Univoque numbers and an avatar of Thue–Morse

by

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1. Introduction. Komornik and Loreti determined in [17] the smallest univoque real number in the interval (1, 2), i.e., the smallest number \( \lambda \in (1, 2) \) such that 1 has a unique expansion \( 1 = \sum_{j \geq 0} a_j/\lambda^{j+1} \) with \( a_j \in \{0, 1\} \) for every \( j \geq 0 \). The word “univoque” in this context seems to have been introduced (with a slightly different meaning) by Daróczy and Kátai in [12, 13], while unique expansions of the real number 1 were characterized by Erdős, Joó, and Komornik in [14]. The first author and Cosnard showed in [4] how the result of [17] parallels (and can be deduced from) their study of a certain set of binary sequences arising in the study of iterations of unimodal continuous functions on the unit interval (see [11, 2, 1]). The relevant sets of binary sequences occurring in [2, 1], resp. [17], can be defined by

\[
\Gamma := \{ A \in \{0, 1\}^N : \forall k \geq 0, \ \overline{A} \leq \sigma^k A \leq A \}, \quad \Gamma_{\text{strict}} := \{ A \in \{0, 1\}^N : \forall k \geq 1, \ \overline{A} < \sigma^k A < A \},
\]

where \( \sigma \) is the shift on sequences and the bar operation replaces 0’s by 1’s and 1’s by 0’s, i.e., if \( A = (A_n)_{n \geq 0} \), then \( \sigma A = (a_{n+1})_{n \geq 0} \) and \( \overline{A} = (1 - a_n)_{n \geq 0} \); furthermore, \( \leq \) denotes the lexicographical order on sequences induced by \( 0 < 1 \), the notation \( A < B \) meaning as usual that \( A \leq B \) and \( A \neq B \). The smallest univoque number in (1, 2) and the smallest nonperiodic sequence in \( \Gamma \) both involve the Thue–Morse sequence (see for example [6] for more on this sequence).

It is tempting to generalize these sets to alphabets with more than two letters.

DEFINITION 1. For \( b \) a positive integer, we will say that the real number \( \lambda > 1 \) is \( \{0, 1, \ldots, b\}\)-univoque if the number 1 has a unique expansion as

\[
1 = \sum_{j \geq 0} a_j \lambda^{-(j+1)}, \quad a_j \in \{0, 1, \ldots, b\} \text{ for all } j \geq 0.
\]

Furthermore, if \( \lambda > 1 \) is \( \{0, 1, \ldots, [\lambda] - 1\}\)-univoque, we will simply say that \( \lambda \) is univoque.

Key words and phrases: beta-expansion, univoque numbers, iteration of continuous functions, Thue–Morse sequence, uniform morphism, automatic sequence, transcendence.

DOI: 10.4064/aa136-4-2

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Remark 1. If \( \lambda > 1 \) is \( \{0, 1, \ldots, b\} \)-univoque for some positive integer \( b \), then \( \lambda \leq b + 1 \). Also note that any integer \( q \geq 2 \) is univoque, since there is exactly one expansion \( 1 = \sum_{j \geq 0} a_j q^{-(j+1)} \) with \( a_j \in \{0, 1, \ldots, q-1\} \), namely \( 1 = \sum_{j \geq 0} (q-1)q^{-(j+1)} \).

Komornik and Loreti studied in [18] the reals \( \lambda \in (1, b+1] \) that are \( \{0, 1, \ldots, b\} \)-univoque. For their study, they introduced admissible sequences on the alphabet \{0, 1, \ldots, b\}. Denote, as above, by \( \sigma \) the shift on sequences, and by \( \overline{t} \) the operation that replaces every \( t \in \{0, 1, \ldots, b\} \) by \( b-t \), i.e., if \( A = (a_n)_{n \geq 0} \), then \( \overline{A} := (b-a_n)_{n \geq 0} \). Also denote by \( \preceq \) the lexicographical order on sequences induced by the natural order on \{0, 1, \ldots, b\}. Then a sequence \( A = (a_n)_{n \geq 0} \) on \{0, 1, \ldots, b\} is admissible if
\[
\forall k \geq 0 \text{ such that } a_k < b, \quad \sigma^{k+1} A < A,
\]
\[
\forall k \geq 0 \text{ such that } a_k > 0, \quad \sigma^{k+1} A > \overline{A}.
\]
(Note that our notation is not exactly the notation of [18], since our sequences are indexed by \( \mathbb{N} \) and not \( \mathbb{N} \setminus \{0\} \).) These sequences have the following property: the map that associates with a real \( \lambda \in (1, b+1] \) the sequence of coefficients \( (a_j)_{j \geq 0} \in \{0, 1, \ldots, b\} \) of the greedy (i.e., lexicographically largest) expansion of 1, \( 1 = \sum_{j \geq 0} a_j \lambda^{-(j+1)} \), is a bijection from the set of \( \{0, 1, \ldots, b\} \)-univoque \( \lambda \)'s to the set of admissible sequences on \{0, 1, \ldots, b\} (see [18]).

