ACTA ARITHMETICA
155.3 (2012)

Arf equivalence II

by

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Dedicated to Andrzej Schinzel
on the occasion of his seventy-fifth birthday

1. Introduction. A *global Artin root number* is a complex number $W(\rho)$ of modulus 1 appearing in the functional equation of an Artin L-series

$$\Lambda(s,\rho) = W(\rho)\cdot\Lambda(1-s,\overline{\rho}),$$

in which ρ is a continuous representation

$$\rho : \mathrm{Gal}(\overline{K}|K) \to \mathrm{Gl}_n(\mathbb{C})$$

of the absolute Galois group of a global field K, where $\overline{\rho}$ denotes the complex-conjugate representation, and where $\Lambda(s,\rho)$ is the extended Artin L-series with gamma factors at the archimedean places of K.

Work of Deligne ([4], [5]), based partly on unpublished work of Langlands, shows that the global Artin root number can be written as a product

$$W(\rho) = \prod W_P(\rho)$$

of other complex numbers of modulus 1, called *local Artin root numbers* (Deligne calls them simply *local constants*). Given ρ, there is one local root number $W_P(\rho)$ for each non-trivial place P of the base field K, and $W_P(\rho) = 1$ for almost all P.

When ρ is a real representation, then each local root number is a fourth root of unity, and, by the theorem of Fröhlich–Queyrut ([6]), the global root number $W(\rho)$ is $+1$. This means that the product of the local root numbers of a real representation is $+1$, so the theorem of Fröhlich–Queyrut is a *reciprocity law* or a *product formula* for local root numbers.

Here is a key example. Let K be a number field. For each element $a \in K^*$ there is the 1-dimensional real representation ρ_a of the absolute Galois group

2010 *Mathematics Subject Classification*: Primary 11E12; Secondary 19F15.
Key words and phrases: Arf invariant, root number, Hilbert symbol, quadratic refinement, Witt equivalence.

DOI: 10.4064/aa155-3-8

G_K given by

$$\rho_a(\sigma) = \frac{\sigma(\sqrt{a})}{\sqrt{a}}$$

for $\sigma \in G_K$. The global root number is 1. For a place P of K, the local root number $W_P(\rho_a)$ depends only on the square-class of a in the local square-class group K_P^*/K_P^{*2}. Tate's explicit formula for $W_P(\rho_a)$ will be recalled in Section 3 below. For P fixed and for a varying over the local square-classes, we collect these local root numbers $W_P(\rho_a)$ into a function

$$r_P : K_P^*/K_P^{*2} \to \mathbb{C}^*,$$

called the *local root number function at* P, defined by

$$r_P(a) = W_P(\rho_a).$$

Tate proved

$$r_P(ab) = (a,b)_P \cdot r_P(a) \cdot r_P(b) \tag{1.1}$$

This has the following interpretation. The local square-class group K_P^*/K_P^{*2} is a multiplicatively written vector space over the finite field $\mathbb{F}_2$ of two elements. The *Hilbert symbol* $(\ ,\ )_P$ at P is a map from two copies of this vector space to the multiplicative group $\{\pm 1\}$, which we identify with the (additive group of the) finite field $\mathbb{F}_2$. With this identification, the Hilbert symbol at P defines a non-degenerate symmetric bilinear form, making the local square-class group into an inner-product space over $\mathbb{F}_2$. Recall that for any inner-product space (V, B) over $\mathbb{F}_2$, a *classical quadratic refinement* is a map $q : V \to \mathbb{F}_2$ satisfying

$$q(v + w) = B(v, w) + q(v) + q(w) \tag{1.2}$$

for all $v, w \in V$. Comparing (1.1) with (1.2) shows that, for the Hilbert symbol, the local root number function is a multiplicative counter-part of a classical refinement.

More precisely, let θ be the unique isomorphism from the additive group $\{0, 1\}$ to the multiplicative group $\{\pm 1\}$ and let $\beta = \theta \circ B$. Then a (*multiplicative*) *quadratic refinement* of an $\mathbb{F}_2$-inner-product space (V, B) is a map $q : V \to \mathbb{C}^*$ satisfying

$$q(v + w) = \beta(v, w) \cdot q(v) \cdot q(w) \tag{1.3}$$

for all $v, w \in V$. Note that we speak of a refinement of (V, B), or more simply of B, rather than referring to β. There are always exactly 2^n multiplicative quadratic refinements of B, where n denotes the dimension of V. These all arise from classical refinements when (V, B) is totally isotropic (a so-called type II space); there are no classical refinements when (V, B) contains a non-zero non-isotropic vector (a type I space). When (V, B) has type II, two classical refinements are isometric precisely when their classical Arf invariants agree. There is a notion of Arf invariant for multiplicative refinements as well, defined in the next section.

Now let (V,B,q) and (V',B',q') be two inner-product spaces, each with a chosen multiplicative refinement. Theorem 4 of [10] states that there is an isometry from q to q' if and only if V and V' have the same dimension, (V,B) and (V',B') have the same type (I or II), and (V,B,q) and (V',B',q') have the same Arf invariant.

We can now formulate what it means for two number fields K, L to have "the same" local root numbers. Let Ω_K denote the collection of all non-trivial places of K and let Ω_L denote the collection of all non-trivial places of L. We say that K and L are *Arf equivalent* if there is a bijection $T : \Omega_K \to \Omega_L$ for which

$$\dim K_P^*/K_P^{*2} = \dim L_{TP}^*/L_{TP}^{*2},$$
$$\mathrm{type}(\ ,\ )_P = \mathrm{type}(\ ,\ )_{TP},$$
$$\mathrm{Arf}(r_P) = \mathrm{Arf}(r_{TP})$$

for every $P \in \Omega_K$.

It would be more precise to say that these equations define K and L to be *locally local root number equivalent*, but it is easier to say *Arf equivalent.* The notion of Arf equivalent number fields was introduced in [10], along with another notion called *globally local root number equivalent.* It was proved there (see Theorem 5 in [10]) that when K and L are globally local root number equivalent then they are also Arf equivalent, which in turn implies that K and L are Witt equivalent (have isomorphic Witt rings). Carpenter [1] and Czogała [3] independently proved that quadratic number fields fall into exactly seven classes up to Witt equivalence. In this paper we show that quadratic fields fall into infinitely many Arf equivalence classes.

