Infinite series identities from modular transformation formulas that stem from generalized Eisenstein series

by

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1. Introduction. B. C. Berndt [3] has found a general class of Eisenstein series and has computed transformation formulas for them. In [5] he has also converted these transformation formulas into transformation formulas for a large class of functions containing the classical Dedekind eta function. By this process, in [4], [5] he has established a number of interesting infinite series identities, many of which are stated in the Notebooks of Ramanujan [11] or are proved by many authors.

In [8], using the latter transformation formulas (Theorem 2.1), the present author also evaluated many infinite series identities, a number of which are given by other authors including Ramanujan and B. C. Berndt, and many of which are new series or generalizations of series given by other authors. In this paper, we give more general infinite series identities. In Section 3, we obtain a few classes of infinite series for an arbitrary positive integer c'. A special case of those series (when $c' = 2$) is considered in Section 4. Some of them have already been found by Berndt [4], [5] and other authors. But many of them are new series involving hyperbolic functions and trigonometric functions. In particular, we obtain some elegant symmetric (in α, β) identities in Section 4.1.

2. Notation and preliminaries. We basically follow the notation in [5]. The branch of the argument for a complex w is defined by $-\pi \leq \arg w < \pi$. We set $e(w) = e^{2\pi i w}$, and $V\tau = V(\tau) = (a\tau + b)/(c\tau + d)$ always denotes a modular transformation with $c > 0$ and $c \equiv 0 \pmod{N}$ for every complex τ. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and the associated vectors R and H are defined by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2),$$

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

For a positive integer N, let λ_N denote the characteristic function of the integers modulo N, i.e.,

$$\lambda_N(m) = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{N}, \\ 0 & \text{otherwise.} \end{cases}$$

For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For real α, x and $\operatorname{Re}(s) > 1$, let

$$(2.1) \qquad \psi(x, \alpha, s) := \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^s}.$$

If x is an integer and α is not an integer, then $\psi(x, \alpha, s) = \zeta(s, \{\alpha\})$, where $\zeta(s, x)$ is the Hurwitz zeta function. The function $\psi(x, \alpha, s)$ can be analytically continued to the entire s-plane [2] except for $s = 1$ when x is an integer. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex number s, define

$$A_N(\tau, s; r, h) := \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e(Nmh_1 + ((Nm+r_1)\tau + r_2)(n - h_2))}{(n-h_2)^{1-s}}.$$

Let

$$H_N(\tau, s; r, h) := A_N(\tau, s; r, h) + e(s/2) A_N(\tau, s; -r, -h).$$

It is easy to see that the logarithm of the classical Dedekind eta function is expressed using H_N. We now state the theorem about transformation formulas for a large class of functions containing the classical Dedekind eta function.

THEOREM 2.1 ([5], [8]). *Let* $Q = \{\tau \in \mathbb{C} \mid \operatorname{Re}(\tau) > -d/c\}$, $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$ *and* $c = Nc'$. *Then for* $\tau \in Q$ *and all* s,

$$(c\tau + d)^{-s} H_N(V\tau, s; r, h) = H_N(\tau, s; R, H)$$

$$- \lambda_N(r_1) e(-r_1 h_1)(c\tau + d)^{-s} \Gamma(s)(-2\pi i)^{-s}$$

$$\times (\psi(h_2, r_2, s) + e(s/2)\psi(-h_2, -r_2, s))$$

$$+ \lambda_N(R_1) e(-R_1 H_1)\Gamma(s)(-2\pi i)^{-s}(\psi(H_2, R_2, s) + e(-s/2)\psi(-H_2, -R_2, s))$$

$$+ (2\pi i)^{-s} L_N(\tau, s; R, H),$$

where

$$L_N(\tau, s; R, H)$$

$$:= \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d))$$

$$\times \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)} \, du,$$

where C is a loop beginning at $+\infty$, running in the upper half-plane, en-circling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(c\tau+d)u} - e(cH_1 + dH_2))\,(e^u - e(-H_2))$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

REMARK 2.2. Note that after evaluation of $L_N(\tau, s; R, H)$ for an inte-ger s, the transformation formula in Theorem 2.1 will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

For computations of the integral in Theorem 2.1 when s is an integer, it is useful to employ the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi)$$

for Bernoulli polynomials $B_n(x)$, $n \geq 0$. The nth Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Recall that $B_{2n+1} = 0$, $n \geq 1$, and that $B_{2n+1}(1/2) = 0$, $n \geq 0$. The following formulas are helpful (see [1]):

$$(2.2) \qquad\qquad B_n(1 - x) = (-1)^n B_n(x),$$

$$(2.3) \qquad\qquad \sum_{j=0}^{c-1} B_n(j/c + x) = c^{1-n} B_n(cx).$$

For the zeta function $\zeta(s)$, we know from [7, p. 194] that

$$(2.4) \qquad\qquad \zeta(2M) = \frac{2^{2M-1}|B_{2M}|\pi^{2M}}{(2M)!}, \quad M > 0,$$

and

$$(2.5) \qquad\qquad \zeta(1 - 2M) = -\frac{B_{2M}}{2M}, \quad M > 0.$$

For brevity, we let

$$D_1(h_2, r_2, s) := -(c\tau + d)^{-s}\Gamma(s)(-2\pi i)^{-s}$$
$$\times (\psi(h_2, r_2, s) + e(s/2)\psi(-h_2, -r_2, s)),$$
$$D_2(H_2, R_2, s) := \Gamma(s)(-2\pi i)^{-s}(\psi(H_2, R_2, s) + e(-s/2)\psi(-H_2, -R_2, s)),$$

