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Mean value theorems for binary Egyptian fractions II

by

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1. Introduction. In the previous memoir of this series (see [HV]) we studied the mean value

$$S(N;a)=\sum_{\substack{n\le N\\(n,a)=1}} R(n;a) \tag{1.1}$$

of the number $R(n;a)$ of positive integer solutions to the Diophantine equation

$$\frac{a}{n}=\frac{1}{x}+\frac{1}{y}. \tag{1.2}$$

Here we extend our investigation to the second moment and some consequences thereof.

THEOREM 1. *For fixed positive integer a, we have, for every $N\in\mathbb{N}$ with $N\ge 2$,*

$$\sum_{\substack{n\le N\\(n,a)=1}}\Big|R(n;a)-\frac{1}{\phi(a)}\sum_{\substack{\chi \bmod a\\ \chi^2=\chi_0}}\bar\chi(-n)\sum_{u\mid n^2}\chi(u)\Big|^2\ll_a N\log^2 N,$$

where $\ll_a$ indicates that the implicit constant depends at most on a, and where χ_0 denotes the principal character modulo a.

In the character sum here the term $\chi=\chi_0$ contributes an amount $d(n^2)$ where d is the divisor function and we can expect that this is the dominant contribution on average. Thus as a consequence of the Erdős–Kac theorem, just as for the divisor function $d(n)$, one can anticipate that $\log R(n;a)$ admits a Gaussian distribution. As a first approximation we establish the normal order of $\log R(n;a)$.

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THEOREM 2. *When a is fixed, the normal order of* $\log R(n;a)$ *as a function of n is* $(\log 3)\log\log n$.

Let

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

Then with a little more work we can establish the full distribution.

THEOREM 3. *For fixed positive integer a, we have*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\left\{ n \le N : \frac{\log R(n;a) - (\log 3)\log\log n}{(\log 3)\sqrt{\log\log n}} \le z \right\} = \Phi(z).$$

For completeness we also establish the mean square of $R(n;a)$ for fixed a. Since $R(n;a)$ resembles quite closely the divisor function $d(n^2)$ in many aspects, we expect that their mean squares share the same order of magnitude. Thus the following theorem can be compared with the asymptotic formula

$$\sum_{n\le N} d^2(n^2) = NP_8(\log N) + O(N^{1-\delta}),$$

which holds for some $\delta > 0$ and with $P_8(\cdot)$ a polynomial of degree 8.

THEOREM 4. *Let a be a fixed positive integer and $\varepsilon > 0$. Then*

$$\sum_{\substack{n\le N\\(n,a)=1}} R(n;a)^2 = NP_8(\log N;a) + O_a(N^{35/54+\varepsilon}),$$

where $P_8(\cdot;a)$ is a degree 8 polynomial with coefficients depending on a, and its leading coefficient is

$$\frac{1}{8!a^2}\prod_{p\mid a}\left(1-\frac{1}{p}\right)^7 \prod_{p\nmid a}\left(1+\frac{6}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right)^6.$$

The error term in the theorem above is closely related to the generalised divisor problem, and in particular depends on a mean value estimate for the ninth moment of Dirichlet L-functions $L(s,\chi)$ modulo a inside the critical strip. As is easily verified, the error can be improved to $O_a(N^{1/2+\varepsilon})$ under the assumption of the generalised Lindelöf Hypothesis.

2. Proof of Theorem 1. We rewrite equation (1.2) in the form

$$(ax-n)(ay-n) = n^2.$$

After the change of variables $u = ax - n$ and $v = ay - n$, it follows that $R(n;a)$ is the number of ordered pairs of natural numbers u, v such that $uv = n^2$ and $u \equiv v \equiv -n \pmod{a}$.