2. Arf equivalence.

In this section we collect several relevant definitions and theorems from [10] and from [12]. These will be assembled in the following section to give our main result.

Throughout this section V denotes a vector space of finite dimension n over the field with two elements $\mathbb{F}_2$, and B denotes a non-degenerate symmetric bilinear form on V. We refer to the pair (V,B) as an *inner-product space.* Finally, let q be a multiplicative quadratic refinement of B as defined in the introduction.

DEFINITION 2.1. Let (V,B) be an inner-product space. Then B is said to be *type II* when B is totally isotropic, and is said to be *type I* otherwise.

Note that when P is a place of a number field K, then the Hilbert symbol $(\ ,\ )_P$ has type II if and only if $(a,a)_P = 1$ for all $a \in K_P^*/K_P^{*2}$. Since

$(a,a)_P = (a,-1)_P$ for all a, it follows that the Hilbert symbol has type II if and only if -1 is a local square in K_P^*.

Now fix an inner-product space (V, B) and let $\beta = \theta \circ B$ where

$$\theta(x) = (-1)^x$$

for $x \in \mathbb{F}_2$.

Definition 2.2 ([10]). A (multiplicative) *refinement* of (V, B) is a map $q : V \to \mathbb{C}^*$ for which $q(v+w) = \beta(v,w) \cdot q(v) \cdot q(w)$ for every $v, w \in V$.

Note that a quadratic refinement is not required to take values in $\{\pm 1\}$. The reader can check from the definitions that B has type II if and only if every quadratic refinement of B takes values in $\{\pm 1\}$.

When V is fixed, we say that q is a *quadratic refinement* of B. Now let q be a refinement of an inner-product space (V, B) and q' a refinement of a second space (V', B'). By definition, (V, B, q) is isometric to (V', B', q') provided that there is a vector-space isomorphism ϕ from V to V' satisfying $q'(\phi(v)) = q(v)$ for all $v \in V$. When there is no danger of confusion, we say q is isometric to q'. An isometry of refinements is necessarily also an isometry of bilinear forms, but not conversely. Since the field of scalars is $\mathbb{F}_2$, the bilinear form invariants are just dimension and type (see Theorems 19 and 20 of [7]). A third invariant is needed for isometry of quadratic refinements.

Definition 2.3 ([10]). The *Arf invariant* of (V, B, q) is

$$\mathrm{Arf}(q) = 2^{-n/2} \sum_{v \in V} q(v).$$

With the Arf invariant we can state

Theorem 2.4 ([10, Theorem 4]). *(V, B, q) is isometric to (V', B', q') if and only if*

$$\dim V = \dim V',$$
$$\mathrm{type}(V, B) = \mathrm{type}(V', B'),$$
$$\mathrm{Arf}(V, B, q) = \mathrm{Arf}(V', B', q').$$

Let K be a number field with a prime P. Then the commutative multiplicative group K_P^*/K_P^{*2} becomes a finite-dimensional vector space over $\mathbb{F}_2$, with multiplication by scalars defined by $a \cdot t = t^a$. The Hilbert symbol can be considered to be a multiplicative bilinear form on K_P^*/K_P^{*2}. The local root number function is the map $r_P : K_P^*/K_P^{*2}$ assigning to each local square-class a the value

$$r_P(a) = W_P(\rho_a)$$

of the local root number of the representation ρ_a discussed in the introduction. Tate's explicit formula for $r_P(a)$ is given in the next section. For now we wish to point out that Tate [12, p. 126] proved that

$$r_P(ab) = (a,b)_P \cdot r_P(a) \cdot r_P(b)$$

for all $a, b \in K_P^*/K_P^{*2}$. This means that the local root number function r_P is a multiplicative refinement of the Hilbert symbol on K_P^*/K_P^{*2}.

The next definition captures the notion that two number fields have local root number functions that are *everywhere isometric.*

DEFINITION 2.5 ([10]). Two number fields K and L are called *Arf equivalent* if there is a bijection T between the set Ω_K of places of K and the set Ω_L of places of L such that

$$\begin{aligned} \dim K_P^*/K_P^{*2} &= \dim L_{TP}^*/L_{TP}^{*2}, \\ \mathrm{type}(\ ,\ )_P &= \mathrm{type}(\ ,\ )_{TP}, \\ \mathrm{Arf}(r_P) &= \mathrm{Arf}(r_{TP}), \end{aligned}$$

for every place $P \in \Omega_K$.

As a direct consequence of the definition above, if K and L are Arf equivalent then there is a bijection T of the places of K and of L so that for every place P of K there is a local square-class isomorphism

$$t_P : K_P^*/K_P^{*2} \to L_{TP}^*/L_{TP}^{*2}$$

that preserves the Hilbert symbol at P:

$$(a,b)_P = (t_P(a), t_P(b))_{TP}. \tag{2.1}$$

It has been proved in [11, p. 370] that (2.1) holds for every P of K and all non-zero elements $a, b \in K$ if and only if the Witt rings $W(K)$ and $W(L)$ are isomorphic, that is, if and only if the number fields K and L are Witt equivalent. This proves

THEOREM 2.6 ([10]). *If two number fields K and L are Arf equivalent, then they are Witt equivalent.*

Returning to the definition of Arf equivalence, it should be pointed out that the local dimensions are given by

$$\dim_{\mathbb{F}_2} K_P^*/K_P^{*2} = \begin{cases} 0 & \text{if } P \text{ is complex,} \\ 1 & \text{if } P \text{ is real,} \\ 2 & \text{if } P \text{ is finite and non-dyadic,} \\ 2 + [K_P : \mathbb{Q}_2] & \text{if } P \text{ is dyadic (contains 2).} \end{cases} \tag{2.2}$$

Thus, if one wishes to construct a bijection of places T leading to an Arf equivalence from K to L then T must map complex places of K to complex places of L, real places of K to real places of L, dyadic places of K to dyadic places of L (and the local dyadic degrees must be preserved), and finite non-dyadic places of K to finite non-dyadic places of L. Moreover, the square-character of -1 at P and at TP must agree for every place P.