and

(2.6) $$\mathcal{Z}_{\pm}(s, x) := \zeta(s, x) \pm \zeta(s, 1 - x).$$

3. Several classes of infinite series. In this section, we study a more general series than those in [8]. Let N be a positive integer, and let r_1 and r_2 be arbitrary real numbers. For any integer n, put

$$r = (r_1, r_2/c), \quad h = (0, 0), \quad s = -n.$$

Let V denote a modular transformation corresponding to

$$\begin{pmatrix} 1 & -1 \\ c & -c+1 \end{pmatrix},$$

where $c = Nc'$ for $c' > 0$. Let $z = c\tau + 1 - c$. Then $V\tau = 1/c - 1/cz$. The condition $\tau \in Q$ gives $\operatorname{Re}(z) > 0$ and $z \in \mathbb{H}$. We also have

$$R_1 = r_1 + r_2, \quad R_2 = -r_1 - r_2 + r_2/c.$$

We consider four cases of N, r_1 and r_2 and, by the same process as in [8], deduce (i)–(iv) below.

(i) If $N \nmid r_1$, then

(3.1) $$H_N(V\tau, -2n; r, 0)$$
$$= \sum_{k=1}^{\infty} \frac{\cosh\left(\frac{\pi i k}{c'}\left(\frac{2(r_1+r_2)}{N} - 2\left[\frac{r_1}{N}\right] - 1 + \left(1 - 2\left\{\frac{r_1}{N}\right\}\right)\frac{1}{z}\right)\right)}{k^{2n+1}\sinh\left(\frac{\pi i k}{c'}\left(\frac{1}{z} - 1\right)\right)},$$

(3.2) $$H_N(V\tau, -2n-1; r, 0)$$
$$= \sum_{k=1}^{\infty} \frac{\sinh\left(\frac{\pi i k}{c'}\left(\frac{2(r_1+r_2)}{N} - 2\left[\frac{r_1}{N}\right] - 1 + \left(1 - 2\left\{\frac{r_1}{N}\right\}\right)\frac{1}{z}\right)\right)}{k^{2n+2}\sinh\left(\frac{\pi i k}{c'}\left(\frac{1}{z} - 1\right)\right)}.$$

(ii) If $N \mid r_1$, then

(3.3) $$H_N(V\tau, -2n; r, 0) = 2\sum_{k=1}^{\infty} \frac{\cos(2\pi r_2 k/c)}{k^{2n+1}\left(e^{\frac{2\pi i k}{c'}\left(\frac{1}{z}-1\right)} - 1\right)},$$

(3.4) $$H_N(V\tau, -2n-1; r, 0) = 2i\sum_{k=1}^{\infty} \frac{\sin(2\pi r_2 k/c)}{k^{2n+2}\left(e^{\frac{2\pi i k}{c'}\left(\frac{1}{z}-1\right)} - 1\right)}.$$

(iii) If $N \nmid R_1 = r_1 + r_2$, then

(3.5) $$H_N(\tau, -2n; R, 0)$$
$$= \sum_{k=1}^{\infty} \frac{\cosh\left(\frac{\pi i k}{c'}\left(\frac{-2r_1}{N} + 2\left[\frac{r_1+r_2}{N}\right] + 1 + \left(2\left\{\frac{r_1+r_2}{N}\right\} - 1\right)z\right)\right)}{k^{2n+1}\sinh\left(\frac{\pi i k}{c'}(1-z)\right)},$$

(3.6) $\quad H_N(\tau, -2n-1; R, 0)$

$$= \sum_{k=1}^{\infty} \frac{\sinh\left(\frac{\pi i k}{c'}\left(\frac{-2r_1}{N} + 2\left[\frac{r_1+r_2}{N}\right] + 1 + \left(2\left\{\frac{r_1+r_2}{N}\right\} - 1\right)z\right)\right)}{k^{2n+2}\sinh\left(\frac{\pi i k}{c'}(1-z)\right)}.$$

(iv) If $N \mid R_1 = r_1 + r_2$, then

$$(3.7) \qquad H_N(\tau, -2n; R, 0) = 2\sum_{k=1}^{\infty} \frac{\cos(2\pi r_2 k/c)}{k^{2n+1}\left(e^{\frac{2\pi i k}{c'}(1-z)} - 1\right)},$$

$$(3.8) \qquad H_N(\tau, -2n-1; R, 0) = 2i\sum_{k=1}^{\infty} \frac{\sin(2\pi r_2 k/c)}{k^{2n+2}\left(e^{\frac{2\pi i k}{c'}(1-z)} - 1\right)}.$$

Next, by the residue theorem, we find that

(3.9) $\quad L_N(\tau, -n; R, 0)$

$$= 2\pi i \sum_{j=1}^{c'} \sum_{k=0}^{n+2} \frac{B_k\left(\frac{j-\{(r_1+r_2)/N\}}{c'}\right)\bar{B}_{n+2-k}\left(\frac{j+\varrho_N/N}{c'}\right)}{k!(n+2-k)!}(-z)^{k-1}.$$

Applying (3.1)–(3.9) to Theorem 2.1, we obtain a general class of infinite series.

