

## Convexity bounds for $L$ -functions

by

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Let the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be absolutely convergent for  $\sigma > 1$ , and extend to a meromorphic function on  $\mathbb{C}$ . Suppose further that  $F(s)$  is regular apart possibly for a pole of order  $m \geq 0$  at  $s = 1$  and that  $(s - 1)^m F(s)$  is then of finite order. We assume finally that  $\Phi(s) = \gamma(s)F(s)$  satisfies the functional equation

$$\Phi(s) = \overline{\Phi(1 - \bar{s})},$$

where

$$\gamma(s) = \eta Q^{s/2} \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j + i\nu_j).$$

Here  $\eta \in \mathbb{C}$ ,  $Q \in \mathbb{R}$  and  $\lambda_j, \mu_j, \nu_j \in \mathbb{R}$  (for  $1 \leq j \leq k$ ) are constants, satisfying

$$|\eta| = 1, \quad Q > 0, \quad \lambda_j > 0, \quad \mu_j > 0.$$

These hypotheses are amongst those required for the “Selberg class” (Selberg [3]).

One can now use the Phragmén–Lindelöf theorem in a standard way to estimate  $F(1/2)$  (or, more generally,  $F(1/2 + it)$ ). For example, if one has  $|a_n| \leq A(\varepsilon)n^\varepsilon$  for any  $\varepsilon > 0$ , then

$$(1) \quad F(1/2) \ll \varepsilon^{-1} A(\varepsilon) C^{1/4+\varepsilon},$$

where the conductor  $C$  is given by

$$(2) \quad C = Q \prod_{j=1}^k (1 + |\nu_j|)^{2\lambda_j}.$$

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2000 *Mathematics Subject Classification*: Primary 11M41.

*Key words and phrases*:  $L$ -function, convexity, Selberg class, Euler product, Jensen’s formula.

Here the implied constant depends on  $m, k$  and the  $\lambda_j$  and  $\mu_j$ , but not on  $\varepsilon, Q$  or the  $\nu_j$ . The aim of this note is to show how one can remove the extraneous  $\varepsilon$  from the exponent  $1/4 + \varepsilon$ , under suitable additional hypotheses. Where one has appropriate information on the coefficients  $a_n$  this can be done by using some form of approximate functional equation, which will require an estimate for a sum of the type  $\sum_{n \leq N} a_n n^{-1/2}$ . However, it is unclear in general how one can bound such sums efficiently.

Our principal result is the following.

**THEOREM.** *Suppose, in addition to the hypotheses above, that  $F(s)$  has an absolutely convergent Euler product for  $\sigma > 1$ , so that*

$$(3) \quad \log F(s) = \sum_n b_n n^{-s}$$

*with the coefficients  $b_n$  supported on the prime powers. Then*

$$F(1/2) \ll C^{1/4} \exp\left\{2 \sum_n |b_n| n^{-3/2}\right\},$$

*with the implied constant depending on  $m, k$  and the  $\lambda_j$  and  $\mu_j$ , but not on  $Q$  or the  $\nu_j$ .*

The condition that  $F(s)$  should have an Euler product is part of the definition of the Selberg class. However, all that we require of the coefficients  $b_n$  is that (3) should be absolutely convergent for  $\sigma > 1$ . If one were to suppose in addition that  $|b_n| \leq cn^{1/3}$ , say, then one would of course have a clean bound  $F(1/2) \ll_c C^{1/4}$ .

Although our theorem refers only to  $F(1/2)$  it is easily modified to estimate  $F(1/2 + it)$  in general. Indeed, if  $F(s)$  is entire it suffices to apply the theorem to  $F_t(s) = F(s + it)$ , for which  $F_t(1/2) = F(1/2 + it)$ . One readily checks here that  $F_t(s)$  satisfies a functional equation of the same form as before, but with the values  $\nu_j$  shifted by  $t$ .

The proof of our theorem makes it clear that the convexity estimate above could only be tight if all small non-trivial zeros of  $F(s)$  were close to the edge of the critical strip. (The terms  $J(\rho)$  below are genuinely positive for zeros in the interior of the strip.)

A result of the type above, but with stronger hypotheses, was described by Soundararajan at the Canadian Number Theory Association meeting in Waterloo, and the present paper is an outgrowth of discussions started there. Under suitable circumstances Soundararajan's approach leads [4] to an estimate in which one saves a further factor of nearly  $\log C$ . A weak subconvexity result of this type is already sufficient for certain applications, see Holowinsky and Soundararajan [1]. The author is grateful to Professor Soundararajan for a number of interesting remarks, and in particular for the reference [2].

To prove our theorem we will use the following lemma, which can be viewed as a variant of Jensen’s formula, modified by a conformal transformation so as to apply to a function in a strip.

LEMMA. *Let  $P$  be the path from  $\pi/2 - i\infty$  to  $\pi/2 + i\infty$  and then from  $-\pi/2 + i\infty$  to  $-\pi/2 - i\infty$ . When  $|\Im(\varrho)| < \pi/2$  define*

$$J(\varrho) = \log |\cot(\varrho/2)|,$$

*and set  $J(\varrho) = 0$  for  $|\Im(\varrho)| \geq \pi/2$ . Let  $f(z)$  be an entire function of finite order, non-vanishing at  $z = 0$ . Then*

$$(4) \quad \frac{1}{2\pi i} \int_P \log |f(z)| \frac{dz}{\sin z} = \log |f(0)| + \sum_{\varrho} J(\varrho),$$

*where  $\varrho$  runs over zeros of  $f(z)$ , counted according to multiplicity.*

Doubtless, the hypotheses of this lemma can be weakened considerably. A related (but different) result is given by Pólya and Szegő [2, pp. 119 & 120].