Now there are two possible generalizations of the result of [17] on the smallest univoque number in (1,2), i.e., the smallest admissible binary sequence. One is to look at the smallest (if any) admissible sequence on the alphabet \{0, 1, \ldots, b\}, as did Komornik and Loreti in [18], the other is to look at the smallest (if any) univoque number in \( (b, b+1) \), as did de Vries and Komornik in [22].

It so happens that the first author has already studied a generalization of the set \( \Gamma \) to the case of more than two letters (see [1, Part 3]). Interestingly enough, unlike the study of \( \Gamma \), this study was unrelated to iterations of continuous functions, being just a tempting formal arithmetico-combinatorial generalization of the study of the set \( \Gamma \) of binary sequences to a similar set of sequences with more than two values.

The purpose of the present paper is threefold:

(1) to show how the 1983 study [1, Part 3, pp. 63–90] gives both the result of Komornik and Loreti in [18] on the smallest admissible sequence on \{0, 1, \ldots, b\}, and the result of de Vries and Komornik in [22] on the smallest univoque number \( \lambda \in (b, b+1) \) where \( b \) is any positive integer;

(2) to bring to light a universal morphism that governs the smallest elements in (1) above, and to show that the infinite sequence generated by this morphism is an avatar of the Thue–Morse sequence;
(3) to prove that the smallest univoque number in \((b, b + 1)\) (where \(b\) is any positive integer) is transcendental.

The paper consists of five sections. In Section 2 we recall some results of [1, Part 3, pp. 63–90] on the generalization of the set \(\Gamma\) to a \((b + 1)\)-letter alphabet. Then we give some properties of the lexicographically least nonperiodic sequence of this set, completing the results of [1, Part 3, pp. 63–90]. In Section 3 we give two corollaries of the properties of this least sequence: one gives the result in [18], the other gives the result in [22]. The transcendence results are proven in the last section.

2. The generalized \(\Gamma\) and \(\Gamma_{\text{strict}}\) sets

Definition 2. Let \(b\) be a positive integer, and \(A\) be a finite ordered set with \(b + 1\) elements \(\alpha_0 < \alpha_1 < \cdots < \alpha_n\). The \(\bar{\alpha}\) operation is defined on \(A\) by \(\bar{\alpha}_j = \alpha_{b-j}\). We extend this operation to \(A^N\) by \((\bar{a}_n)_{n \geq 0} := (\bar{a}_n)_{n \geq 0}\). Let \(\sigma\) be the shift on \(A^N\), defined by \(\sigma((a_n)_{n \geq 0}) := (a_{n+1})_{n \geq 0}\).

We define

\[
\Gamma(A) := \{ A = (a_n)_{n \geq 0} \in A^N : a_0 = \max A, \forall k \geq 0, \bar{A} \leq \sigma^k A \leq A \},
\]

\[
\Gamma_{\text{strict}}(A) := \{ A = (a_n)_{n \geq 0} \in A^N : a_0 = \max A, \forall k \geq 1, \bar{A} < \sigma^k A < A \}.
\]

Remark 2. The set \(\Gamma(A)\) was introduced by the first author in [1, Part 3, p. 63]. Note that there is a misprint in the definition on p. 66 in [1]: \(a_{\beta-i}\) should be changed into \(a_{\beta-1-i}\) as confirmed by the rest of the text.