3.1. Case of $N \nmid r_1$ and $N \nmid (r_1 + r_2)$. In this case, $\lambda_N(r_1) = \lambda_N(R_1) = 0$, and by Theorem 2.1, we see that, for any integer n and $z \in \mathbb{H}$ with $\mathrm{Re}(z) > 0$,

$$(3.10) \quad z^n H_N(V\tau, -n; r, 0) = H_N(\tau, -n; R, 0) + (2\pi i)^n L_N(\tau, -n; R, 0).$$

Putting (3.1), (3.2), (3.5), (3.6) and (3.9) into (3.10) and employing Remark 2.2, we find that, for any integer n and $z \in \mathbb{H}$,

$$(3.11) \quad z^{2n} \sum_{k=1}^{\infty} \frac{\cosh\left(\frac{\pi i k}{c'}\left(\frac{2(r_1+r_2)}{N} - 2\left[\frac{r_1}{N}\right] - 1 + \left(1 - 2\left\{\frac{r_1}{N}\right\}\right)\frac{1}{z}\right)\right)}{k^{2n+1}\sinh\left(\frac{\pi i k}{c'}\left(\frac{1}{z} - 1\right)\right)}$$

$$= \sum_{k=1}^{\infty} \frac{\cosh\left(\frac{\pi i k}{c'}\left(\frac{-2r_1}{N} + 2\left[\frac{r_1+r_2}{N}\right] + 1 + \left(2\left\{\frac{r_1+r_2}{N}\right\} - 1\right)z\right)\right)}{k^{2n+1}\sinh\left(\frac{\pi i k}{c'}(1-z)\right)}$$

$$+ (2\pi i)^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j-\{(r_1+r_2)/N\}}{c'}\right)\bar{B}_{2n+2-k}\left(\frac{j+\varrho_N/N}{c'}\right)}{k!(2n+2-k)!}(-z)^{k-1},$$

and

$$(3.12) \qquad z^{2n+1} \sum_{k=1}^{\infty} \frac{\sinh\left(\frac{\pi i k}{c'}\left(\frac{2(r_1+r_2)}{N} - 2\left[\frac{r_1}{N}\right] - 1 + \left(1 - 2\left\{\frac{r_1}{N}\right\}\right)\frac{1}{z}\right)\right)}{k^{2n+2}\sinh\left(\frac{\pi i k}{c'}\left(\frac{1}{z} - 1\right)\right)}$$

$$= \sum_{k=1}^{\infty} \frac{\sinh\left(\frac{\pi ik}{c'}\left(\frac{-2r_1}{N} + 2\left[\frac{r_1+r_2}{N}\right] + 1 + \left(2\left\{\frac{r_1+r_2}{N}\right\} - 1\right)z\right)\right)}{k^{2n+2}\sinh\left(\frac{\pi ik}{c'}(1-z)\right)}$$

$$+ (2\pi i)^{2n+2}\sum_{j=1}^{c'}\sum_{k=0}^{2n+3}\frac{B_k\left(\frac{j-\{(r_1+r_2)/N\}}{c'}\right)\bar{B}_{2n+3-k}\left(\frac{j+\varrho_N/N}{c'}\right)}{k!(2n+3-k)!}(-z)^{k-1}.$$

THEOREM 3.1. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Suppose that* r, r' *are real numbers such that* r, $r + r'$ *are not integers. Then, for any integer* n *and for any positive integer* c',

$$\alpha^{-n}\sum_{k=1}^{\infty}\frac{\cosh((2(r+r')-2[r]-1)\pi ik/c' + (1-2\{r\})\alpha k/c')}{k^{2n+1}\sinh((\alpha-i\pi)k/c')}$$

$$= (-\beta)^{-n}\sum_{k=1}^{\infty}\frac{\cosh((-2r+2[r+r']+1)\pi ik/c' + (1-2\{r+r'\})\beta k/c')}{k^{2n+1}\sinh((\beta+i\pi)k/c')}$$

$$- 2^{2n+1}\sum_{j=1}^{c'}\sum_{k=0}^{2n+2}\frac{B_k\left(\frac{j-\{r+r'\}}{c'}\right)\bar{B}_{2n+2-k}\left(\frac{j+\varrho_N/N}{c'}\right)}{k!(2n+2-k)!}(-\pi i)^k\alpha^{n-k+1},$$

and

$$\alpha^{-n-1/2}\sum_{k=1}^{\infty}\frac{\sinh((2(r+r')-2[r]-1)\pi ik/c' + (1-2\{r\})\alpha k/c')}{k^{2n+2}\sinh((\alpha-i\pi)k/c')}$$

$$= (-\beta)^{-n-1/2}\sum_{k=1}^{\infty}\frac{\sinh((-2r+2[r+r']+1)\pi ik/c' + (1-2\{r+r'\})\beta k/c')}{k^{2n+2}\sinh((\beta+i\pi)k/c')}$$

$$- 2^{2n+2}\sum_{j=1}^{c'}\sum_{k=0}^{2n+3}\frac{B_k\left(\frac{j-\{r+r'\}}{c'}\right)\bar{B}_{2n+3-k}\left(\frac{j+\varrho_N/N}{c'}\right)}{k!(2n+3-k)!}(-\pi i)^k\alpha^{n-k+3/2}.$$

Proof. Put $z = \pi i/\alpha$ in (3.11) and (3.12), and replace r_1/N and r_2/N by r and r', respectively. Then the desired results follow. ∎

We now assume that r is a real number with $0 < r < 1$ and $r' = 0$ in Theorem 3.1. Then it is easy see that

$$\left\{\frac{j+\varrho_N/N}{c'}\right\} = \left\{\frac{j+c'\{-Nr\}+Nc'r-r}{c'}\right\} = \left\{\frac{j-r}{c'}\right\}.$$

Thus, from Theorem 3.1 we obtain the following theorem.