Under the assumption that $(n,a) = 1$, $R(n;a)$ can be reduced further to counting the number of divisors u of n^2 with $u \equiv -n \pmod{a}$. Now the residue class $u \equiv -n \pmod{a}$ is readily isolated via the orthogonality of the Dirichlet characters χ modulo a. Thus we have

$$R(n;a) = \frac{1}{\phi(a)} \sum_{\chi \bmod a} \bar{\chi}(-n) \sum_{u \mid n^2} \chi(u), \tag{2.1}$$

where the condition $(n,a)=1$ is taken care of by the character $\bar{\chi}(n)$.

Hence

$$\sum_{\substack{n\le N\\(a,n)=1}} \Bigg| R(n;a) - \frac{1}{\phi(a)} \sum_{\substack{\chi \bmod a\\ \chi^2=\chi_0}} \bar{\chi}(-n) \sum_{u\mid n^2} \chi(u) \Bigg|^2$$
$$\ll_a \sum_{\substack{n=1\\(a,n)=1}}^{\infty} e^{-n/N} \Bigg| \sum_{\substack{\chi \bmod a\\ \chi^2\ne\chi_0}} \bar{\chi}(-n) \sum_{u\mid n^2} \chi(u) \Bigg|^2$$
$$= \sum_{\substack{\chi_1 \bmod a\\ \chi_1^2\ne\chi_0}} \sum_{\substack{\chi_2 \bmod a\\ \chi_2^2\ne\chi_0}} \bar{\chi}_1\chi_2(-1) \sum_{n=1}^{\infty} \sum_{u\mid n^2} \sum_{v\mid n^2} \chi_1(u)\bar{\chi}_2(v)\bar{\chi}_1\chi_2(n) e^{-n/N},$$

where χ_0 denotes the principal character modulo a. In order to evaluate the sum over n, we analyze the Dirichlet series

$$f_{\chi_1,\chi_2}(s) := \sum_{n=1}^{\infty} \sum_{u\mid n^2} \sum_{v\mid n^2} \chi_1(u)\bar{\chi}_2(v)\bar{\chi}_1\chi_2(n) n^{-s}.$$

The condition $u \mid n^2$ can be written as $u_1u_2^2 \mid n^2$ with u_1 squarefree, i.e. $u_1u_2 \mid n$, and likewise for $v \mid n^2$. Thus

$$f_{\chi_1,\chi_2}(s) = \sum_{m=1}^{\infty} \frac{\bar{\chi}_1\chi_2(m)}{m^s} \sum_{d=1}^{\infty} \frac{F(d)}{d^s}, \tag{2.2}$$

where

$$F(d) = \sum_{\substack{u_1,u_2,v_1,v_2\\ d=[u_1u_2,v_1v_2]}} \mu^2(u_1)\mu^2(v_1)\chi_1(u_1u_2^2)\bar{\chi}_2(v_1v_2^2)\bar{\chi}_1\chi_2(d).$$

The function F is multiplicative and so the inner sum above is

$$\prod_p \Big(1 + \sum_{k=1}^{\infty} F(p^k) p^{-ks}\Big), \tag{2.3}$$

where

$$F(p^k) = \sum_{\substack{u_1,u_2,v_1,v_2\\ [u_1u_2,v_1v_2]=p^k}} \mu^2(u_1)\mu^2(v_1)\chi_1(u_1u_2^2)\bar{\chi}_2(v_1v_2^2)\bar{\chi}_1\chi_2(p^k). \tag{2.4}$$

In particular we have

$$F(p) = \chi_0(p) + \sum_{\chi \in \mathcal{X} \setminus \{\bar{\chi}_1\chi_2\}} \chi(p),$$

where $\mathcal{X} = \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\bar{\chi}_2, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_1\bar{\chi}_2, \bar{\chi}_1\chi_2\}$ (and the entries are considered to be formally distinct), and

$$|F(p^k)| \le 8k.$$

Thus the Dirichlet series f converges absolutely for $\sigma > 1$ and

$$f_{\chi_1,\chi_2}(s) = G_{\chi_1,\chi_2}(s) L(s, \chi_0) \prod_{\chi \in \mathcal{X}} L(s, \chi), \tag{2.5}$$