Remark 3. A sequence belongs to \(\Gamma_{\text{strict}}(A)\) if and only if it belongs to \(\Gamma(A)\) and is nonperiodic. Indeed, \(\sigma^k A = A\) if and only if \(A\) is \(k\)-periodic; if \(\sigma^k A = \bar{A}\), then \(\sigma^{2k} A = A\), and the sequence \(A\) is \(2k\)-periodic.

Remark 4. If the set \(A := \{i, i+1, \ldots, i+z\}\), where \(i\) and \(z\) are integers, is equipped with the natural order, then for any \(x \in A\), we have \(\bar{x} = 2i+z-x\). Indeed, following Definition 2 above, we write \(\alpha_0 := i, \alpha_1 := i+1, \ldots, \alpha_z := i+z\). Hence, for any \(j \in [0, z]\), we have \(\bar{\alpha}_j = \alpha_{z-j}\), which can be rewritten \(\bar{i+j} = i+z-j\), i.e., for any \(x \in A\), we have \(\bar{x} = i+z-(x-i) = 2i+z-x\).

A first result is that the sets \(\Gamma_{\text{strict}}(A)\) are closely linked to the set of admissible sequences whose definition was recalled in the introduction.

Proposition 1. Let \(A = (a_n)_{n \geq 0}\) be a sequence in \(\{0, 1, \ldots, b\}^N\) such that \(a_0 = t \in [0, b]\) and \(A \neq b b b \ldots\). Then \(A\) is admissible if and only if \(2t > b\) and \(A \in \Gamma_{\text{strict}}(\{b-t, b-t+1, \ldots, t\})\). (The order on \(\{b-t, b-t+1, \ldots, t\}\) is induced by the order on \(N\). From Remark 4 the \(\bar{\cdot}\) operation is given by \(\bar{j} = b - j\).)
Proof. First suppose that $2t > b$ and $A \in \Gamma_{\text{strict}}(\{b-t, b-t+1, \ldots, t\})$. Then, for all $k \geq 1$, $\overline{A} < \sigma^k A < A$, which clearly implies that $A$ is admissible. Conversely, suppose that $A$ is admissible. We thus have

\[
\forall k \geq 1 \text{ such that } a_{k-1} < b, \quad \sigma^k A < A,
\]

\[
\forall k \geq 1 \text{ such that } a_{k-1} > 0, \quad \sigma^k A > \overline{A}.
\]

We first prove that if $A$ is not a constant sequence, then

\[
\forall k \geq 1, \quad \overline{A} < \sigma^k A < A.
\]

We only prove that $\sigma^k A < A$; the remaining inequalities are proved in a similar way. If $a_{k-1} < b$, then $\sigma^k A < A$. If $a_{k-1} = b$, there are two cases: either

- $a_0 = a_1 = \cdots = a_{k-1} = b$; then if $a_k < b$ we clearly have $\sigma^k A < A$; if $a_k = b$, then the sequence $\sigma^k A$ begins with some block of $b$'s followed by a letter $< b$, thus it begins with a block of $b$'s shorter than the initial block of $b$'s in $A$, hence $\sigma^k A < A$; or
- there exists an index $\ell$ with $1 < \ell < k$ such that $a_{\ell-1} < b$ and $a_\ell = a_{\ell+1} = \cdots = a_{k-1} = b$. As $A$ is admissible, we have $\sigma^\ell A < A$. It thus suffices to prove that $\sigma^k A \leq \sigma^\ell A$. This is clearly the case if $a_k < b$. On the other hand, if $a_k = b$, the sequence $\sigma^k A$ begins with a block of $b$'s which is shorter than the initial block of $b$'s in $\sigma^\ell A$, hence $\sigma^k A \leq \sigma^\ell A$.

Now, since $a_0 = t$ and $\sigma^k A < A$ for all $k \geq 1$, we have $a_k \leq t$ for all $k \geq 0$. Similarly, since $\sigma^k A > \overline{A}$ for all $k \geq 1$, we have $a_k \geq b-t$ for all $k \geq 1$. Finally, $A > \overline{A}$ implies that $t = a_0 \geq b-t$. Thus $2t \geq b$ and $A \in \Gamma(\{b-t, b-t+1, \ldots, t\})$. Now, if $b = 2t$, then $\{b-t, b-t+1, \ldots, t\} = \{t\}$ and $\overline{t} = t$. This implies that $A = t \ t \ t \ \ldots$, which is not an admissible sequence. \hfill \qed

Remark 5. For $b = 1$, this (easy) result is given without proof in [14] and proved in [4].