THEOREM 3.2. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be any real number with* $0 < r < 1$. *Then, for any integer* n *and for any positive integer* c',

$$(3.13) \quad \alpha^{-n} \sum_{k=1}^{\infty} \frac{\cosh((1-2r)(\alpha-i\pi)k/c')}{k^{2n+1}\sinh((\alpha-i\pi)k/c')}$$

$$= (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{\cosh((1-2r)(\beta+i\pi)k/c')}{k^{2n+1}\sinh((\beta+i\pi)k/c')}$$

$$- 2^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j-r}{c'}\right) B_{2n+2-k}\left(\frac{j-r}{c'}\right)}{k!(2n+2-k)!} (-\pi i)^k \alpha^{n-k+1},$$

and

$$(3.14) \quad \alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{\sinh((1-2r)(\alpha-i\pi)k/c')}{k^{2n+2}\sinh((\alpha-i\pi)k/c')}$$

$$= (-\beta)^{-n-1/2} \sum_{k=1}^{\infty} \frac{\sinh((1-2r)(\beta+i\pi)k/c')}{k^{2n+2}\sinh((\beta+i\pi)k/c')}$$

$$- 2^{2n+2} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k\left(\frac{j-r}{c'}\right) B_{2n+3-k}\left(\frac{j-r}{c'}\right)}{k!(2n+3-k)!} (-\pi i)^k \alpha^{n-k+3/2}.$$

COROLLARY 3.3.

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}((\alpha-i\pi)k/c')}{k^{2n+1}}$$

$$= (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}((\beta+i\pi)k/c')}{k^{2n+1}}$$

$$- 2^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{2j-1}{2c'}\right) B_{2n+2-k}\left(\frac{2j-1}{2c'}\right)}{k!(2n+2-k)!} (-\pi i)^k \alpha^{n-k+1}.$$

Proof. Put $r = 1/2$ in (3.13). ∎

COROLLARY 3.4.

$$\sum_{k=1}^{\infty} \frac{\cosh(\pi k/c')\sin(\pi k/c')}{k^{4n+1}(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$

$$= \frac{(2\pi)^{4n+1}}{2} \sum_{j=1}^{c'} \sum_{k=0}^{4n+2} \frac{B_k\left(\frac{2j-1}{2c'}\right) B_{4n+2-k}\left(\frac{2j-1}{2c'}\right)}{k!(4n+2-k)!} (-i)^{k-1}.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.3 and replace n by $2n$. Use

$$\operatorname{csch}(x-iy) - \operatorname{csch}(x+iy) = \frac{2i\cosh(x)\sin(y)}{\cosh(2x) - \cos(2y)}. \quad ∎$$

COROLLARY 3.5.

$$\sum_{k=1}^{\infty} \frac{\sinh(\pi k/c')\cos(\pi k/c')}{k^{4n+3}(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$

$$= -\frac{(2\pi)^{4n+3}}{2} \sum_{j=1}^{c'} \sum_{k=0}^{4n+4} \frac{B_k\left(\frac{2j-1}{2c'}\right)B_{4n+4-k}\left(\frac{2j-1}{2c'}\right)}{k!(4n+4-k)!} (-i)^k.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.3 and replace n by $2n+1$. Use

$$\operatorname{csch}(x - iy) + \operatorname{csch}(x + iy) = \frac{2\sinh(x)\cos(y)}{\cosh(2x) - \cos(2y)}. \quad \blacksquare$$

COROLLARY 3.6.

$$\alpha \sum_{k=1}^{\infty} k\operatorname{csch}((\alpha - i\pi)k/c') + \beta \sum_{k=1}^{\infty} k\operatorname{csch}((\beta + i\pi)k/c') = -c'/2.$$

Proof. Put $n = -1$ in Corollary 3.3. $\blacksquare$

COROLLARY 3.7.

$$\sum_{k=1}^{\infty} \frac{k\sinh(\pi k/c')\cos(\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = -\frac{c'}{2\pi}.$$

Proof. Put $n = -1$ in Corollary 3.5. $\blacksquare$

COROLLARY 3.8.

$$\sum_{k=1}^{\infty} \frac{\operatorname{csch}((\alpha - i\pi)k/c')}{k} = \sum_{k=1}^{\infty} \frac{\operatorname{csch}((\beta + i\pi)k/c')}{k} - \frac{\beta - \alpha}{12c'} + \frac{\pi i}{6}\left(c' - \frac{1}{c'}\right).$$

Proof. Put $n = 0$ in Corollary 3.3. It is easy to see that

$$\sum_{j=1}^{c'} B_1\left(\frac{2j-1}{2c'}\right)^2 = \sum_{j=1}^{c'} \left(\frac{2j-1}{2c'} - \frac{1}{2}\right)^2 = \frac{1}{12}\left(c' - \frac{1}{c'}\right). \quad \blacksquare$$