Carpenter and Czogała have independently given the following classification of quadratic number fields up to Witt equivalence:

THEOREM 2.7 ([1] and [3]). *There are exactly seven Witt equivalence classes of quadratic number fields, represented by* $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\pm 2})$, $\mathbb{Q}(\sqrt{\pm 7})$, $\mathbb{Q}(\sqrt{\pm 17})$. *For a square free integer* $n \neq 1$, *the quadratic number field* $\mathbb{Q}(\sqrt{n})$ *is Witt equivalent to* $\mathbb{Q}(\sqrt{d})$, *where* $d = -1$ *if* $n = -1$ *and*

$$d = \begin{cases} \operatorname{sign}(n) \cdot 2 & \text{if } |n| \equiv 2, 3, 5, 6 \pmod 8, \\ \operatorname{sign}(n) \cdot 7 & \text{if } |n| \equiv 7 \pmod 8, \\ \operatorname{sign}(n) \cdot 17 & \text{if } |n| \equiv 1 \pmod 8. \end{cases} \tag{2.3}$$

Since Arf equivalent fields are Witt equivalent, a set of pairwise non-Witt equivalent fields is also a set of pairwise non-Arf equivalent fields. It follows at once that there are at least seven Arf equivalence classes in quadratic number fields. In the next section we will see that there are in fact infinitely many Arf equivalence classes of quadratic fields. For this, it is crucial to find local root numbers.

3. Computing local root numbers. We begin with Tate's formula for the local root numbers $r_P(a)$.

LEMMA 3.1 (Tate's formula, [12]). *Let P be a place of a number field K. Then*

(a) *If $K_P = \mathbb{C}$, then $r_P(a) = 1$.*
(b) *If $K_P = \mathbb{R}$, then*

$$r_P(a) = \begin{cases} 1 & \text{if } a \text{ is a square in } \mathbb{R}^*, \\ -i & \text{otherwise.} \end{cases} \tag{3.1}$$

(c) *If P is a non-Archimedean place, then*

$$r_P(a) = \mathcal{N}(f_a^{-1/2}) \sum_{x \in O_{K_P}^* \bmod^* f_a} \overline{\alpha}_a(d^{-1}x)\psi_{K_P}(d^{-1}x),$$

where $\mathcal{N}$ denotes the absolute norm, f_a is the conductor of a, $dO_{K_P} = f_a D_{K_P}$, and ψ_{K_P} is the map $K_P \to \mathbb{C}^*$, called the canonical character, *which is the composition*

$$K_P \xrightarrow{\alpha} \mathbb{Q}_p \xrightarrow{\beta} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\gamma} \mathbb{Q}/\mathbb{Z} \xrightarrow{\delta} \mathbb{R}/\mathbb{Z} \xrightarrow{\epsilon} \mathbb{C}^*,$$

in which α is the trace map, β, γ, δ are the natural maps and $\epsilon(x + \mathbb{Z}) = e^{2\pi x i}$. The conductor f_a satisfies $\alpha_a(1 + f_a) = 1$ and $1 + f_a = U_{K_P}$ if and only if $f_a = O_{K_P}$.

From now on, m is a positive square free integer and $e = \pm 1$. The results in the following lemma, which we will derive from Tate's formula, are also listed in [2].

LEMMA 3.2. *Let $K := \mathbb{Q}(\sqrt{em})$ be a quadratic field. Suppose an odd rational prime p is split in K, meaning $pO_K = PP'$, where P and P' are distinct prime ideals in O_K. Then*

(a) *$r_P(\epsilon) = 1$, where ϵ is a non-square unit in K_P.*
(b) *$r_P(p) = 1$ if $p \equiv 1 \pmod 4$ and $r_P(p) = -i$ if $p \equiv 3 \pmod 4$.*

Proof. Note $[K_P : \mathbb{Q}_p] = e(P|p) \cdot f(P|p) = 1$. So write $K_P^*/K_P^{*2} = \{1, \epsilon, p, \epsilon p\}$. The local absolute different D_{K_P} equals O_{K_P} since P is unramified (see p. 62 in [9]).

(a) Define a map $\alpha_\epsilon : K_P \to \{1, -1\}$ by $\alpha_\epsilon(x) = (\epsilon, x)_p$. Then $\alpha_\epsilon(U_{K_P}) = 1$. So $f_\epsilon = O_{K_P}$. This implies $\mathcal{N} f_\epsilon = 1$. Thus $f_\epsilon D_{K_P} = (1)$. So by Tate's formula,

$$\begin{aligned} r_P(\epsilon) &= \mathcal{N} f_\epsilon^{-1/2} \sum_{x \in U_{K_P} \bmod^* f_\epsilon} \alpha_\epsilon(x) \psi_{K_P}(x) \\ &= 1^{-1/2} \sum_{x=1} \alpha_\epsilon(x) \psi_{K_P}(x) = (\epsilon, 1)_P = 1. \end{aligned}$$

(b) Define $\alpha_p : K_P^* \to \{1, -1\}$ by $\alpha_p(x) = (p, x)_p$. Then $f_p = P$ since $\alpha_p(1+P) = 1$ by Hensel's lemma. So $\mathcal{N} f_p$ equals $p^{f(P|p)} = p$ and $f_p D_{K_P} = P = (p)$. Therefore

$$\begin{aligned} r_P(p) &= \mathcal{N} f_p^{-1/2} \sum_{x \in U_{K_P} \bmod^* f_p} \alpha_p(p^{-1}x) \psi_{K_P}(p^{-1}x) \\ &= \frac{1}{\sqrt{p}} (p, p)_p \sum_{x \in U_{K_P} \bmod^* f_p} \alpha_p(x) \psi_{K_P}\left(\frac{x}{p}\right) \\ &= \frac{1}{\sqrt{p}} (p, -1)_p \sum_{j \in \mathbb{F}_p^*} (p, x)_p \, e^{2\pi j i/p} = \frac{1}{\sqrt{p}} (p, -1)_p \sum_{j \in \mathbb{F}_p^*} \left(\frac{x}{p}\right) e^{2\pi j i/p}. \end{aligned}$$

