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On some new congruences for generalized Bernoulli numbers

by

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Dedicated to Professor Andrzej Schinzel on the occasion of his 75th birthday with great respect

1. Introduction. We present two types of results. We show that the celebrated conjecture for the classical Euler numbers [7, Problem B45] proved by P. Yuan [25] (based on results of G.-D. Liu [13]) and generalized by W.-P. Zhang and Z.-F. Xu [26], as well as the main results of [22] (based on Yamamoto's nice paper [24]) on congruences between sums of special values of L-functions and Euler numbers are in fact consequences of a Kummer type congruence which is an exercise in L. Washington's book [23].

Let $T_{r,k}(n) = \sum_{i=1}^{[n/r]}(\chi_n(i)/i^k)$, where χ_n is the trivial character modulo n. The main result of the paper is a new congruence for the sum $T_{4,2}(n) \bmod n^2$ for odd $n > 3$. The congruence follows from an identity proved in [19] which was earlier exploited in [16] and [6]. The congruence generalizes a pretty congruence obtained by Z.-H. Sun [17] for this sum for an odd prime n. The identity from [19] was applied by T. Cai [1] to prove some new congruences for the sum $T_{2,1}(n) \bmod n^2$ for odd $n > 1$.

1.1. A generalized Kummer congruence for generalized Bernoulli numbers. Let p be a fixed prime and let $\psi \neq 1$ be a Dirichlet character not of the second kind at a fixed prime number p. Then for $n \geq 1$ the numbers $(1/n)B_{n,\psi\omega^{-n}}$ are p-integral (except when $\chi = \omega$, $p = 2$, $n = 1$), and if $m \equiv n \pmod{p^\alpha}$ for $\alpha \geq 0$, then we have a version of the generalized Kummer congruence

$$(1)\qquad (1-\psi\omega^{-m}(p)p^{m-1})\frac{B_{m,\psi\omega^{-m}}}{m} \equiv (1-\psi\omega^{-n}(p)p^{n-1})\frac{B_{n,\psi\omega^{-n}}}{n} \pmod{p^{\alpha+1}}$$

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(see [23, Exercise 7.5, p. 141]). It is easy to check that (1) contains the classical Kummer congruences. For other versions of the Kummer congruences, see [5], [4], [3], [11] and [15].

Throughout the paper, ϕ is the Euler ϕ-function, ω is the Teichmüller character at p and $B_{s,\chi}$ denotes the sth generalized Bernoulli number attached to the Dirichlet character χ.

1.2. Applications of the generalized Kummer congruence. In [25] and [26] the authors proved the following incongruence for the classical Euler numbers E_n:

$$E_{\phi(p^\alpha)/2} \not\equiv 0 \,(\operatorname{mod} p^\alpha), \tag{2}$$

where $p \equiv 1 \pmod 4$ is a prime number. This was conjectured for $\alpha = 1$ in [7] and was proved in [13] for $p \equiv 5 \pmod 8$. In [13] and [25] the case $\alpha = 1$ was only considered.

An elementary proof presented in [25] is based on a lemma of [13] (see [25, Lemma 2.1]) and uses some pretty identities for Euler numbers.

It is worth pointing out that (2) is an immediate consequence of (1). Let us rewrite (1) as

$$(1 - \chi(p)p^{k-1})\frac{B_{k,\chi}}{k} \equiv (1 - \chi^*(p)p^{k+m-1})\frac{B_{k+m,\chi^*}}{k+m} \pmod{p^\alpha} \tag{3}$$

and refer to it as the *generalized Kummer congruence* (cf. [20]). Here χ and χ^* are Dirichlet characters satisfying $\chi = \chi^*\omega^m$ for an integer m such that $p^{\alpha-1} \,|\, m$ ($\alpha \geq 1$ an integer), and for $\chi\omega^m \neq 1$ not being a character of the second kind at p.