COROLLARY 3.9.

$$\sum_{k=1}^{\infty} \frac{\cosh(\pi k/c')\sin(\pi k/c')}{k(\cosh(2\pi k/c') - \cos(2\pi k/c'))} = \frac{\pi}{12}\left(c' - \frac{1}{c'}\right).$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.8. $\blacksquare$

COROLLARY 3.10.

$$\alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{\operatorname{sech}\left((\alpha - i\pi)\frac{k}{2c'}\right)}{k^{2n+2}}$$

$$= (-\beta)^{-n-1/2} \sum_{k=1}^{\infty} \frac{\operatorname{sech}\left((\beta + i\pi)\frac{k}{2c'}\right)}{k^{2n+2}}$$

$$- 2^{2n+3} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k\left(\frac{4j-1}{4c'}\right) B_{2n+3-k}\left(\frac{4j-1}{4c'}\right)}{k!(2n+3-k)!} (-\pi i)^k \alpha^{n-k+3/2}.$$

Proof. Let $r = 1/4$ in (3.14). ∎

COROLLARY 3.11.

$$\alpha^{1/2}\left(\sum_{k=1}^{\infty} \operatorname{sech}\left((\alpha - i\pi)\frac{k}{2c'}\right) + \frac{1}{2}\right)$$

$$= (-\beta)^{1/2}\left(\sum_{k=1}^{\infty} \operatorname{sech}\left((\beta + i\pi)\frac{k}{2c'}\right) + \frac{1}{2}\right).$$

Proof. Let $n = -1$ in Corollary 3.10. We see that

$$-2\sum_{j=1}^{c'}\sum_{k=0}^{1} \frac{B_k\left(\frac{4j-1}{4c'}\right) B_{1-k}\left(\frac{4j-1}{4c'}\right)}{k!(1-k)!} (-\pi i)^k \alpha^{-k+1/2} = -\frac{1}{2}\alpha^{1/2} + \frac{1}{2}(-\beta)^{1/2}. \quad \blacksquare$$

3.2. Case of $N \mid r_1$ and $N \nmid (r_1 + r_2)$. By Theorem 2.1, for any integer n and $z \in \mathbb{H}$ with $\operatorname{Re}(z) > 0$,

$$z^n H_N(V\tau, -n; r, 0) = H_N(\tau, -n; R, 0) + (2\pi i)^n L_N(\tau, -n; R, 0)$$
$$+ \lim_{s \to -n} D_1(0, r_2/c, s),$$

where $D_1(0, r_2/c, s)$ is defined in Section 2. Since

$$\frac{\varrho_N}{N} = c'\left\{-r_2 + \frac{r_2}{Nc'}\right\} + (Nc' - 1)\left\{\frac{r_2}{N}\right\}$$
$$= -c'\left[-r_2 + \frac{r_2}{Nc'}\right] - (Nc' - 1)\left[\frac{r_2}{N}\right],$$

we find that

$$\left\{\frac{j + \frac{\varrho_N}{N}}{c'}\right\} = \left\{\frac{j - c'\left[-r_2 + \frac{r_2}{Nc'}\right] - Nc'\left[\frac{r_2}{N}\right] + \left[\frac{r_2}{N}\right]}{c'}\right\} = \left\{\frac{j + \left[\frac{r_2}{N}\right]}{c'}\right\}.$$

Recall the notation of $\mathscr{Z}_{\pm}(s, x)$ defined in (2.6), i.e.,

$$\mathscr{Z}_{\pm}(s, x) := \zeta(s, x) \pm \zeta(s, 1-x).$$

Then, using Lemma 2.2 in [8] and employing Remark 2.2, we have, for any $z \in \mathbb{H}$,

$$(3.15) \quad z^{2n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi r_2 k/c)}{k^{2n+1}(e^{\frac{2\pi i k}{c'}(\frac{1}{z}-1)} - 1)}$$

$$= \sum_{k=1}^{\infty} \frac{\cosh\left(\frac{\pi i k}{c'}\left(2\left[\frac{r_2}{N}\right] + 1 + \left(2\{\frac{r_2}{N}\} - 1\right)z\right)\right)}{k^{2n+1}\sinh\left(\frac{\pi i k}{c'}(1-z)\right)}$$

$$+ (2\pi i)^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j - \{r_2/N\}}{c'}\right) \bar{B}_{2n+2-k}\left(\frac{j + [r_2/N]}{c'}\right)}{k!(2n+2-k)!} (-z)^{k-1} + J_0(n),$$

where

$$J_0(n) := \begin{cases} -(2\pi i z)^{2n}(-2n-1)! \, \mathcal{Z}_+(-2n, \{r_2/c\}) & \text{if } n < 0, \\ \log(1 - e^{-2\pi i r_2/c}) & \text{if } n = 0, \\ -z^{2n}\psi(-r_2/c, 0, 2n+1) & \text{if } n > 0, \end{cases}$$

and

$$(3.16) \quad z^{2n+1} \sum_{k=1}^{\infty} \frac{2i\sin(2\pi r_2 k/c)}{k^{2n+2}(e^{\frac{2\pi i k}{c'}(\frac{1}{z}-1)} - 1)}$$