So if $p \equiv 1 \pmod 4$, then, by the known value of the Gaussian sum,

$$r_P(p) = \frac{1}{\sqrt{p}} \cdot 1 \cdot \left(\frac{1}{p}\right) \cdot \sqrt{p} = 1,$$

while if $p \equiv 3 \pmod 4$, then similarly

$$r_P(p) = \frac{1}{\sqrt{p}} \cdot (-1) \cdot \left(\frac{1}{p}\right) \cdot i \cdot \sqrt{p} = -i. \blacksquare$$

LEMMA 3.3. *Let $K := \mathbb{Q}(\sqrt{em})$ be a quadratic field. Suppose an odd rational prime p is inert in K, meaning $pO_K = P$, where P is a prime ideal in O_K. Then*

(a) *$r_P(\epsilon) = 1$, where ϵ is a non-square unit in K_P.*
(b) *$r_P(p) = -1$.*

Proof. (a) Reread the proof of Lemma 3.2(a).

(b) Define $\alpha_p : K_P^* \to \{1,-1\}$ by $\alpha_p(x) = (p,x)_P$. Then $\alpha_p(1+P)=1$ by Hensel's lemma. So the conductor f_p equals P and $\mathcal{N} f_p = p^{f(P|p)} = p^2$. The local absolute different D_{K_P} equals O_{K_P} since P is unramified. Thus

$$f_p D_{K_P} = P = (p).$$

So by Tate's formula,

$$\begin{aligned}
r_P(p) &= \mathcal{N} f_p^{-1/2} \sum_{x\in U_{K_P} \bmod^* f_p} \alpha_p(p^{-1}x)\psi_{K_P}(p^{-1}x) \\
&= \frac{1}{p}(p,p)_P \sum_{x\in U_{K_P} \bmod^* f_p} \alpha_p(x)\psi_{K_P}\left(\frac{x}{p}\right) \\
&= \frac{1}{p}(p,-1)_P\Big\{ \sum_{x\in \mathbb{F}_{p^2}^*,\, \mathrm{Tr}(x)=0} \alpha_p(x) + \sum_{j=1}^{p-1}\Big(\sum_{x\in \mathbb{F}_{p^2}^*,\, \mathrm{Tr}(x)=j} \alpha_p(x)\Big) e^{2\pi j i/p}\Big\}.
\end{aligned}$$

If $p\equiv 1 \pmod 4$, then

$$r_P(p) = \frac{1}{p}\cdot 1 \cdot \Big\{-(p-1) + \sum_{j=1}^{p-1} e^{\frac{2\pi j i}{p}}\Big\} = \frac{1}{p}\cdot\{-(p-1)+(-1)\} = -1.$$

If $p\equiv 3 \pmod 4$, then

$$r_P(p) = \frac{1}{p}\cdot(-1)\cdot \Big\{(p-1) - \sum_{j=1}^{p-1} e^{\frac{2\pi j i}{p}}\Big\} = \frac{1}{p}\cdot(-1)\cdot\{(p-1)-(-1)\} = -1. \ \blacksquare$$

Lemma 3.4. *Let $K := \mathbb{Q}(\sqrt{em})$ be a quadratic field. Then $r_P(\epsilon) = -1$ for every non-dyadic ramified prime P in K, where ϵ is a non-square unit in K_P^*.*

Proof. Define a map $\alpha_\epsilon(x) := (\epsilon, x)_P$ on K_P^*. Then $\alpha_\epsilon(U_{K_P}) = 1$. So $f_\epsilon = O_{K_P}$. Therefore $\mathcal{N} f_\epsilon = 1$. The absolute different D_{K_P} is clearly $\varPi O_{K_P}$, where $\varPi = \sqrt{em}$. So by Tate's formula,

$$\begin{aligned}
r_P(\epsilon) &= \frac{1}{\sqrt{\mathcal{N} f_\epsilon}} \sum_{x\in U_{K_P} \bmod^* f_\epsilon} \alpha_\epsilon(\varPi^{-1}x)\psi_{K_P}(\varPi^{-1}x) \\
&= \frac{1}{\sqrt{1}} \sum_{x=1} \alpha_\epsilon(\varPi^{-1}x)\psi_{K_P}(\varPi^{-1}x) \\
&= \alpha_\epsilon(\varPi)\cdot\psi_{K_P}(1/\varPi) = (\epsilon, \varPi)_P = -1. \ \blacksquare
\end{aligned}$$

4. Arf invariants and Arf equivalence classes in quadratic number fields

Theorem 4.1. *Suppose $K := \mathbb{Q}(\sqrt{em})$ is a quadratic field. Let P be a non-dyadic split prime or a non-dyadic inert prime in K, where $P\cap\mathbb{Z} = (p)$. Then* $\mathrm{Arf}(r_P) = 1$.

Proof. (1) Suppose P is split in K. Write $K_P^*/K_P^{*2} = \{1, \epsilon, p, \epsilon p\}$. Then

$$r_P(\epsilon p) = (\epsilon, p)_P \cdot r_P(\epsilon) \cdot r_P(p) = (-1) \cdot 1 \cdot r_P(p) = -r_P(p),$$

by Lemma 3.2.