We need another definition from [1].

Definition 3. Let $b$ be a positive integer, and $\mathcal{A}$ be a finite ordered set with $b+1$ elements $\alpha_0 < \alpha_1 < \cdots < \alpha_b$. We suppose that $\mathcal{A}$ is equipped with a bar operation as in Definition 2. Let $A = (a_n)_{n \geq 0}$ be a periodic sequence of smallest period $T$, and with $a_{T-1} < \max \mathcal{A}$. Let $a_{T-1} = \alpha_j$ (thus $j < b$). Then $\Phi(A)$ is the $2T$-periodic sequence beginning with $a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_{j+1} \ \overline{a_0} \ \overline{a_1} \ \ldots \ \overline{a_{T-2}} \ \alpha_{b-j-1}$, i.e.,

$\Phi((a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_j)^\infty) := (a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_{j+1} \ \overline{a_0} \ \overline{a_1} \ \ldots \ \overline{a_{T-2}} \ \alpha_{b-j-1})^\infty$.

We first prove the following easy claim.
PROPOSITION 2. The smallest element of $\Gamma(b-t, b-t+1, \ldots, t)$ (where $2t > b$) is the 2-periodic sequence $(t \, b-t) \infty = (t \, b-t \, t \, (b-t) \, t \, \ldots)$. 

Proof. Since any sequence $A = (a_n)_{n \geq 0}$ in $\Gamma(b-t, b-t+1, \ldots, t)$ begins in $t$, and satisfies $\sigma A \geq \overline{A}$, it must satisfy $a_0 = t$ and $a_1 \geq b-t$. Now if $a_0 = t$ and $a_1 = b-t$, then $A$ must be the 2-periodic sequence $(t \, b-t) \infty$ ([1, Lemma 2b, p. 73]). Since this periodic sequence trivially belongs to $\Gamma(b-t, b-t+1, \ldots, t)$, it is its smallest element. □

Denoting as usual by $\Phi^s$ the $s$th iterate of $\Phi$, we state the following theorem which is a particular case of the theorem on pp. 72–73 of [1] about the smallest elements in certain subintervals of $\Gamma(0,1, \ldots, b)$, and of the definition of $q$-mirror sequences given in [1, Section II, 1, p. 67] (here $q = 2$).

THEOREM 1 ([1]). Define $P := (t \, (b-t)) \infty = (t \, b-t \, t \, (b-t) \, t \, \ldots)$. The smallest nonperiodic sequence in $\Gamma(b-t, b-t+1, \ldots, t)$ (i.e., the smallest element of $\Gamma_{\text{strict}}(b-t, b-t+1, \ldots, t)$) is the sequence $M := \lim_{s \to \infty} \Phi^s(P)$, that actually takes the (not necessarily distinct) values $b-t, b-t+1, t-1, t$. Furthermore, this sequence $M = (m_n)_{n \geq 0} = t \ b-t+1 \ b-t \ t \ b-t \ t-1 \ \ldots$ can be recursively defined by

$$\forall k \geq 0, \ m_{2^{k}-1} = t,$$

$$\forall k \geq 0, \ m_{2^{k+1}-1} = b + 1 - t,$$

$$\forall k \geq 0, \forall j \in [0, 2^{k+1} - 2], \ m_{2^{k+1}+j} = \overline{m_j}.$$

It was proven in [1] that the sequence $\lim_{s \to \infty} \Phi^s((t \, (b-t)) \infty$ is 2-automatic (for more about automatic sequences, see [7]). The second author noted that this sequence is actually a fixed point of a uniform morphism of length 2 as soon as the cardinality of the set $\{b-t, b-t+1, \ldots, b\}$ is at least 4, i.e., $2t \geq b+3$. (Recall that we always have $t \geq b-t$, i.e., $2t \geq b$.) More precisely, we have Theorem 2 below, where the Thue–Morse sequence pops up, as in [1] and in [18], but also as in [2] and [17]. Before stating this theorem we give a definition.