$$= \sum_{k=1}^{\infty} \frac{\sinh\left(\frac{\pi i k}{c'}\left(2\left[\frac{r_2}{N}\right] + 1 + \left(2\{\frac{r_2}{N}\} - 1\right)z\right)\right)}{k^{2n+2}\sinh\left(\frac{\pi i k}{c'}(1-z)\right)}$$

$$+ (2\pi i)^{2n+2} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k\left(\frac{j - \{r_2/N\}}{c'}\right) \bar{B}_{2n+3-k}\left(\frac{j + [r_2/N]}{c'}\right)}{k!(2n+3-k)!} (-z)^{k-1} + \mathcal{J}_0(n),$$

where

$$\mathcal{J}_0(n) := \begin{cases} (2\pi i z)^{2n+1}(-2n-2)! \, \mathcal{Z}_-(-2n-1, \{r_2/c\}) & \text{if } n < -1, \\ (2\pi i z)^{-1}(\pi\cot(\pi\{r_2/c\}) - \pi i) & \text{if } n = -1, \\ z^{2n+1}\psi(-r_2/c, 0, 2n+2) & \text{if } n \geq 0. \end{cases}$$

THEOREM 3.12. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be any real number which is not an integer. Then for any integer* n *and for any positive integer* c',

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi r k/c')}{k^{2n+1}(e^{(\alpha - i\pi)2k/c'} - 1)}$$

$$= (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{\cosh((2[r]+1)\pi i k/c' + (1 - 2\{r\})\beta k/c')}{k^{2n+1}\sinh((\beta + i\pi)k/c')}$$

$$-2^{2n+1}\sum_{j=1}^{c'}\sum_{k=0}^{2n+2}\frac{B_k\left(\frac{j-\{r\}}{c'}\right)\bar{B}_{2n+2-k}\left(\frac{j+[r]}{c'}\right)}{k!(2n+2-k)!}(-\pi i)^k\alpha^{n-k+1}+J_1(n),$$

where

$$J_1(n):=\begin{cases}-(-4\beta)^n(-2n-1)!\,\mathcal{Z}_+(-2n,\{r/c'\})&\text{if }n<0,\\\log(1-e^{-2\pi i r/c'})&\text{if }n=0,\\-\alpha^{-n}\psi(-r/c',0,2n+1)&\text{if }n>0.\end{cases}$$

Proof. Put $z=\pi i/\alpha$ and $r_2/N=r$ in (3.15). ∎

THEOREM 3.13. *Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. Let r be any real number that is not an integer. Then for any integer n and for any positive integer c',*

$$\alpha^{-n-1/2}\sum_{k=1}^{\infty}\frac{2i\sin(2\pi rk/c')}{k^{2n+2}(e^{(\alpha-i\pi)2k/c'}-1)}$$

$$=(-\beta)^{-n-1/2}\sum_{k=1}^{\infty}\frac{\sinh((2[r]+1)\pi ik/c'+(1-2\{r\})\beta k/c')}{k^{2n+2}\sinh((\beta+i\pi)k/c')}$$

$$-2^{2n+2}\sum_{j=1}^{c'}\sum_{k=0}^{2n+3}\frac{B_k\left(\frac{j-\{r\}}{c'}\right)\bar{B}_{2n+3-k}\left(\frac{j+[r]}{c'}\right)}{k!(2n+3-k)!}(-\pi i)^k\alpha^{n-k+3/2}+\mathcal{J}_1(n),$$

where

$$\mathcal{J}_1(n):=\begin{cases}(-4\beta)^{n+1/2}(-2n-2)!\,\mathcal{Z}_-(-2n-1,\{r/c'\})&\text{if }n<-1,\\-\frac{i}{2}\alpha^{1/2}\cot(\pi\{r/c'\})-\frac{1}{2}\alpha^{1/2}&\text{if }n=-1,\\\alpha^{-n-1/2}\psi(-r/c',0,2n+2)&\text{if }n\geq 0.\end{cases}$$

Proof. Put $z=\pi i/\alpha$ and $r_2/N=r$ in (3.16). ∎

PROPOSITION 3.14. *Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. Then for any integer n and any positive integer c',*

$$\alpha^{-n}\sum_{k=1}^{\infty}\frac{2\cos(\pi k/c')}{k^{2n+1}(e^{(\alpha-i\pi)2k/c'}-1)}$$

$$=(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{\cos(\pi k/c')}{k^{2n+1}\sinh((\beta+i\pi)k/c')}$$

$$-2^{2n+1}\sum_{j=1}^{c'}\sum_{k=0}^{2n+2}\frac{B_k\left(\frac{2j-1}{2c'}\right)\bar{B}_{2n+2-k}\left(\frac{j}{c'}\right)}{k!(2n+2-k)!}(-\pi i)^k\alpha^{n-k+1}+J_2(n),$$

where

$$J_2(n):=\begin{cases}-(-4\beta)^n(-2n-1)!\,\mathcal{Z}_+(-2n,1/2c')&\text{if }n<0,\\\log(1-e^{-\pi i/c'})&\text{if }n=0,\\-\alpha^{-n}\psi(-1/2c',0,2n+1)&\text{if }n>0.\end{cases}$$