(1a) If $p \equiv 1 \pmod 4$, then by Lemma 3.2,

$$r_P(a) = \begin{cases} 1 & \text{if } a = 1, \epsilon, p, \\ -1 & \text{if } a = \epsilon p. \end{cases} \tag{4.1}$$

Therefore

$$\begin{aligned} \mathrm{Arf}(r_P) &= 2^{-2/2} \sum_{a \in K_P^*/K_P^{*2}} r_P(a) = \frac{1}{2}\{r_p(1) + r_p(\epsilon) + r_p(p) + r_p(\epsilon p)\} \\ &= \frac{1}{2}\{1 + 1 + 1 + (-1)\} = 1. \end{aligned}$$

(1b) If $p \equiv 3 \pmod 4$, then by Lemma 3.2,

$$r_P(a) = \begin{cases} 1 & \text{if } a = 1, \epsilon, \\ -i & \text{if } a = p, \\ i & \text{if } a = \epsilon p. \end{cases} \tag{4.2}$$

Therefore

$$\begin{aligned} \mathrm{Arf}(r_P) &= 2^{-2/2} \sum_{a \in K_P^*/K_P^{*2}} r_P(a) = \frac{1}{2}\{r_p(1) + r_p(\epsilon) + r_p(p) + r_p(\epsilon p)\} \\ &= \frac{1}{2}\{1 + 1 + (-i) + i\} = 1. \end{aligned}$$

(2) Suppose P is an inert prime in K. Then $P = (p)$. Write $K_P^*/K_P^{*2} = \{1, \epsilon, p, \epsilon p\}$. Then by Lemma 3.3,

$$r_P(a) = \begin{cases} 1 & \text{if } a = 1, \epsilon, \epsilon p, \\ -1 & \text{if } a = p. \end{cases} \tag{4.3}$$

Therefore

$$\begin{aligned} \mathrm{Arf}(r_P) &= 2^{-2/2} \sum_{a \in K_P^*/K_P^{*2}} r_P(a) = \frac{1}{2}\{r_P(1) + r_P(\epsilon) + r_P(p) + r_P(\epsilon p)\} \\ &= \frac{1}{2}\{1 + 1 + (-1) + 1\} = 1. \ \blacksquare \end{aligned}$$

THEOREM 4.2. *Suppose $K := \mathbb{Q}(\sqrt{em})$ is a quadratic field. Let P be a non-dyadic ramified prime in K with $P \cap \mathbb{Z} = (p)$, where $p \equiv 3 \pmod 4$. Then* $\mathrm{Arf}(r_P) = i$ *or* $-i$.

Proof. Since p is ramified, p must divide n, and $pO_K = P^2$. Let $\Pi = \sqrt{em}$. Write $K_P^*/K_P^{*2} = \{1, \epsilon, \Pi, \epsilon\Pi\}$. Define $\alpha_\Pi : K_P^* \to \{1, -1\}$ by $\alpha_\Pi(x) = (\Pi, x)_P$. Then $\alpha_\Pi(1 + P) = 1$ by Hensel's lemma. So $f_\Pi = P$. Thus

$\mathcal{N} f_{\Pi} = p$. Clearly, $D_{K_P} = \Pi O_{K_P}$. Therefore

$$f_{\Pi} D_{K_P} = P^2 = pO_{K_P}.$$

By Tate's formula,

$$\begin{aligned} r_P(\Pi) &= \frac{1}{\sqrt{p}} \sum_{x \in U_{K_P} \bmod^* f_{\Pi}} \alpha_{\Pi}(p^{-1}x)\psi_{K_P}(p^{-1}x) \\ &= \frac{1}{\sqrt{p}}(\Pi, p)_P \sum_{x \in U_{K_P} \bmod^* P} \alpha_{\Pi}(px)\psi_{K_P}(p^{-1}x) \\ &= \frac{1}{\sqrt{p}}(\Pi, p)_P \sum_{x \in \mathbb{F}_p^*} \left(\frac{x}{p}\right) e^{4\pi xi/p} = i \cdot (\Pi, p)_P \cdot \left(\frac{2}{p}\right) \end{aligned}$$

So we get $r_P(\Pi) = i$ or $-i$. By the way,

$$r_P(\epsilon\Pi) = (\epsilon, \Pi)_P \cdot r_P(\epsilon) \cdot r_P(\Pi) = (-1) \cdot (-1) \cdot r_P = r_P(\Pi)$$

by Lemma 3.4. So

$$\begin{aligned} \mathrm{Arf}(r_P) &= \frac{1}{2} \sum_{a \in K_P^*/K_P^{*2}} r_P(a) = \frac{1}{2}\{r_P(1) + r_P(\epsilon) + r_P(\Pi) + r_P(\epsilon\Pi)\} \\ &= \frac{1}{2}\{1 + (-1) + r_P(\Pi) + r_P(\epsilon\Pi)\} = r_P(\Pi) = e_{K_P} \cdot i, \end{aligned}$$

where $e_{K_P} = 1$ or -1. ■

THEOREM 4.3. *There are infinitely many Arf equivalence classes in quadratic number fields.*

Proof. It is enough to show that two different quadratic number fields $K := \mathbb{Q}(\sqrt{p_1 \cdots p_n})$ and $L := \mathbb{Q}(\sqrt{q_1 \cdots p_l})$ are not Arf equivalent, where $n \neq l$, $p_j, q_h \equiv 3 \pmod 4$ for each j and h, $p_1, \ldots, p_n$ [resp. $q_1, \ldots, q_l$] are pairwise distinct primes respectively. There are n distinct non-dyadic ramified primes $P_1, \ldots, P_n$ in K and l distinct non-dyadic ramified primes $Q_1, \ldots, Q_l$ in L. Write $P_j \cap \mathbb{Z} = (p_j)$ for $j = 1, \ldots, n$ and $Q_h \cap \mathbb{Z} = (q_h)$ for $h = 1, \ldots, l$. By Theorem 4.2, Arf $(r_{P_j}) = e_{K_j} \cdot i$ and Arf $(r_{Q_h}) = e_{L_h} \cdot i$ for each j and h, where $e_{K_j}, e_{L_h} \in \{1, -1\}$. An Arf equivalence from K to L, if it existed, can not match a non-dyadic prime P in K with a dyadic prime Q in L or vice versa since $|K_P^*/K_P^{*2}| \neq |L_Q^*/L_Q^{*2}|$. And a (hypothetical) Arf equivalence cannot match an Archimedean place with a non-Archimedean place by a similar dimension argument. So there should be a one-to-one correspondence between the sets of non-dyadic non-Archimedean places of K and L for the two number fields to be Arf equivalent. On the other hand there is no non-dyadic unramified prime with Arf invariant i or $-i$ by Theorem 4.1. So we should have a bijection between non-dyadic ramified primes of K

and non-dyadic ramified primes of L for K and L to be Arf equivalent. But this is impossible since $n \neq l$. Therefore K and L are not Arf equivalent. ∎

Although there are infinitely many Arf equivalence classes in quadratic fields, there are interesting infinite subsets of the set of all quadratic fields that fall into finitely many Arf equivalence classes. Here is one example.