Given the discriminant d of a quadratic field, let χ_d denote its quadratic character (Kronecker symbol). It was proved in [3] that the numbers $B_{n,\chi_d}/n$ are rational integers unless $d = -4$ or $d = \pm p$, where p is an odd prime such that $2n/(p-1)$ is an odd integer. If $d = -4$ and n is odd, then

$$B_{n,\chi_{-4}} = -\frac{n}{2}E_{n-1}, \tag{4}$$

where E_s for s even are odd integers and called the *Euler numbers*. If $d = \pm p$, then the numbers B_{n,χ_d} have p in their denominators and $pB_{n,\chi_d} \equiv p - 1$ $(\operatorname{mod} p^{\operatorname{ord}_p(n)+1})$.

1.2.1. *First application of (3).* We shall deduce (2) from (3). Set in (3) $m = \phi(p^\alpha)/2$, $k = 1$ and $\chi^* = \chi_{-4}$. By (4) and $\omega^m = \chi_p$ congruence (3) implies the well-known congruence

$$E_{\phi(p^\alpha)/2} \equiv -2B_{1,\chi} \pmod{p^\alpha},$$

where $\chi = \chi_{-4p}$. Hence, by the formula $B_{1,\chi} = -h(-4p)$ (see [21, p. 28]), we obtain the congruence

$$E_{\phi(p^\alpha)/2} \equiv 2h(-4p) \pmod{p^\alpha},$$

where for the discriminant d of a quadratic field, $h(d)$ denotes the class number of this field.

On the other hand, by the famous Dirichlet formula, we have

$$h(-4p) = -2 \sum_{a=1}^{(p-1)/4} \chi_p(a)$$

(see [21, p. 40]) and consequently $0 < h(-4p) < p/2$, which gives (2) for every α at once.

1.2.2. *Second application of (3).* One of the most important properties of generalized Bernoulli numbers is that they give the values of Dirichlet L-functions at non-positive integers. Namely, we have

$$L(1-m,\chi) = -\frac{B_{m,\chi}}{m}, \tag{5}$$

where $m \geq 1$ (see [23, Theorem 4.2]). Given a Dirichlet character χ modulo M assume that $\chi(-1) = (-1)^\delta$, where $\delta = 0$ or 1, and denote by $\tau(\chi)$ the normalized Gauss sum attached to χ. By the functional equation for L-series, we can rewrite (5) in the form

$$L(m,\chi) = (-1)^{(m-\delta+2)/2} \frac{(2\pi)^m \tau(\chi)}{2i^\delta m! M^m} B_{m,\bar{\chi}} \tag{6}$$

if $m \equiv \delta \pmod 2$.

In Theorem 2.1 of [22] (see also other main results in [22, Section 2]) the authors applied Yamamoto's Theorem [24, pp. 275–289] to find the residue modulo a prime power of a linear combination of the values of the Dirichlet L-function $L(s,\chi)$ at positive integral arguments s such that s and χ are of the same parity, in terms of Euler numbers.

Note that the main results in [22] are again consequences of the generalized Kummer congruence (3). For example, we give simpler proofs of the first two congruences (2.21) and (2.22) of Theorem 2.1 in [22]. Set in (3) again $\chi = \chi^*\omega^m$, where $\chi^* = \chi_{-4}$ and $m > 0$ is a multiple of $\phi(p^\alpha)/2$. We consider the case when $m/(\phi(p^\alpha)/2)$ is odd, i.e., $\omega^m = \chi_p$ if $p \equiv 1 \pmod 4$ and $\omega^m = \chi_{-p}$ if $p \equiv 3 \pmod 4$. Assume that $0 \leq l \leq \phi(p^\alpha)/2$ and l is even, resp. odd if $p \equiv 1$ resp. $3 \pmod 4$.