Proof. Put $r = 1/2$ in Theorem 3.12. ∎

PROPOSITION 3.15. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Then for any integer* n *and any positive integer* c',

$$\alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{2\sin(\pi k/c')}{k^{2n+2}(e^{(\alpha-i\pi)2k/c'} - 1)}$$

$$= (-\beta)^{-n-1/2} \sum_{k=1}^{\infty} \frac{\sin(\pi k/c')}{k^{2n+2}\sinh((\beta + i\pi)k/c')}$$

$$+ i2^{2n+2} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k\left(\frac{2j-1}{2c'}\right)\bar{B}_{2n+3-k}\left(\frac{j}{c'}\right)}{k!(2n+3-k)!} (-\pi i)^k \alpha^{n-k+3/2} + \mathcal{J}_2(n),$$

where

$$\mathcal{J}_2(n) := \begin{cases} (-1)^n (4\beta)^{n+1/2}(-2n-2)!\, \mathcal{Z}_-(-2n-1, 1/2c') & \text{if } n < -1, \\ -\frac{1}{2}\alpha^{1/2}\cot(\pi/2c') + \frac{i}{2}\alpha^{1/2} & \text{if } n = -1, \\ -i\alpha^{-n-1/2}\psi(-1/2c', 0, 2n+2) & \text{if } n \geq 0. \end{cases}$$

Proof. Put $r = 1/2$ in Theorem 3.13. ∎

COROLLARY 3.16.

$$\alpha \sum_{k=1}^{\infty} \frac{2k\cos(\pi k/c')}{e^{(\alpha-i\pi)2k/c'} - 1} = -\beta \sum_{k=1}^{\infty} \frac{k\cos(\pi k/c')}{\sinh((\beta + i\pi)k/c')} - \frac{c'}{2}.$$

Proof. Let $n = -1$ in Proposition 3.14. ∎

COROLLARY 3.17.

$$\alpha^{1/2} \sum_{k=1}^{\infty} \frac{2\sin(\pi k/c')}{e^{(\alpha-i\pi)2k/c'} - 1}$$

$$= (-\beta)^{1/2} \sum_{k=1}^{\infty} \frac{\sin(\pi k/c')}{\sinh((\beta + i\pi)k/c')} - \frac{1}{2}\alpha^{1/2}\cot\left(\frac{\pi}{2c'}\right).$$

Proof. Let $n = -1$ in Proposition 3.15. ∎

COROLLARY 3.18. *Let* r *be any real number with* $0 < r < 1$. *For any positive integer* M,

$$\alpha^M \sum_{k=1}^{\infty} \frac{k^{2M-1}\cos(2\pi rk/c')\sin(2\pi k/c')}{\cosh(2\alpha k/c') - \cos(2\pi k/c')}$$

$$= -(-\beta)^M \sum_{k=1}^{\infty} \frac{k^{2M-1}\cosh(2\beta rk/c')\sin(2\pi k/c')}{\cosh(2\beta k/c') - \cos(2\pi k/c')}.$$

Proof. Let $0 < r < 1$ and consider the imaginary parts in Theorem 3.12 in the case of $n = -M < 0$. ∎

COROLLARY 3.19. *Let r be any real number with $0 < r < 1$. For any positive integer M,*

$$\alpha^{M-1/2} \sum_{k=1}^{\infty} \frac{k^{2M-2} \sin(2\pi rk/c') \sin(2\pi k/c')}{\cosh(2\alpha k/c') - \cos(2\pi k/c')}$$

$$= (-1)^{M+1} \beta^{M-1/2} \sum_{k=1}^{\infty} \frac{k^{2M-2} \sinh(2\beta rk/c') \sin(2\pi k/c')}{\cosh(2\beta k/c') - \cos(2\pi k/c')}.$$

Proof. Let $0 < r < 1$ and consider the real parts in Theorem 3.13 in the case of $n = -M < 0$. ∎

3.3. Case of $N \mid r_1$ and $N \mid (r_1 + r_2)$. By Theorem 2.1, for any integer n and $z \in \mathbb{H}$ with $\mathrm{Re}(z) > 0$, we have

$$z^n H_N(V\tau, -n; r, 0) = H_N(\tau, -n; R, 0) + (2\pi i)^n L_N(\tau, -n; R, 0)$$
$$+ \lim_{s \to -n} (D_1(0, r_2/c, s) + D_2(0, -r_1 - r_2 + r_2/c, s)),$$

where $D_1(0, r_2/c, s)$, $D_2(0, -r_1 - r_2 + r_2/c, s)$ are defined in Section 2. By the given conditions on r_1 and r_2,

$$\frac{\varrho_N}{N} = c'\left\{-r_1 - r_2 + \frac{r_2}{Nc'}\right\} - (1 - Nc')\left\{\frac{r_1 + r_2}{N}\right\} = c'\left\{\frac{r_2}{Nc'}\right\}.$$