Let $\mathcal{K}$ be the set of all quadratic fields of the form $\mathbb{Q}(\sqrt{ep})$, where p is a rational positive prime. We will show two quadratic fields $\mathbb{Q}(\sqrt{ep})$ and $\mathbb{Q}(\sqrt{e'p'})$ in $\mathcal{K}$ are Arf equivalent if and only if $e = e'$ and $p \equiv p' \pmod 8$.

THEOREM 4.4. *Let $K := \mathbb{Q}(\sqrt{ep})$ be a quadratic field in $\mathcal{K}$, where p is an odd rational prime with $pO_K = P^2$ for a prime ideal in O_K. Then*

$$\mathrm{Arf}(r_P) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8, \\ -ei & \text{if } p \equiv 3 \pmod 8, \\ -1 & \text{if } p \equiv 5 \pmod 8, \\ ei & \text{if } p \equiv 7 \pmod 8. \end{cases} \tag{4.4}$$

Proof. We will argue for the case when $K := \mathbb{Q}(\sqrt{-p})$, where $p \equiv 3 \pmod 8$. Other cases are very similar. Write $K_P^*/K_P^{*2} = \{1, \epsilon, \Pi, \epsilon\Pi\}$, where $\Pi = \sqrt{-p}$. First we will find $r_P(\Pi)$. Define $\alpha_\Pi : K_P^* \to \{1, -1\}$ by $\alpha_\Pi(x) = (\sqrt{-p}, x)_P$. Then $\alpha_\Pi(1 + P) = 1$ by Hensel's lemma. So $f_\Pi = P$. Thus $\mathcal{N} f_\Pi = p$. Moreover $D_{K_P} = \Pi O_{K_P}$. So $f_\Pi D_{K_P} = (p)$. On the other hand,

$$U_{K_P}/(1 + P) \cong \bar{K}_P^* \cong \mathbb{F}_p^*,$$

where $\bar{K}_P$ is the residue class field of K_P. By Tate's formula,

$$\begin{aligned} r_P(\Pi) &= \mathcal{N} f_\Pi^{-1/2} \sum_{x \in U_{K_P} \bmod^* f_\Pi} \alpha_\Pi(p^{-1}x)\psi_{K_P}(p^{-1}x) \\ &= \frac{1}{\sqrt{p}} \sum_{x \in U_{K_P} \bmod^* f_\Pi} \alpha_\Pi(px)\psi_{K_P}\left(\frac{x}{p}\right) \\ &= \frac{1}{\sqrt{p}} (\sqrt{-p}, p)_P \sum_{x \in \mathbb{F}_p^*} (\sqrt{-p}, x)_P e^{4\pi xi/p} \\ &= \frac{1}{\sqrt{p}} (\sqrt{-p}, -(\sqrt{-p})^2)_P \sum_{x \in \mathbb{F}_p^*} (p, x)_p e^{4\pi xi/p} \\ &= \frac{1}{\sqrt{p}} (\sqrt{-p}, -1)_P \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{4\pi xi/p} = \frac{1}{\sqrt{p}} \cdot (p, -1)_p \cdot i \cdot \sqrt{p} \cdot \left(\frac{2}{p}\right) \\ &= \frac{1}{\sqrt{p}} \cdot (-1) \cdot i \cdot \sqrt{p} \cdot (-1) = i. \end{aligned}$$

So by Lemma 3.4,

$$r_P(\epsilon\Pi) = (\epsilon, \Pi)_P \cdot r_P(\epsilon) \cdot r_P(\Pi) = (-1) \cdot (-1) \cdot i = i.$$

Therefore,

$$\begin{aligned}\mathrm{Arf}(r_P) &= 2^{-2/2} \sum_{a\in K_P^*/K_P^{*2}} r_P(a) = \frac{1}{2}\{r_P(1)+r_P(\epsilon)+r_P(\Pi)+r_P(\epsilon\Pi)\}\\ &= \frac{1}{2}\{1+(-1)+i+i\} = i. \ \blacksquare\end{aligned}$$

Recall $\mathcal{K}$ is the set of all quadratic fields of the form $\mathbb{Q}(\sqrt{ep})$, where p is a rational positive prime. The following result is taken from [8].

Theorem 4.5. *There are exactly ten Arf equivalence classes in* $\mathcal{K}$. *They are represented by* $\mathbb{Q}(\sqrt{d})$ *for* $d = \pm 2, \pm 3, \pm 5, \pm 7, \pm 17$. *A quadratic field* $\mathbb{Q}(\sqrt{n})$ *in* $\mathcal{K}$, *where* $|n|$ *is a rational prime, is Arf equivalent to* $\mathbb{Q}(\sqrt{d})$ *with* d *determined as follows:*

$$d = \begin{cases} \mathrm{sign}(n)\cdot 2 & \text{if } |n|\equiv 2 \pmod 8,\\ \mathrm{sign}(n)\cdot 3 & \text{if } |n|\equiv 3 \pmod 8,\\ \mathrm{sign}(n)\cdot 5 & \text{if } |n|\equiv 5 \pmod 8,\\ \mathrm{sign}(n)\cdot 7 & \text{if } |n|\equiv 7 \pmod 8,\\ \mathrm{sign}(n)\cdot 17 & \text{if } |n|\equiv 1 \pmod 8.\end{cases} \tag{4.5}$$

Proof. It is clear that a real quadratic field in $\mathcal{K}$ is not Arf equivalent to an imaginary quadratic field in $\mathcal{K}$. In $\mathbb{Q}(\sqrt{2})$ there is no non-dyadic ramified prime, so there are no non-dyadic non-Archimedean places with Arf invariants $\pm i$ or -1. It follows from Theorem 4.4 that $\mathbb{Q}(\sqrt{2})$ is not Arf equivalent to either $\mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(\sqrt{5})$. By Theorems 4.1 and 4.4 we see that $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ are not Arf equivalent. Similarly, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-5})$ are in pairwise distinct Arf equivalence classes. So the ten representations above are different Arf equivalence classes in $\mathcal{K}$ by Theorem 2.7.