where $G_{\chi_1,\chi_2}(s)$ is a function which is analytic in the region $\Re s > 1/2$ and satisfies

$$G(s) \ll 1 \quad (\sigma \ge 1/2 + \delta)$$

for any fixed $\delta > 0$. As χ_1, χ_2 are not characters of order 1 or 2, $f_{\chi_1,\chi_2}(s)$ has a triple pole at $s = 1$ when $\chi_1 = \chi_2$ or $\chi_1\chi_2 = \chi_0$, and a simple pole otherwise. By Corollary 1.17 and Lemma 10.15 of [MV], for fixed a,

$$L(s, \chi) - \frac{E(\chi)\phi(a)}{a(s-1)} \ll 2 + |t|$$

uniformly for $\sigma \ge 1/2$, where $E(\chi)$ is 1 when $\chi = \chi_0$ and 0 otherwise. Hence by (5.25) of [MV],

$$\sum_{n=1}^{\infty} \sum_{u|n^2} \sum_{v|n^2} \chi_1(u)\bar{\chi}_2(v)\bar{\chi}_1\chi_2(n) e^{-n/N} = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} f_{\chi_1,\chi_2}(s) N^s \Gamma(s)\, ds,$$

where $\theta > 1$. Since the gamma function decays exponentially fast on any vertical line we may move the vertical path to the 3/4-line picking up the residue of the integrand at $s = 1$. The residue contributes an amount

$$\ll N(\log N)^2$$

and the new vertical path contributes $\ll N^{3/4}$. This completes the proof of Theorem 1. ∎

3. Proof of Theorem 2. By Theorem 1, we expect that for almost all n with $(a, n) = 1$, $R(n; a)$ is close to

$$\frac{1}{\phi(a)} \sum_{\substack{\chi \bmod a \\ \chi^2 = \chi_0}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u).$$

Thus we need to examine the contribution from the characters modulo a of order 1 and 2. For general a, there may be many quadratic characters modulo a. Nevertheless we believe that the major contribution to the sum above comes from the principal character, and this is of size

$$\frac{d(n^2)}{\phi(a)}.$$

Thus, for fixed a, $\log R(n;a)$ should have the normal order of $\log d(n^2)$, namely $(\log 3)\log\log n$. When $(n,a)>1$ we have

$$R(n;a) = R(n/(n,a); a/(n,a)) \tag{3.1}$$

and so we can expect that the general case follows from the special case $(n,a)=1$.

Before embarking on the proof of Theorem 2, we state a lemma. We define, for any quadratic character χ,

$$\Omega_\chi(n) = \mathrm{card}\{p,k : k\ge 1,\ p^k \mid n,\ \chi(p^k)=1\}.$$

LEMMA 5. *Suppose that χ is a quadratic character to a fixed modulus a and that $N \ge 3$. Then*

$$\sum_{n\le N}\Big(\Omega_\chi(n) - \frac{1}{2}\log\log N\Big)^2 \ll N\log\log N,$$
$$\sum_{1<n\le N}\Big(\Omega_\chi(n) - \frac{1}{2}\log\log n\Big)^2 \ll N\log\log N.$$

Proof. The proof follows in the same way as Turán's theorem (see Theorem 2.12 of [MV]) on observing that

$$\sum_{\substack{p\le N\\ \chi(p)=1}} \frac{1}{p} = \frac{1}{2}\log\log N + O(1),$$

and this is readily deduced from Corollary 11.18 of [MV]. ∎

It is an immediate consequence of the above lemma that $\Omega_\chi(n)$ has normal order $\frac{1}{2}\log\log n$. In particular, for any fixed $\varepsilon>0$, for almost all n,