Thus, by (3), since $l + m \geq \alpha$, we obtain the following congruence modulo p^α:

$$E_{l+m} \equiv -2(1-\chi^*(p)p^{l+m})\frac{B_{l+m,\chi^*}}{l+m} \equiv \begin{cases} -2\dfrac{B_{l+1,\chi_{-4p}}}{l+1} & \text{if } p \equiv 1 \pmod 4, \\ -2\dfrac{B_{l+1,\chi_{4p}}}{l+1} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Note that the characters χ_{-4p} and χ_{4p} are odd, resp. even if $p \equiv 1$ resp. 3 (mod 4) and the numbers

$$\frac{B_{2r+1,\chi_{-4p}}}{2r+1}, \qquad \frac{B_{2r,\chi_{4p}}}{2r}$$

are integers. Therefore the last congruence implies

$$-2\sum_{r=0}^{l/2}\binom{l}{2r}p^{\alpha(l-2r)}\frac{B_{2r+1,\chi_{-4p}}}{2r+1} \equiv E_{l+m} \pmod{p^\alpha}$$

if $p \equiv 1 \pmod 4$, and

$$-2\sum_{r=1}^{(l+1)/2}\binom{l}{2r-1}p^{\alpha(l-2r+1)}\frac{B_{2r,\chi_{4p}}}{2r} \equiv E_{l+m} \pmod{p^\alpha}$$

if $p \equiv 3 \pmod 4$. Using formula (6) and Gauss' famous result on the value of Gauss' sum for a quadratic character (see [21, p. 17]), we obtain congruences (2.21) and (2.22) of [22] at once. Note that obviously

$$\sum_{r=0}^{(l-2)/2}\binom{l}{2r}p^{\alpha(l-2r)}\frac{B_{2r+1,\chi_{-4p}}}{2r+1} \equiv 0 \pmod{p^\alpha}$$

if $p \equiv 1 \pmod 4$, and

$$\sum_{r=1}^{(l-1)/2}\binom{l}{2r-1}p^{\alpha(l-2r+1)}\frac{B_{2r,\chi_{4p}}}{2r} \equiv 0 \pmod{p^\alpha}$$

if $p \equiv 3 \pmod 4$, and these parts of the congruences, in fact, are only glued to the generalized Kummer congruences (3).

1.3. Applications of an identity for generalized Bernoulli numbers. Let $n > 1$ be odd and let $\chi_n = \chi_{0,n}$ be the trivial Dirichlet character modulo n. For $r \geq 2$ prime to n denote by $q_r(n)$ the Euler quotient, i.e.,

$$q_r(n) = \frac{r^{\phi(n)} - 1}{n}.$$

T. Cai [1] applied an identity proved in [19] to generalize a classical congruence proved by E. Lehmer [10] for p prime,

$$\sum_{i=1}^{(p-1)/2}\frac{1}{i} \equiv -2q_2(p) + pq_2^2(p) \pmod{p^2}.$$

Cai obtained a more general congruence for n odd,

$$\sum_{i=1}^{(n-1)/2}\frac{\chi_n(i)}{i} \equiv -2q_2(n) + nq_2^2(n) \pmod{n^2}. \tag{7}$$

See [1, Theorem 1]. We shall recall Cai's elegant and short (half a page) elementary proof of (7).

Before we sketch the proof of Theorem 1 of [1] we recall an identity proved in [19]. Let χ be a Dirichlet character modulo M, N a positive integral multiple of M, and r (>1) a positive integer prime to N. Then for any integer $m \geq 0$ we have

$$(m+1)r^m \sum_{0<n<N/r} \chi(n)n^m = -B_{m+1,\chi}r^m + \frac{\overline{\chi}(r)}{\phi(r)} \sum_{\psi} \overline{\psi}(-N) B_{m+1,\chi\psi}(N), \tag{8}$$

where the sum on the right hand side is taken over all Dirichlet characters ψ modulo r. Here $B_{s,\chi}(X) = \sum_{i=0}^{s} \binom{s}{i} B_{s-i,\chi} X^i$ denotes the sth generalized Bernoulli polynomial attached to χ.