Suppose two quadratic fields $K := \mathbb{Q}(\sqrt{ep})$ and $L := \mathbb{Q}(\sqrt{eq})$ in $\mathcal{K}$ are given, where $p \equiv q \pmod 8$. It is clear that $-1 \notin K^{*2}$ since $ep/(-1)$ is not a square in $\mathbb{Q}$. By the Chebotarev density theorem there are infinitely many primes P and P' such that $-1 \notin K_P^{*2}$ and $-1 \in K_{P'}^{*2}$ respectively. This indicates there are infinitely many type I and type II spaces occurring among K_P. By the same argument there are infinitely many type I and type II spaces occurring among L_Q. So we have a bijection T_1 between the sets of non-dyadic split and non-dyadic inert places of K and L satisfying $\mathrm{Arf}(r_P) = \mathrm{Arf}(r_{T_1P})$, $\mathrm{type}[(\ ,\ )_P] = \mathrm{type}[(\ ,\ )_{T_1P}]$, and $|K_P^*/K_P^{*2}| = |L_{T_1P}^*/L_{T_1P}^{*2}|$ for every non-dyadic unramified prime P of K by Theorem 4.1.

Suppose P and Q are non-dyadic ramified primes in K and L respectively. Then $P\cap\mathbb{Z} = (p)$ and $Q\cap\mathbb{Z} = (q)$. Now suppose $p\equiv 1 \pmod 4$. Then $q \equiv 1 \pmod 4$ since $p\equiv q \pmod 8$. So $-1 \in K_P^{*2}$ and $-1 \in L_Q^{*2}$ since -1 is a square in $\mathbb{Q}_p$ and $\mathbb{Q}_q$ respectively. Suppose $p \equiv 3 \pmod 4$. Then $q\equiv 3 \pmod 4$ since $p \equiv q \pmod 8$. So $-1 \notin K_P^{*2}$ and $-1 \notin L_Q^{*2}$ since -1 is not a square in $\mathbb{Q}_p$ and $\mathbb{Q}_q$ respectively. This means $(\ ,\ )_P$ and $(\ ,\ )_Q$ have the same type. It is also clear $\dim[K_P^*/K_P^{*2}] = \dim[L_Q^*/L_Q^{*2}]$. By Theorem 4.4, $\mathrm{Arf}(r_P) = \mathrm{Arf}(r_Q)$. We denote the correspondence by T_2, i.e.,

$$\{P\} \xrightarrow{T_2} \{Q\}.$$

Suppose P is a dyadic place in K and Q is a dyadic place in L. Then the local degrees $[K_P : \mathbb{Q}_2]$ and $[L_Q : \mathbb{Q}_2]$ are 3 or 4, depending on p and q. In particular, K_P and L_Q are exactly the same fields since $p \equiv q \pmod 8$. This implies that the local square-classes are all the same. So local square-classes K_P^*/K_P^{*2} and L_Q^*/L_Q^{*2} are also exactly the same. Thus there is a bijection T_3 between the sets of dyadic places of K and L such that $\mathrm{Arf}(r_P) = \mathrm{Arf}(r_{T_3P})$, $\dim[K_P^*/K_P^{*2}] = \dim[L_{T_3P}^*/L_{T_3P}^{*2}]$, and $\mathrm{type}[(\ ,\ )_P] = \mathrm{type}[(\ ,\ )_{T_3P}]$ for every non-dyadic ramified prime P of K.

It is obvious that there exists a bijection T_4 between Archimedean places of K and L such that $\mathrm{Arf}(r_P) = \mathrm{Arf}(r_{T_4P})$, $\dim[K_P^*/K_P^{*2}] = \dim[L_{T_4P}^*/L_{T_4P}^{*2}]$, and $\mathrm{type}[(\ ,\ )_P] = \mathrm{type}[(\ ,\ )_{T_4P}]$ for every dyadic prime P of K.

We define a map

$$T : \Omega_K \to \Omega_L,$$

where Ω_K and Ω_L are sets of places of K and L respectively, by

$$T(P) = \begin{cases} T_1(P) & \text{if } P \text{ is a non-dyadic split or inert place in } K, \\ T_2(P) & \text{if } P \text{ is a non-dyadic ramified place in } K, \\ T_3(P) & \text{if } P \text{ is a dyadic place in } K, \\ T_4(P) & \text{if } P \text{ is an Archimedean place in } K. \end{cases} \tag{4.6}$$

Then for every place P of K, we have $\mathrm{Arf}(r_P) = \mathrm{Arf}(r_{TP})$, $\dim[K_P^*/K_P^{*2}] = \dim[L_{TP}^*/L_{TP}^{*2}]$, and $\mathrm{type}[(\ ,\ )_P] = \mathrm{type}[(\ ,\ )_{TP}]$. Therefore K and L are Arf equivalent. ∎

THEOREM 4.6. *Two quadratic number fields* $K := \mathbb{Q}(\sqrt{p_1\cdots p_m})$ *and* $L := \mathbb{Q}(\sqrt{q_1\cdots q_m})$ *are Arf equivalent if* $p_j \equiv q_j \pmod 8$ *and* $\left(\frac{p_j'}{p_j}\right) = \left(\frac{q_j'}{q_j}\right)$ *for each* j, *where* $p_j' = (\prod_{k=1}^m p_k)/p_j$ *and* $q_j' = (\prod_{l=1}^m q_l)/q_j$ *for* $j = 1, \ldots, m$.

Proof. Note that p_j and p_j' are in the same square-class of K_{P_j} since $p_jp_j' = (\sqrt{p_1\cdots p_m})^2$ in K_{P_j}. It is also true that q_j and q_j' are in the same square-class of L_{Q_j}. It was proved that there is a one to one correspondence among non-dyadic split or non-dyadic inert places in K and L having the same dimensions, same types, and same Arf invariants in the proof of Theorem 4.5. It is also clear there is a bijection among the dyadic places in K and L having the same dimensions, same types, and same Arf invariants since the corresponding completions are exactly the same for dyadic split, inert, ramified primes since $p_1\cdots p_m \equiv q_1\cdots q_m \pmod 8$. The same is clear for the Archimedean places. So all we have to show is to find a bijection between the set of non-dyadic ramified primes in K and the set of non-dyadic ramified primes in L having the same types and same Arf invariants.