DEFINITION 4. The “universal” morphism $\Theta$ is defined on $\{e_0, e_1, e_2, e_3\}$ by

$$\Theta(e_3) := e_3 e_1, \quad \Theta(e_2) := e_3 e_0, \quad \Theta(e_1) := e_0 e_3, \quad \Theta(e_0) := e_0 e_2.$$

Note that this morphism has an infinite fixed point beginning in $e_3$,

$$\Theta^\infty(e_3) = \lim_{k \to \infty} \Theta^k(e_3) = e_3 \ e_1 \ e_0 \ e_3 \ e_0 \ e_2 \ e_3 \ e_1 \ e_0 \ e_2 \ldots.$$
Theorem 2. Let \((\varepsilon_n)_{n \geq 0}\) be the Thue–Morse sequence defined by \(\varepsilon_0 = 0\) and \(\varepsilon_{2k} = \varepsilon_k\) and \(\varepsilon_{2k+1} = 1 - \varepsilon_k\) for all \(k \geq 0\). Then the smallest nonperiodic sequence \(M = (m_n)_{n \geq 0}\) in \(\Gamma(\{b - t, b - t + 1, \ldots, t\})\) satisfies
\[
\forall n \geq 0, \quad m_n = \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1.
\]
Using the morphism \(\Theta\) introduced in Definition 4 above we thus have:

- if \(2t \geq b + 3\), then \(M\) is the fixed point beginning in \(t\) of the morphism deduced from \(\Theta\) by renaming \(e_0, e_1, e_2, e_3\) respectively \(b - t, b - t + 1, t - 1, t\) (note that the condition \(2t \geq b + 3\) implies that these four numbers are distinct);
- if \(2t = b + 2\) (thus \(b - t + 1 = t - 1\)), then \(M\) is the pointwise image of the fixed point beginning in \(e_3\) of the morphism \(\Theta\) under the map \(g\) defined by \(g(e_3) := t, g(e_2) = g(e_1) := t - 1, g(e_0) := b - t\);
- if \(2t = b + 1\) (thus \(b - t = t - 1\) and \(b - t + 1 = t\)), then \(M\) is the pointwise image of the fixed point beginning in \(e_3\) of the morphism \(\Theta\) under the map \(h\) defined by \(h(e_3) = h(e_1) := t, h(e_2) = h(e_0) := t - 1\).

Proof. Let us first prove that the sequence \(M = (m_n)_{n \geq 0}\) is equal to the sequence \((u_n)_{n \geq 0}\), where \(u_n := \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1\). It suffices to prove that \((u_n)_{n \geq 0}\) satisfies the recursive relations defining \((m_n)_{n \geq 0}\) that are given in Theorem 1. Recall that \(\varepsilon_n\) is equal to the parity of the sum of the binary digits of \(n\) (see [6] for example). Hence, for all \(k \geq 0\), \(\varepsilon_{2k-1} = 0\), \(\varepsilon_{2k+1} = 1\), and \(\varepsilon_{2k} = \varepsilon_{2k+1} = 1\). This implies that for all \(k \geq 0\), \(u_{2k-1} = t\) and \(u_{2k+1} = b + 1 - t\). Furthermore, for all \(k \geq 0\) and \(j \in [0,2^k+1 - 2]\), we have \(\varepsilon_{2k+1+j} = 1 - \varepsilon_j\) and \(\varepsilon_{2k+1+j+1} = 1 - \varepsilon_{j+1}\). Hence \(u_{2k+1+j} = b - u_j = u_j\).

To show how the “universal” morphism \(\Theta\) enters the picture, we study the sequence \((v_n)_{n \geq 0}\) with values in \(\{0, 1\}^2\) defined by \(v_n := (\varepsilon_n, \varepsilon_{n+1})\) for all \(n \geq 0\). Since \(v_{2n} = (\varepsilon_n, 1 - \varepsilon_n)\) and \(v_{2n+1} = (1 - \varepsilon_n, \varepsilon_{n+1})\) for all \(n \geq 0\), we clearly have

- if \(v_n = (0, 0)\), then \(v_{2n} = (0, 1)\) and \(v_{2n+1} = (1, 0)\),
- if \(v_n = (0, 1)\), then \(v_{2n} = (0, 1)\) and \(v_{2n+1} = (1, 1)\),
- if \(v_n = (1, 0)\), then \(v_{2n} = (1, 0)\) and \(v_{2n+1} = (0, 0)\),
- if \(v_n = (1, 1)\), then \(v_{2n} = (1, 0)\) and \(v_{2n+1} = (0, 1)\).

