\alpha)})^{2/dR_{\alpha}} - 1 \geq 0.$$ From the quadratic formula we deduce that

$$M(\alpha) = e^{dh(\alpha)} \geq \left(\frac{2^{\beta} + \sqrt{4^{\beta} + 4}}{2} \right)^{dR_{\alpha}/2}.$$  

3. Proof of Theorem 2. If $\alpha$ does not have a real Galois conjugate let $\gamma \equiv \alpha \xi(\alpha)$, and if $\alpha$ has a real Galois conjugate, $\tau$, let $\gamma = \tau$. Since $\alpha$ does not have all its conjugates on the archimedean unit circle, we can assume that $\gamma \neq \pm 1$. Let $H_{Q(\gamma)}$ denote the subgroup of $G$ that fixes the field $Q(\gamma)$. Let $N_{G}(H_{Q(\gamma)}) = \{x \in G : xH_{Q(\gamma)}x^{-1} = H_{Q(\gamma)}\}$. From Galois theory we recall that $\{G : N_{G}(H_{Q(\gamma)})\}$ is the number of subfields of $K$ that are distinct from and conjugate to $Q(\gamma)$. We have

$$\frac{|C_{G}(\xi)|}{|C_{G}(\xi) \cap N_{G}(H_{Q(\gamma)})|} \geq \frac{|C_{G}(\xi)|}{|N_{G}(H_{Q(\gamma)})|} = \frac{1}{n} \cdot \frac{|G|}{|N_{G}(H_{Q(\gamma)})|}.$$ Consequently, at least $1/n$ of the elements of the orbit of $Q(\gamma)$ under $G/N_{G}(H_{Q(\gamma)})$ are the images of $Q(\gamma)$ by elements of $C_{G}(\xi)$ so that at least $1/n$ of the Galois conjugates of $\gamma$ are real: $R_{\gamma} \geq 1/n$. It then follows from Theorem 1 that

$$h(\alpha) \geq \log \left(\frac{2^{1-n} + \sqrt{4^{1-n} + 4}}{2} \right)^{1/(2\theta(\alpha)n)}.$$  

4. An application to Lehmer’s problem

Corollary 3. For $n \in \mathbb{N}$ let $H_{n} \equiv (2^{1-n} + \sqrt{4^{1-n} + 4})/2$. Let $K/Q$ be a Galois extension of finite degree. Let $C(Aut(K/Q))$ be the center of $Aut(K/Q)$. Let $n \equiv |Aut(K/Q) : C(Aut(K/Q))|$. Let $\alpha \in \mathcal{O}_{K}$ be different from the roots of unity such that $K$ is the Galois closure of $Q(\alpha)$. Let $a \in (1, \infty)$. If $[K : Q] \geq (4n^{2} \log a)/(\log H_{n})$ then $M(\alpha) \geq a$.  


Proof. Let $G \equiv \text{Aut}(\mathbb{K}/\mathbb{Q})$. Let $H_{\mathbb{Q}(\alpha)}$ be the subgroup of $G$ that fixes the field $\mathbb{Q}(\alpha)$. By Galois theory we have $C(G) \cap H_{\mathbb{Q}(\alpha)} = \{1\}$ from which it follows that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \geq |G|/n$. By Theorem 2 we have

$$h(\alpha) \geq \log H_n^{1/4n}.$$ 

Suppose that

$$[\mathbb{K} : \mathbb{Q}] = |G| \geq \frac{4n^2 \log a}{\log H_n}.$$

Then

$$\log(M(\alpha)) = [\mathbb{Q}(\alpha) : \mathbb{Q}] \cdot h(\alpha) \geq \log H_n^{[G]/4n^2} \geq \log a. \quad \blacktriangleleft$$

References


Department of Mathematics
The University of Texas at Austin
1 University Station, C1200
Austin, TX 78712, U.S.A.
E-mail: jgarza@math.utexas.edu

Received on 17.12.2006
and in revised form on 20.3.2007 (5346)