Let P_j be a non-dyadic ramified prime in K and let Q_j be a non-dyadic ramified prime in L with $P_j \cap \mathbb{Z} = (p_j)$ and $Q_j \cap \mathbb{Z} = (q_j)$. Write $K^*_{P_j}/K^{*2}_{P_j} = \{1, \epsilon_{K_j}, \Pi_K, \epsilon_{K_j}\Pi_K\}$, $L^*_{Q_j}/L^{*2}_{Q_j} = \{1, \epsilon_{L_j}, \Pi_L, \epsilon_{L_j}\Pi_L\}$, where ϵ_{K_j} and ϵ_{L_j} are non-square units in K_{P_j} and L_{Q_j} respectively and $\Pi_K = \sqrt{p_1 \cdots p_m}$ and $\Pi_L = \sqrt{q_1 \cdots q_m}$. As we have seen in Lemma 3.4, $r_{P_j}(\epsilon_{K_j}) = r_{Q_j}(\epsilon_{L_j}) = -1$. Therefore $\mathrm{Arf}(r_{P_j}) = r_{P_j}(\Pi_K)$ and $\mathrm{Arf}(r_{Q_j}) = r_{Q_j}(\Pi_K)$. So we will show that $\mathrm{Arf}(r_{P_j}) = \mathrm{Arf}(r_{Q_j})$ and type[(, $)_{P_j}$]=type[(, $)_{Q_j}$] for each j. Define a map $\alpha_{\Pi_K}(x) = (\Pi_K, x)_{P_j}$ for $x \in K^*_{P_j}$. Then $\alpha(1+P_j) = 1$ by Hensel's lemma. So $f_{\Pi_K} = P_j$. Hence $\mathcal{N} f_{\Pi_K} = p_j^{f(P_j|p_j)} = p_j$. The local absolute different $D_{K_{P_j}}$ equals $\Pi_K O_{K_{P_j}}$. So $f_{\Pi_K} D_{K_{P_j}} = p_j O_{K_{P_j}}$. By Tate's formula,

$$\begin{aligned}
r_{P_j}(\Pi_K) &= \frac{1}{\sqrt{p_j}} \sum_{x \in O^*_{K_{P_j}} \bmod^* P_j} \alpha_{\Pi_j}(p_j^{-1}x) \cdot \psi_{K_{P_j}}(p_j^{-1}x) \\
&= \frac{1}{\sqrt{p_j}} (\Pi_K, p_j)_{P_j} \sum_{x \in K^*_{P_j} \bmod^* P_j} \alpha_{\Pi_j}(x) \cdot \psi_{K_{P_j}}(p_j^{-1}x) \\
&= \frac{1}{\sqrt{p_j}} (\Pi_K, p'_j)_{P_j} \sum_{x \in \mathbb{F}^*_{p_j}} \left(\frac{x}{p_j}\right) \cdot e^{4\pi x i/p_j} \\
&= \frac{1}{\sqrt{p_j}} \left(\frac{p'_j}{p_j}\right) \sum_{x \in \mathbb{F}^*_{p_j}} \left(\frac{x}{p_j}\right) \cdot e^{4\pi x i/p_j}
\end{aligned}$$

since $(\Pi_K, p_j)_{P_j} = (\Pi_K, p'_j)_{P_j} = (p_j, p'_j)_{p_j} = \left(\frac{p'_j}{p_j}\right)$. By a similar argument we get

$$r_{Q_j}(\Pi_L) = \frac{1}{\sqrt{q_j}} \left(\frac{q'_j}{q_j}\right) \sum_{x \in \mathbb{F}^*_{q_j}} \left(\frac{x}{q_j}\right) \cdot e^{4\pi x i/q_j}.$$

(1) Suppose $p_j \equiv 1 \pmod 8$ or $p_j \equiv 5 \pmod 8$. Then $q_j \equiv 1 \pmod 8$ or $q_j \equiv 5 \pmod 8$ respectively by the assumption. Then using the known value of the Gaussian sum we obtain

$$\begin{aligned}
\mathrm{Arf}(r_{P_j}) = r_{P_j}(\Pi_K) &= \frac{1}{\sqrt{p_j}} \cdot \left(\frac{p'_j}{p_j}\right) \cdot \sqrt{p_j} \cdot \left(\frac{2}{p_j}\right) = \left(\frac{p'_j}{p_j}\right) \cdot \left(\frac{2}{p_j}\right) \\
&= \left(\frac{q'_j}{q_j}\right) \cdot \left(\frac{2}{q_j}\right) = r_{Q_j}(\Pi_L) = \mathrm{Arf}(r_{Q_j}).
\end{aligned}$$

Observe that type[(, $)_{P_j}$] = type[(, $)_{Q_j}$] = type II since -1 is a square in $\mathbb{Q}_{p_j}$ and $\mathbb{Q}_{q_j}$.

(2) The reasoning is similar for the cases $p_j \equiv 3 \pmod 8$ or $p_j \equiv 7 \pmod 8$. In these cases we also have the same Arf invariants and type I spaces.

So by (1) and (2) we get a bijection among non-dyadic ramified places of K and L with the same types and same Arf invariants. Therefore K and L are Arf equivalent. ■

The following table summarizes the discussion above about Arf invariants of quadratic fields at non-dyadic primes P:

place P	complex	real	non-dyadic split or inert	non-dyadic ramified
$\dim_{\mathbb{F}_2} K_P^*/K_P^{*2}$	0	1	2	2
type$[(\ ,\ )_P]$	II	I	I or II	I or II
$\mathrm{Arf}(r_P)$	1	$(1-i)/\sqrt{2}$	1	± 1 or $\pm i$

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Received on 3.10.2011 (6847)