$$3^{\Omega_\chi(n)} < 3^{(1/2+\varepsilon)\log\log n}.$$

Now, for any quadratic character χ modulo a, let

$$g_\chi(n) = \sum_{u\mid n^2} \chi(u).$$

This is

$$\prod_{p^k \| n} (1+\chi(p)+\chi^2(p)+\cdots+\chi^{2k}(p)).$$

When $\chi(p) = -1$ the general factor is 1, and when $\chi(p) = 1$ it is $2k+1$. Hence

$$0 < g_\chi(n) \le 3^{\Omega_\chi(n)}.$$

Thus for any fixed $\varepsilon > 0$, for every quadratic character modulo a, for almost all n,

$$g_\chi(n) < (\log n)^{\frac{1}{2}\log 3+\varepsilon}. \tag{3.2}$$

Let

$$r(n;a) = \frac{1}{\phi(a/(n,a))} \sum_{\substack{\chi \bmod a/(n,a)\\ \chi^2=\chi_0}} \bar{\chi}(-n/(n,a)) g_\chi(n/(n,a)).$$

Since $R(n;a) = R(n/(n,a); a/(n,a))$, it follows by Theorem 1 that

$$\sum_{n\le N} (R(n;a) - r(n;a))^2 = \sum_{d|a} \sum_{\substack{m\le N/d\\ (m,a/d)=1}} \left(R\left(m;\frac{a}{d}\right) - r\left(m;\frac{a}{d}\right)\right)^2 \ll N(\log N)^2.$$

Hence, for any fixed $\varepsilon > 0$, for almost all n we have

$$|R(n;a) - r(n;a)| < (\log n)^{1+\varepsilon}.$$

Therefore, by (3.2), for almost all n,

$$\left| R(n;a) - \frac{d((n/(a,n))^2)}{\phi(a/(a,n))} \right| < (\log n)^{1+2\varepsilon}. \tag{3.3}$$

Now $3 \le d(p^{2k}) = 1 + 2k \le 3^k$. Hence

$$3^{\omega(n)-\omega(a)} \le d((n/(a,n))^2) \le 3^{\Omega(n)} \tag{3.4}$$

and it follows that

$$(\log n)^{\log 3-\varepsilon} < \frac{d((n/(a,n))^2)}{\phi(a/(a,n))} < (\log n)^{\log 3+\varepsilon}$$

for almost all n. Theorem 2 now follows.

4. Proof of Theorem 3. By (3.3) and (3.4), for every fixed $\varepsilon > 0$, for almost all n,

$$\frac{3^{\omega(n)}}{\phi(a/(a,n))} - (\log n)^{1+\varepsilon} < R(n;a) < 3^{\Omega(n)} + (\log n)^{1+\varepsilon}.$$

Moreover, for almost all n we have $\Omega(n) \ge \omega(n) > (1-\varepsilon)\log\log n$. Hence for any δ with $0 < \delta < \log 3 - 1$ we have, for almost all n,

$$3^{\omega(n)-\omega(a)-\log\phi(a/(a,n))} \exp(-(\log n)^{-\delta}) < R(n;a) < 3^{\Omega(n)} \exp((\log n)^{-\delta})$$

and so

$$3^{\omega(n)}\exp(-\varepsilon\sqrt{\log\log n}) < R(n;a) < 3^{\Omega(n)}\exp(\varepsilon\sqrt{\log\log n}).$$

Let

$$S(N;z) = \operatorname{card}\left\{n \le N : \frac{\log R(n;a) - (\log 3)\log\log n}{\log 3\sqrt{\log\log n}} \le z\right\},$$
$$S_-(N;z) = \operatorname{card}\left\{n \le N : \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} \le z\right\},$$
$$S_+(N;z) = \operatorname{card}\left\{n \le N : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \le z\right\},$$

Then for a non-negative monotonic function $\eta(n)$ tending to 0 sufficiently slowly as $N \to \infty$ we have

$$-\eta(N)N + S_-(N;z-\varepsilon) < S(N;z) < \eta(N)N + S_+(N;z+\varepsilon).$$

Hence, by the Erdős–Kac theorem (see, for example Theorem 7.21 and Exercise 7.4.4 of [MV]),

$$\Phi(z-\varepsilon) \le \liminf_{N\to\infty} N^{-1}S(N;z) \le \limsup_{N\to\infty} N^{-1}S(N;z) \le \Phi(z+\varepsilon).$$

The theorem now follows from the continuity of Φ.