This exactly means that \((v_n)_{n \geq 0}\) is the fixed point beginning in \((0, 1)\) of the 2-morphism
\[
\begin{align*}
(0, 0) & \to (0, 1)(1, 0), \\
(0, 1) & \to (0, 1)(1, 1), \\
(1, 0) & \to (1, 0)(0, 0), \\
(1, 1) & \to (1, 0)(0, 1).
\end{align*}
\]
We may define $e_0 := (1,0), e_1 := (1,1), e_2 := (0,0), e_3 := (0,1)$. Then the above morphism can be written

$$e_3 \to e_3e_1, \quad e_2 \to e_3e_0, \quad e_1 \to e_0e_3, \quad e_0 \to e_0e_2,$$

which is the morphism $\Theta$. The above construction shows that the sequence $(v_n)_{n \geq 0}$ is a fixed point of $\Theta$.

Now, define the map $\omega$ on $\{0,1\}^2$ by

$$\omega((x,y)) := y - (2t - b - 1)x + t - 1.$$ 

We have $\omega(v_n) = m_n$ for all $n \geq 0$. Thus

- if $2t \geq b + 3$, the sequence $(m_n)_{n \geq 0}$ takes exactly four distinct values, namely $b - t, b - t + 1, t - 1, t$. This implies that $(m_n)_{n \geq 0}$ is the fixed point beginning in $t$ of the morphism obtained from $\Theta$ by renaming the letters, i.e., $e_3 \to t, e_2 \to (t-1), e_1 \to (b - t + 1), e_0 \to (b - t)$. The morphism can thus be written $t \to t(b - t + 1), (t - 1) \to t(b - t), (b - t + 1) \to (b - t) t, (b - t) \to (b - t)(t - 1)$;

- if $2t = b + 2$ (resp. $2t = b + 1$) the sequence $(m_n)_{n \geq 0}$ takes exactly three (resp. two) values, namely $b - t, t - 1, t$ (resp. $t - 1, t$). It is still the pointwise image under $\Theta$ of the sequence $(v_n)_{n \geq 0}$. Renaming the fixed point of $\Theta$ under $g$ (resp. $h$) as in the statement of Theorem 2 only takes into account that the integers $b - t, b - t + 1, t - 1, t$ are not distinct.

**Remark 6.** The reason for the choice of indices for $e_3, e_2, e_1, e_0$ is that the order of indices is the same as the natural order on the integers $t, t - 1, b - t + 1, b - t$ to which they correspond when $2t \geq b + 3$. In particular, if $b = t = 3$, the morphism reads: $3 \to 31, 2 \to 30, 1 \to 03, 0 \to 02$. Interestingly enough, though not surprisingly, this morphism also occurs (up to renaming the letters once more) in the study of infinite square-free sequences on a 3-letter alphabet. Namely, in [9], Berstel proves that the square-free Istrail sequence [15], originally defined (with no mention of the Thue–Morse sequence) as the fixed point of the (nonuniform) morphism $0 \to 12, 1 \to 102, 2 \to 0$, is actually the pointwise image of the fixed point beginning in 1 of a 2-morphism $\Theta'$ on the 4-letter alphabet $\{0,1,2,3\}$ under the map $0 \to 0, 1 \to 1, 2 \to 2, 3 \to 0$. The morphism $\Theta'$ is given by

$$\Theta'(0) = 12, \quad \Theta'(1) = 13, \quad \Theta'(2) = 20, \quad \Theta'(3) = 21.$$ 