This useful identity was applied in a couple of papers to obtain some rather deep results on class numbers of imaginary quadratic fields and other higher generalized Bernoulli numbers.

The most spectacular are the results of Schinzel et al. [16] on the class numbers of imaginary quadratic fields which can be represented as single sums of Kronecker symbols, and results in [16] on the cases when short sums of Kronecker symbols vanish.

Identity (8) was also used to show in an elementary way an extension of Gauss' congruence $h(d) \equiv 0 \pmod{2^{\nu-1}}$, where ν is the number of distinct prime factors of the discriminant d of a quadratic field, to a similar congruence for generalized Bernoulli numbers, $(B_{k,\chi_d}/k) \equiv 0 \pmod{2^{\nu-1}}$. See [6] or [21]. See also some deep generalizations of the above congruence in [9].

We shall use Cai's techniques to prove further congruences similar to those given in [10], [12] or [17]. For other results of the same type, not using (8), see [2] where almost all remaining congruences from [10] were extended.

1.3.1. *Some auxiliary notation.* If the character χ modulo M is induced from a character χ_1 modulo some divisor of M then

$$B_{s,\chi} = B_{s,\chi_1} \prod_{p \mid M} (1 - \chi_1(p)p^{s-1}), \tag{9}$$

where the product is taken over all primes p dividing M.

In this paper we shall consider congruences for the sums

$$T_{r,k}(n) = \sum_{0<i<n/r} \frac{\chi_n(i)}{i^k}$$

modulo odd powers n^s.

If $(i,n)=1$, then by Euler's theorem we have $i^{\phi(n)} \equiv 1 \pmod{n}$, and more generally,

$$i^{\phi(n)n^s} \equiv 1 \pmod{n^{s+1}} \quad \text{for } s \geq 0. \tag{10}$$

For r prime to n and integers $s,k \geq 0$ we denote

$$S_{r,k,s}(n) = \sum_{0<i<n/r} \chi_n(i) i^{n^s\phi(n)-k}.$$

We have

$$T_{r,k}(n) \equiv S_{r,k,s}(n) \pmod{n^s}. \tag{11}$$

We consider two specific types of such congruences:

(i) $r=2$, $k=1$, $s=1$ (see [1]);
(ii) $r=4$, $k=2$, $s=1$.

1.3.2. *Sketch of Cai's proof.* We consider the case when $r=2$, $k=1$, $s=1$. Set in (8) $r=2$, $\chi=\chi_n$, $N=M=n$, $m=n\phi(n)-1$ (and so $m+1=n\phi(n)$). Note that there is only one character modulo $r=2$, $\chi_{0,2}$, and so by (9) we have

$$S_{2,1,1}(n) = -\frac{B_{n\phi(n)}}{n\phi(n)} \prod_{p|n}(1-p^{n\phi(n)-1}) + \frac{2^{1-n\phi(n)}}{n\phi(n)} B_{n\phi(n),\chi_{0,2n}}(n).$$

Hence, again by (9), we obtain

$$S_{2,1,1}(n) \equiv \frac{1-2^{n\phi(n)}}{2^{n\phi(n)-1}} \frac{B_{n\phi(n)}}{n\phi(n)} \prod_{p|n}(1-p^{n\phi(n)-1}) \pmod{n^2}, \tag{12}$$

which follows from the von Staudt and Clausen theorem and the congruence

$$\frac{2^{1-n\phi(n)}}{n\phi(n)} \sum_{i=1}^{n\phi(n)} \binom{n\phi(n)}{i} n^i B_{n\phi(n)-i} \prod_{p|2n}(1-p^{n\phi(n)-i-1}) \equiv 0 \pmod{n^2}$$

since $B_{n\phi(n)-i}=0$ if i is odd unless $n\phi(n)-i=1$, in which case $i \geq 3$.