5. Proof of Theorem 4. By a similar discussion to that in §2, we can show that the generating Dirichlet series for $R(n;a)^2$ is

$$\sum_{\substack{n=1\\(n,a)=1}}^{\infty} \frac{R(n;a)^2}{n^s} = \frac{1}{\phi(a)^2}\sum_{\chi_1,\chi_2 \bmod a} \bar{\chi}_1\chi_2(-1) f_{\chi_1,\chi_2}(s),$$

where $f_{\chi_1,\chi_2}(s)$ is analytic in the region $\Re s > 1/2$ and is given by (2.5). Here $f_{\chi_1,\chi_2}(s)$ has a pole at 1 of order at least 1, and as high as 9 exactly when χ_1 and χ_2 are equal to the principal character χ_0. Now on applying Perron's formula we have, for $\theta = 1 + 1/\log(2N)$,

$$\sum_{\substack{n\le N\\(n,a)=1}} R(n;a)^2 = \sum_{\chi_1,\chi_2 \bmod a} \frac{\bar{\chi}_1\chi_2(-1)}{2\pi i\phi(a)^2}\int_{\theta-iT}^{\theta+iT} f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\,ds + O_a(N^{1+\varepsilon}/T). \tag{5.1}$$

Since we are shooting for the asymptotics for the mean square, smoothing factors of the kind used in Section 2 are best avoided. Since the integrand includes a product of nine L-functions, we cannot expect to be able to move the vertical integral path too close to the 1/2-line, in the current state of knowledge. Nevertheless, the following result of Meurman [Me1] which extends Heath-Brown's theorem [H-B] on the twelfth power moment of the

Riemann zeta function to Dirichlet L-functions, provides a starting point for the analysis.

Lemma 6.

$$\sum_{\chi \bmod a} \int_{-T}^{T} |L(1/2+it,\chi)|^{12}\, dt \ll a^3 T^{2+\varepsilon},$$

where $\varepsilon > 0$, $a \geq 1$ *and* $T \geq 2$.

Then, adapting the argument of Chapter 8 of Ivić [Iv] for the Riemann zeta function to Dirichlet L-functions establishes the following.

Lemma 7.

$$\int_{-T}^{T} |L(35/54+it,\chi)|^{9}\, dt \ll_a T^{1+\varepsilon},$$

where $\varepsilon > 0$, χ *is a fixed Dirichlet character modulo* $a \geq 1$ *and* $T \geq 2$.

If one utilises the sharpest estimates for the underlying exponential sums, Lemma 7 is susceptible to slight improvements.

Now, we move the vertical integral path in (5.1) to the 35/54-line, picking up the residue of the integrand at 1. Thus

$$\int_{\theta-iT}^{\theta+iT} f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\, ds = \int_{\theta-iT}^{35/54-iT} f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\, ds + \int_{35/54+iT}^{\theta+iT} f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\, ds + \int_{35/54-iT}^{35/54+iT} f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\, ds + \mathrm{Res}_{s=1}\Big(f_{\chi_1,\chi_2}(s)\frac{N^s}{s}\Big).$$

Here, in order to deal with the contribution from the horizontal paths, we cannot afford to use the crude convexity bounds on Dirichlet L-functions, due to the large number of L-functions in the integrand. Fortunately, a sharper bound has been established by Pan & Pan in Theorem 24.2.1 of [PP]:

Lemma 8. *Let* $l \geq 3$, $L = 2^{l-1}$ *and* $\sigma_l = 1 - l(2L-2)^{-1}$. *When* $\sigma \geq \sigma_l$,

$$L(\sigma+it,\chi) \ll_a |t|^{1/(2L-2)} \log|t|$$

holds uniformly for $|t| \geq 2$.