The reader will note immediately that $\Theta'$ is another avatar of $\Theta$ obtained by renaming letters as follows: $0 \to 2, 1 \to 3, 2 \to 0, 3 \to 1$. This, in particular, shows that the sequence $(m_n)_{n \geq 0}$, in the case where $2t = b + 2$, is the fixed point of the nonuniform morphism $t \to t(t - 1)(b - t), (t - 1) \to t(b - t), (b - t) \to (t - 1)$, i.e., an avatar of Istrail’s square-free sequence. Furthermore, it follows from [9] that this sequence on three letters cannot be the fixed point...
of a uniform morphism. A last remark is that the square-free Braunholtz sequence on three letters given in [10] (see also [9, p. 18-07]) is exactly our sequence \((m_n)_{n \geq 0}\) when \(t = b = 2\), i.e., the sequence 2 1 0 2 0 1 2 1 0 1 2 0 \ldots

3. Small admissible sequences and small univoque numbers with given integer part

3.1. Small admissible sequences with values in \(\{0, 1, \ldots, b\}\). In [18] the authors are interested in the smallest admissible sequence with values in \(\{0, 1, \ldots, b\}\), where \(b\) is an integer \(\geq 1\). They prove in particular the following result, which is an immediate corollary of our Theorem 2.

**Corollary 1** (Theorems 4.3 and 5.1 of [18]). Let \(b\) be an integer \(\geq 1\). The smallest admissible sequence with values in \(\{0, 1, \ldots, b\}\) is the sequence \((z + \varepsilon_{n+1})_{n \geq 0}\) if \(b = 2z + 1\), and \((z + \varepsilon_{n+1} - \varepsilon_n)_{n \geq 0}\) if \(b = 2z\).

**Proof.** Let \(A = (a_n)_{n \geq 0}\) be the smallest (nonconstant) admissible sequence with values in \(\{0, 1, \ldots, b\}\). Since \(A > \overline{A}\), we must have \(a_0 \geq \overline{a}_0 = b - a_0\).

Thus, if \(b = 2z + 1\) we have \(a_0 \geq z + 1\). We also have, for all \(i \geq 0\), \(\overline{a}_0 \leq a_i \leq a_0\). Now the smallest element of \(\Gamma(\{b-z-1, b-z, \ldots, z-1, z+1\})\) is the smallest admissible sequence on \(\{0, 1, \ldots, b\}\) that begins in \(z + 1\). Hence this is the smallest admissible sequence with values in \(\{0, 1, \ldots, b\}\). Theorem 2 shows that this sequence is \((m_n)_{n \geq 0}\) with \(m_n = \varepsilon_{n+1} + z\) for all \(n \geq 0\).

If \(b = 2z\), we have \(a_0 \geq z\). But if \(a_0 = z\), then \(\overline{a}_0 = z\), and the condition of admissibility implies that \(a_n = z\) for all \(n \geq 0\) and \((a_n)_{n \geq 0}\) would be the constant sequence \((z z z \ldots)\). Hence we must have \(a_0 \geq z + 1\). Now the smallest element of \(\Gamma(\{b-z-1, b-z, \ldots, z-1, z+1\})\) is the smallest admissible sequence on \(\{0, 1, \ldots, b\}\) that begins in \(z + 1\). Hence this is the smallest admissible sequence with values in \(\{0, 1, \ldots, b\}\). Theorem 2 implies that this sequence is \((m_n)_{n \geq 0}\) with \(m_n = \varepsilon_{n+1} - \varepsilon_n + z\) for all \(n \geq 0\).

3.2. Small univoque numbers with given integer part. We are interested here in the univoque numbers \(\lambda\) in an interval \((b, b + 1]\) with \(b\) a positive integer. This set was studied in [16], where it was proved to have Lebesgue measure 0. Since \(1 = \sum_{j \geq 0} a_j \lambda^{-(j+1)}\) and \(\lambda \in (b, b+1]\), and \(a_0 \leq b\), the fact that the expansion of 1 is unique, hence equal to the greedy expansion, implies that \(a_0 = b\). In other words, we study the admissible sequences with values in \(\{0, 1, \ldots, b\}\) that begin in \(b\), i.e., the set \(\Gamma_{\text{strict}}(\{0, 1, \ldots, b\})\).