Now Cai's congruence (7) follows from the formula $2^{\phi(n)} = nq_2(n)+1$, (12), (11), and from the congruence

$$\frac{n}{\phi(n)} B_{n\phi(n)} \prod_{p|n}\left(1-p^{n\phi(n)-1}\right) \equiv 1 \pmod{n^2},$$

which follows from $pB_{n\phi(n)} \equiv p-1 \pmod{p^{2\operatorname{ord}_p(n)}}$, $p \mid n$ ([1]).

([1]) We have corrected two minor but relevant inaccuracies in Cai's original proof in [1].

2. Main result. The idea exploited in [1] to use identity (8) to extend classical congruences for the sums $T_{r,k}(n)$ seems to be very efficient. Identity (8) allows us to obtain almost automatically many new interesting congruences of Lerch [12], Lehmer [10] or Sun [17] types. Usually the proofs using (8) are much easier, more unified and much shorter than those applying other methods.

For congruences proved recently in [1], [2], [14], [17] or [18] we can often give much shorter proofs. Only Jakubec's congruence [8] seems to resist our methods so far.

The general scheme of reasoning is uniform. We take a classical congruence modulo powers of primes and applying identity (8) to the sums $S_{r,k,s}(n)$, after some elementary transformations, we obtain similar congruences modulo the same powers but of odd natural numbers. Usually in this way we obtain new, non-trivial congruences of a more complicated form.

2.1. The case when $r = 4$, $k = 2$, $s = 1$

Theorem. *In the above notation, if $n > 3$ is an odd natural number, then*

$$\sum_{0<i<n/4} \frac{\chi_n(i)}{i^2} \equiv 8\Big(nB_{n\phi(n)-2}\prod_{p|4n}(1-p^{n\phi(n)-3}) + \frac{1}{2}(-1)^{(n-1)/2}E_{n\phi(n)-2}\prod_{p|n}(1-(-1)^{(p-1)/2}p^{n\phi(n)-2})\Big) \pmod{n^2}.$$

Proof. Using (8), we shall determine the sum $S_{4,2,1}(n)$ modulo n^2 ($n > 1$ odd) and next substitute it into (11). In this case we put in (8) $m = n\phi(n)-2$ and $\chi = \chi_n$ (and so $m+1 = n\phi(n)-1$). Thus $m+1$ is odd and m is even. Therefore we obtain, from (8) and (9),

$$(n\phi(n)-1)4^{n\phi(n)-2}S_{4,2,1}(n) = -B_{n\phi(n)-1}\prod_{p|n}(1-p^{n\phi(n)-2})4^{n\phi(n)-2} + U/2,$$

where

$$U = \sum_{\psi \bmod 4} \psi(-n)B_{n\phi(n)-1,\chi_{0,n}\psi}(n).$$

Note that $n\phi(n)-1 > 1$ and $n > 1$ are odd, and so $B_{n\phi(n)-1} = 0$. Hence,

$$(13)\qquad (n\phi(n)-1)4^{n\phi(n)-2}S_{4,2,1} = U/2.$$

Also note that there are only two characters modulo 4, $\chi_{0,4}$ and χ_{-4}. Therefore we can divide the sum U into two parts,

$$(14)\qquad U = U_1 + U_2,$$

where

$$U_1 = B_{n\phi(n)-1,\chi_{0,n}\chi_{0,4}}(n) \quad \text{and} \quad U_2 = -(-1)^{(n-1)/2} B_{n\phi(n)-1,\chi_{0,n}\chi_{-4}}(n)$$

because $\chi_{0,4}(-n) = 1$ and $\chi_{-4}(-n) = -(-1)^{(n-1)/2}$. Hence by (9) we have

$$U_1 = \sum_{i=0}^{n\phi(n)-1} \binom{n\phi(n)-1}{i} n^i B_{n\phi(n)-1-i} \prod_{p|n} (1 - p^{n\phi(n)-2-i})$$

and

$$U_2 = (-1)^{(n+1)/2} \sum_{i=0}^{n\phi(n)-1} \binom{n\phi(n)-1}{i} n^i B_{n\phi(n)-1-i,\chi_{-4}} \times \prod_{p|n} (1 - (-1)^{(p-1)/2} p^{n\phi(n)-2-i})$$

because $\chi_{-4}(p) = (-1)^{(p-1)/2}$ for p odd.