When $l = 3$ we obtain

$$L(\sigma+it,\chi) \ll_a |t|^{1/6} \log|t|$$

uniformly for $|t| \geq 2$ and $\sigma \geq 1/2$, and when $l = 4$,

$$L(\sigma+it,\chi) \ll_a |t|^{1/14} \log|t|$$

uniformly for $\sigma \ge 5/7$. Thus, by the convexity principle for Dirichlet series,

$$L(\sigma + it, \chi) \ll_a |t|^{\mu(\sigma)+\varepsilon}$$

uniformly for $|t| \ge 2$ and $\sigma \ge 1/2$, where

$$\mu(\sigma) = \begin{cases} \dfrac{1}{6} - \dfrac{4}{9}\left(\sigma - \dfrac{1}{2}\right) & \text{when } 1/2 \le \sigma \le 5/7, \\ \dfrac{1-\sigma}{4} & \text{when } 5/7 < \sigma \le 1, \\ 0 & \text{when } 1 < \sigma. \end{cases}$$

We note that $\mu(35/54) = 49/486 < 1/9$ and $\mu(5/7) = 1/14$.

Now the horizontal paths contribute

$$\ll \int_{35/54}^{1+\varepsilon} N^\sigma |f_{\chi_1,\chi_2}(\sigma + iT)| T^{-1}\, d\sigma,$$

and this is

$$\ll \max_{35/54 \le \sigma \le 1+\varepsilon} N^\sigma T^{9\mu(\sigma)-1+\varepsilon},$$

and by the piecewise linearity of σ and $\mu(\sigma)$ this is

$$\ll N^{1+\varepsilon}T^{-1} + N^{5/7}T^{9\mu(5/7)-1+\varepsilon} + N^{35/54}T^{9\mu(35/54)-1+\varepsilon}.$$

When $T = N$ this is

$$\ll N^{35/54+\varepsilon}.$$

On the other hand, by Lemma 7 the vertical path also contributes

$$\ll N^{35/54+\varepsilon}.$$

The main term comes from the residual contributions, which, in the case that $\chi_1 = \chi_2 = \chi_0$, is $NP_8(\log N; a)$ where $P_8(\cdot\,; a)$ is a polynomial of degree 8 whose coefficients depend on a. Notice that for other choices of χ_1 and χ_2, the residual contribution gives a polynomial of $\log N$ of lower degree than above.

For the leading coefficient, we need more precise information about f_{χ_0,χ_0}. By (2.2)–(2.4) we have

$$\begin{aligned} f_{\chi_0,\chi_0} &= L(s,\chi_0) \prod_{p\nmid a} \left(1 + \sum_{k=1}^{\infty} \frac{8k}{p^{ks}}\right) \\ &= L(s,\chi_0)^9 \prod_{p\nmid a} (1 + 6p^{-s} + p^{-2s})(1 - p^{-s})^6, \end{aligned}$$

from which the leading coefficient is readily deduced. This completes the proof of Theorem 4. ∎

In conclusion we remark that a concomitant argument will give

$$\sum_{\substack{n\le N\\(n,a)=1}} R(n;a)^k = NP_{3^k-1}(\log N;a) + O_a(N^{\alpha_k+\varepsilon})$$

for any $\varepsilon > 0$, where $P_{3^k-1}(\cdot\,;a)$ is a polynomial of degree 3^k-1 whose coefficients depend on a and α_k is a constant that depends on the best 3^kth power moment estimates for $L(s,\chi)$ in the critical strip and the quantity $\mu(\sigma)$ defined above. This question is closely related to the generalised divisor problem, and one is referred to Chapter 13 in Ivić [Iv] for more details.

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