A corollary of Theorem 2 is that, for any positive integer \(b\), there exists a smallest univoque number belonging to \((b, b + 1]\). This result was obtained in [22] (see the penultimate remark in that paper); it generalizes the result obtained for \(b = 1\) in [17].
Corollary 2. For any positive integer $b$, there exists a smallest univoque number in $(b, b+1)$. It is the solution of the equation $1 = \sum_{n \geq 0} d_n \lambda^{-n-1}$, where $d_n := \varepsilon_{n+1} - (b-1)\varepsilon_n + b - 1$ for all $n \geq 0$.

Proof. It suffices to apply Theorem 2 with $t = b$. ■

4. Transcendence results. We now prove, mimicking the proof given in [3], that numbers $\lambda$ such that the $\lambda$-expansion of 1 is given by the sequence $(m_n)_{n \geq 0}$ are transcendental. This generalizes the transcendence results of [3] and [18].

Theorem 3. Let $b$ be an integer $\geq 1$ and $t \in [0, b]$ be an integer such that $2t \geq b + 1$. Define the sequence $(m_n)_{n \geq 0}$ as in Theorem 2 by

$$m_n := \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1 \text{ for all } n \geq 0,$$

thus $(m_n)_{n \geq 0}$ begins with $t b - t + 1 b - t t - 1 \ldots$. Then the number $\lambda \in (1, b+1)$ defined by

$$1 = \sum_{n \geq 0} m_n \lambda^{-n-1}$$

is transcendental.

Proof. Define the $\pm 1$ Thue–Morse sequence $(r_n)$ by $r_n := (-1)^{\varepsilon_n}$. We clearly have $r_n = 1 - 2\varepsilon_n$ (recall that $\varepsilon_n$ is 0 or 1). It is also immediate that the function $F$ defined for the complex numbers $X$ with $|X| < 1$ by

$$F(X) = \sum_{n \geq 0} r_n X^n$$

satisfies $F(X) = \prod_{k \geq 0} (1 - X^{2^k})$ (see, e.g., [6]). Since

$$2m_n = 2\varepsilon_{n+1} - 2(2t - b - 1)\varepsilon_n + 2t - 2 = b - r_{n+1} + (2t - b - 1)r_n$$

we have, for $|X| < 1$,

$$2X \sum_{n \geq 0} m_n X^n = ((2t - b - 1)X - 1)F(X) + 1 + \frac{bX}{1 - X}.$$

Taking $X = 1/\lambda$ where $1 = \sum_{n \geq 0} m_n \lambda^{-n-1}$, we get the equation

$$2 = ((2t - b - 1)\lambda^{-1} - 1)F(1/\lambda) + 1 + \frac{b}{\lambda - 1}.$$

Now, if $\lambda$ were algebraic, then this equation shows that $F(1/\lambda)$ would be an algebraic number. But, since $1/\lambda$ would then be an algebraic number in $(0, 1)$, the quantity $F(1/\lambda)$ would be transcendental from a result of Mahler [19], giving a contradiction. ■

Remark 7. In particular the \{0, 1, \ldots, b\}-univoque number corresponding to the smallest admissible sequence with values in \{0, 1, \ldots, b\} is transcendental, as proved in [18] (Theorems 4.3 and 5.9). Also the smallest univoque number belonging to $(b, b + 1)$ is transcendental.

5. Conclusion. There are many papers dealing with univoque numbers. We just mention here the study of univoque Pisot numbers. The authors together with K. G. Hare determined in [5] the smallest univoque Pisot number, which happens to have algebraic degree 14. Note that the number
corresponding to the sequence of Proposition 2 is the larger real root of the polynomial \(X^2 - tX - (b - t + 1)\), hence a Pisot number (which is unitary if \(t = b\)). Also note that for any \(b \geq 2\), the real number \(\beta\) such that the \(\beta\)-expansion of 1 is \(b1^\infty\) is a univoque Pisot number in \((b, b+1)\). It would be interesting to determine the smallest univoque Pisot number in \((b, b+1)\): the case \(b = 1\) was addressed in [5], but the proof uses heavily the fine structure of Pisot numbers in \((1, 2)\) (see [8, 20, 21]). A similar study of Pisot numbers in \((b, b+1)\) would certainly help.

Acknowledgments. The authors thank M. de Vries and V. Komornik for their remarks on a previous version of this paper.

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Received on 9.11.2007
and in revised form on 5.10.2008