Therefore,

$$U_1 \equiv (n\phi(n)-1) n B_{n\phi(n)-2} \prod_{p|n} (1 - p^{n\phi(n)-3}) \pmod{n^2}$$

since in view of the von Staudt and Clausen theorem and (9) for $i \geq 3$ odd we have

$$\binom{n\phi(n)-1}{i} n^i B_{n\phi(n)-1-i} \prod_{p|n} (1 - p^{\phi(n)n-2-i}) \equiv 0 \pmod{n^2}$$

and $B_{n\phi(n)-1-i,\chi_{0,n}\chi_{0,4}} = 0$ if i is even because $\chi_{0,n}\chi_{0,4}$ is even.

On the other hand, by (4) and (9),

$$U_2 \equiv \frac{1}{2}(-1)^{(n-1)/2} (n\phi(n)-1) E_{n\phi(n)-2} \prod_{p|n} (1 - (-1)^{(p-1)/2} p^{\phi(n)n-2}) \pmod{n^2}$$

because for even $i \geq 2$,

$$\begin{aligned}(-1)^{(n+1)/2} &\binom{n\phi(n)-1}{i} n^i B_{n\phi(n)-1-i,\chi_{-4}} \prod_{p|n} (1 - (-1)^{(p-1)/2} p^{\phi(n)n-2-i}) \\ &= \frac{1}{2}(-1)^{(n-1)/2} n^i (n\phi(n)-3) E_{n\phi(n)-4} \prod_{p|n} (1 - (-1)^{(p-1)/2} p^{\phi(n)n-4}) \\ &\equiv 0 \pmod{n^2}\end{aligned}$$

and $B_{n\phi(n)-1-i,\chi_{0,n}\chi_{-4}} = 0$ for i odd because $\chi_{0,n}$ is even and χ_{-4} is odd.

Finally, by (14), we obtain the following congruence modulo n^2:

$$U \equiv (n\phi(n)-1)\times$$
$$\Big(nB_{n\phi(n)-2}\prod_{p\mid n}(1-p^{n\phi(n)-3})+\frac{1}{2}(-1)^{(n-1)/2}E_{n\phi(n)-2}\prod_{p\mid n}(1-\chi_{-4}(p)p^{n\phi(n)-2})\Big).$$

Putting this into (13), using congruence (10) for $s=1$ and dividing the above congruence by $n\phi(n)-1$ (which is prime to n) we obtain the conclusion in view of (11). ∎

COROLLARY (see [17, Corollary 3.8, p. 296]). *If $p>3$ is an odd prime* (²), *then*

$$\sum_{i=1}^{[p/4]}\frac{1}{i^2}\equiv(-1)^{(p-1)/2}(8E_{p-3}-4E_{2p-4})+\frac{14}{3}pB_{p-3}\pmod{p^2}.$$

Proof. We use the Theorem for $n=p$ prime. The corollary follows from the classical Kummer congruence for ordinary Bernoulli numbers and from the following congruence for Euler numbers:

$$E_{p(p-1)-2}\equiv 2E_{p-3}-E_{2p-4}\pmod{p^2}. \tag{15}$$

Since $p-3\not\equiv 0 \pmod{p-1}$, by the classical Kummer congruence we obtain

$$\frac{B_{p(p-1)-2}}{p(p-1)-2}\equiv\frac{B_{p-3}}{p-3}\pmod{p}$$

and so $B_{p(p-1)-2}\equiv\frac{2}{3}B_{p-3} \pmod{p}$. Hence and in view of Euler's theorem $2^{p(p-1)}\equiv 1 \pmod{p^2}$, we have

$$8pB_{p(p-1)-2}(1-2^{p(p-1)-3})\equiv 8p\frac{2}{3}B_{p-3}(1-2^{-3})\equiv\frac{14}{3}pB_{p-3}\pmod{p^2}.$$