We observe that  $J(a + ib) = J(-a + ib)$ , and that  $J(\varrho) \geq 0$  for all  $\varrho$ . It follows that

$$\log |f(0)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |f(\pi/2 + it)f(-\pi/2 - it)| \frac{dt}{\cosh t}.$$

Thus after a simple change of variable we find that if  $G$  is entire of finite order then

$$\log |G(1/2)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(1 + \delta + i\kappa t)G(-\delta + i\kappa t)| \frac{dt}{\cosh t}$$

where we have set  $\kappa = (1 + 2\delta)/\pi$  for convenience. We shall apply this with  $G(s) = F(s)(s - 1)^m$  and  $0 < \delta < 1$ . The contribution on the right hand side from terms involving  $\log |(s - 1)^m|$  is  $O(m)$ . Applying the functional equation, and noting that

$$\int_{-\infty}^{\infty} \frac{dt}{\cosh t} = \pi,$$

leads to a bound

$$\begin{aligned} \log |F(1/2)| &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \log |F(1 + \delta + i\kappa t)| \frac{dt}{\cosh t} + \left(\frac{1}{4} + \frac{\delta}{2}\right) \log Q \\ &\quad + \sum_{j=1}^k \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left| \frac{\Gamma(\alpha_j + i\nu_j(t))}{\Gamma(\beta_j + i\nu_j(t))} \right| \frac{dt}{\cosh t} + O(m), \end{aligned}$$

where we have written

$$\alpha_j = \lambda_j(1 + \delta) + \mu_j, \quad \beta_j = -\lambda_j\delta + \mu_j, \quad \nu_j(t) = \nu_j + \lambda_j\kappa t$$

for convenience. However,

$$\log \left| \frac{\Gamma(\alpha + i\nu)}{\Gamma(\beta + i\nu)} \right| = (\alpha - \beta) \log(1 + |\nu|) + O(1)$$

uniformly for  $1 \ll \alpha, \beta \ll 1$ , and

$$\int_{-\infty}^{\infty} \log(1 + |\nu + \lambda\kappa t|) \frac{dt}{\cosh t} = \pi \log(1 + |\nu|) + O(1).$$

We therefore conclude that

$$\log |F(1/2)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \log |F(1 + \delta + i\kappa t)| \frac{dt}{\cosh t} + \left( \frac{1}{4} + \frac{\delta}{2} \right) \log C + O(1),$$

where  $C$  is given by (2), and the implied constant depends on  $m, k$  and the  $\lambda_j$  and  $\mu_j$ , but not on  $\delta, Q$  or the  $\nu_j$ .

It is now easy to deduce (1), but to do better we shall need the Euler product. Since

$$\int_{-\infty}^{\infty} n^{-i\kappa t} \frac{dt}{\cosh t} = \frac{2\pi}{n^{\pi\kappa/2} + n^{-\pi\kappa/2}} = \frac{2\pi}{n^{1/2+\delta} + n^{-1/2-\delta}}$$

we find that

$$\log |F(1/2)| \leq 2 \sum_n \Re(b_n)(n^{3/2+2\delta} + n^{1/2})^{-1} + \left( \frac{1}{4} + \frac{\delta}{2} \right) \log C + O(1).$$

The theorem now follows on allowing  $\delta$  to tend downwards to zero.

We end by giving a sketch proof of the lemma. Consider the integral

$$\frac{1}{2\pi i} \int_P \log |1 - z/\varrho| \frac{dz}{\sin z}.$$

This vanishes if  $|\Im(\varrho)| \geq \pi/2$ , and otherwise is  $J(\varrho)$ . To see this one sets  $\varrho = a + ib$  and integrates around the path  $P$  supplemented by a loop integral from  $\pi/2 + ib$  around  $\varrho$  and back. The function  $J(\varrho)$  then arises as

$$J(\varrho) = 2 \cosh b \int_a^{\pi/2} \frac{\sin x}{\cosh(2b) - \cos(2x)} dx.$$

The lemma now follows in the case in which  $f(z)$  is a polynomial. Moreover, if  $g(z)$  is also a polynomial then

$$\frac{1}{2\pi i} \int_P \log |\exp(g(z))| \frac{dz}{\sin z} = \Re(g(0)).$$

Now, since  $f(z)$  is a function of finite order there is a positive integer  $M$  such that the sum  $\sum |\varrho|^{-M}$  is convergent, and such that

$$f(z) = \exp(h(z)) \prod_{\varrho} E_M(z/\varrho),$$

where  $h(z)$  is a suitable polynomial and

$$E_M(w) = (1 - w) \exp\left\{\sum_{j=1}^M w^j/j\right\}.$$

We may then write  $f(z) = f_1(z; N)f_2(z; N)$  with

$$f_1(z; N) = \exp(g(z)) \prod_{|\varrho| \leq N} E_M(z/\varrho), \quad f_2(z; N) = \prod_{|\varrho| > N} E_M(z/\varrho).$$

We have already shown that the lemma holds for the function  $f_1(z; N)$ . Moreover, a crude estimate, considering separately the cases  $|z| \leq |\varrho|/2$ ,  $|\varrho|/2 < |z| < 2|\varrho|$  and  $|z| \geq 2|\varrho|$ , shows that

$$\int_P \log |E_M(z/\varrho)| \frac{dz}{\sin z} \ll_M |\varrho|^{-M}.$$

Thus the contribution to (4) corresponding to  $\log |f_2(z; N)|$  tends to zero as  $N$  goes to infinity, and the lemma follows.

### References

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