Thus to prove the Corollary we should prove (15). This is more complicated. Applying the identity

$$E_{2n}=-4^{2n+1}\frac{B_{2n+1}(1/4)}{2n+1},\qquad n\ge 0,$$

(see [17, Lemma 2.5]) for $n=p(p-1)-1$, $n=2p-3$ and $n=p-2$ and using the congruence

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b}\equiv k\frac{B_{p-1+b}(x)}{p-1+b}-(k-1)\frac{B_b(x)}{b}\pmod{p^2},\qquad b>2,$$

(see [17, Lemma 2.6]) with $x=1/4$, $k=p-1$ and $b=p-2$, we obtain

$$\frac{B_{p(p-1)-1}(1/4)}{p(p-1)-1}\equiv(p-1)\frac{B_{2p-3}(1/4)}{2p-3}-(p-2)\frac{B_{p-2}(1/4)}{p-2}\pmod{p^2},$$

(²) Note that in [17], $p>5$ is required.

and so

$$(16)\quad E_{p(p-1)-2} \equiv 4^{p(p-3)+2}(p-1)E_{2p-4} - 4^{p(p-2)+1}(p-2)E_{p-3} \pmod{p^2}.$$

Now it suffices to use the classical congruence

$$(17)\qquad E_{n+p-1} \equiv E_n \pmod{p}$$

(see for example [4, (4.1), p. 36]) and an elementary congruence

$$(18)\qquad a^{s(p-1)} \equiv 1 + spq_a(p) \pmod{p^2}$$

(see for example [10, p. 354]).

We have $p(p-3)+2 = (p-1)(p-2)$ and $p(p-2)+1 = (p-1)^2$ and so by (18) we obtain

$$4^{p(p-3)+2} \equiv 1 + (p-2)pq_4(p) \pmod{p^2},$$
$$4^{p(p-2)+1} \equiv 1 + (p-1)pq_4(p) \pmod{p^2}.$$

Substituting the above congruences into (16) gives

$$\begin{aligned} E_{p(p-1)-2} \equiv\ & (p-1)(1+(p-2)pq_4(p))E_{2p-4} \\ & - (p-2)(1+(p-1)pq_4(p))E_{p-3} \pmod{p^2}, \end{aligned}$$

and so

$$E_{p(p-1)-2} \equiv (2E_{p-3} - E_{2p-4}) + p(1+2q_4(p))(E_{2p-4} - E_{p-3}) \pmod{p^2}.$$

Now it is sufficient to use (17), and (15) follows at once. ∎

3. Concluding remarks. It is not too difficult to find similar congruences in the twin case when $r = 4$, $k = 1$, $s = 1$. It suffices to put in (8) $m = n\phi(n) - 1$ which is odd and then $m + 1 = n\phi(n)$ is even. The rest of proofs is almost the same. The resulting congruences modulo n^2 are completely new and we leave it to the reader to write them out as an exercise. Their reduction modulo n gives an extension of Lerch's classical congruence proved for p prime:

$$\sum_{0<i<p/4} \frac{1}{i} \equiv -3q_2(p) \pmod{p}$$

(see [12, congruence (10), p. 475]. We can obtain a more general congruence for n odd:

$$\sum_{0<i<n/4} \frac{\chi_n(i)}{i} \equiv -3q_2(n) \pmod{n}.$$

Also one can consider other cases with $r \mid 24$ (then the group $(\mathbb{Z}/r\mathbb{Z})^*$ has exponent 2 and all its characters are quadratic). For $r = 3$ we obtain new congruences for Ernvall's [4] D-numbers. Especially interesting are new congruences obtained for $r = 8$ for the generalized Bernoulli numbers attached to the Dirichlet characters χ_8 and χ_{-8}.

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