Hence employing Lemmas 2.2 and 2.3 of [8] together with Remark 2.2, we find that, for any $z \in \mathbb{H}$,

$$(3.17) \quad z^{2n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi r_2 k/c)}{k^{2n+1}(e^{\frac{2\pi i k}{c'}(\frac{1}{z}-1)} - 1)} = \sum_{k=1}^{\infty} \frac{2\cos(2\pi r_2 k/c)}{k^{2n+1}(e^{\frac{2\pi i k}{c'}(1-z)} - 1)}$$

$$+ (2\pi i)^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k(j/c')\bar{B}_{2n+2-k}(j/c' + \{r_2/c\})}{k!(2n+2-k)!} (-z)^{k-1} + J_3(n),$$

where if $c \nmid r_2$, then

$$J_3(n) := \begin{cases} (1 - z^{2n})(2\pi i)^{2n}(-2n-1)! \, \mathscr{Z}_+(-2n, \{r_2/c\}) & \text{if } n < 0, \\ -2\pi i \bar{B}_1(r_2/c) & \text{if } n = 0, \\ \psi(r_2/c, 0, 1 + 2n) - z^{2n}\psi(-r_2/c, 0, 1 + 2n) & \text{if } n > 0, \end{cases}$$

and if $c \mid r_2$, then

$$J_3(n) := \begin{cases} (1 - z^{2n})\zeta(1 + 2n) & \text{if } n < 0, \\ \pi i - \log z & \text{if } n = 0, \\ (1 - z^{2n})\zeta(1 + 2n) & \text{if } n > 0. \end{cases}$$

For $c \nmid r_2$, we also deduce that, for any $z \in \mathbb{H}$,

$$(3.18) \quad z^{2n+1} \sum_{k=1}^{\infty} \frac{2i \sin(2\pi r_2 k/c)}{k^{2n+2}(e^{\frac{2\pi i k}{c'}(\frac{1}{z}-1)} - 1)} = \sum_{k=1}^{\infty} \frac{2i \sin(2\pi r_2 k/c)}{k^{2n+2}(e^{\frac{2\pi i k}{c'}(1-z)} - 1)}$$

$$+ (2\pi i)^{2n+2} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k(j/c')\bar{B}_{2n+3-k}(j/c' + \{r_2/c\})}{k!(2n+3-k)!} (-z)^{k-1} + \mathcal{J}_3(n),$$

where

$$\mathcal{J}_3(n) := \begin{cases} (z^{2n+1}-1)(2\pi i)^{2n+1}(-2n-2)! \, \mathcal{Z}_-(-2n-1, \{r_2/c\}) & \text{if } n < -1, \\ \frac{1}{2i}(z^{-1}-1)\cot(\pi r_2/c) - \frac{1}{2}(z^{-1}+1) & \text{if } n = -1, \\ \psi(r_2/c, 0, 2+2n) + z^{2n+1}\psi(-r_2/c, 0, 2+2n) & \text{if } n \geq 0. \end{cases}$$

THEOREM 3.20. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Suppose that* r *is an integer and* c' *is a positive integer with* $c' \nmid r$. *For any integer* n,

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi rk/c')}{k^{2n+1}(e^{2k(\alpha-i\pi)/c'} - 1)} = (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi rk/c')}{k^{2n+1}(e^{2k(\beta+i\pi)/c'} - 1)}$$

$$- 2^{2n+1} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k(j/c')\bar{B}_{2n+2-k}(j/c' + \{r/c'\})}{k!(2n+2-k)!} \alpha^{n-k+1}(-\pi i)^k + J_4(n),$$

where

$$J_4(n) := \begin{cases} 2^{2n}(\alpha^n - (-\beta)^n)(-2n-1)! \, \mathcal{Z}_+(-2n, \{r/c'\}) & \text{if } n < 0, \\ -2\pi i \bar{B}_1(r/c') & \text{if } n = 0, \\ (-\beta)^{-n}\psi(r/c', 0, 1+2n) - \alpha^{-n}\psi(-r/c', 0, 1+2n) & \text{if } n > 0. \end{cases}$$

Proof. Put $z = \pi i/\alpha$ and $r_2/N = r$ in (3.17). ∎

THEOREM 3.21. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Suppose that* r *is an integer and* c' *is a positive integer with* $c' \nmid r$. *For any integer* n,

$$\alpha^{-n-1/2} \sum_{k=1}^{\infty} \frac{2i \sin(2\pi rk/c')}{k^{2n+2}(e^{2k(\alpha-i\pi)/c'} - 1)}$$

$$= (-\beta)^{-n-1/2} \sum_{k=1}^{\infty} \frac{2i \sin(2\pi rk/c')}{k^{2n+2}(e^{2k(\beta+i\pi)/c'} - 1)}$$

$$- 2^{2n+2} \sum_{j=1}^{c'} \sum_{k=0}^{2n+3} \frac{B_k(j/c')\bar{B}_{2n+3-k}(j/c' + \{r/c'\})}{k!(2n+3-k)!} (-\pi i)^k \alpha^{n-k+3/2} + \mathcal{J}_4(n),$$

where

$$\mathcal{J}_4(n) := \begin{cases} 2^{2n+1}((-\beta)^{n+1/2} - \alpha^{n+1/2})(-2n-2)! \\ \quad \times \mathcal{Z}_-(-2n-1, \{r/c'\}) & \text{if } n < -1, \\ \frac{1}{2i}(\alpha^{1/2} - (-\beta)^{1/2}) \cot(\pi r/c') - \frac{1}{2}(\alpha^{1/2} + (-\beta)^{1/2}) & \text{if } n = -1, \\ (-\beta)^{-n-1/2}\psi(r/c', 0, 2+2n) \\ \quad + \alpha^{-n-1/2}\psi(-r/c', 0, 2+2n) & \text{if } n \geq 0. \end{cases}$$

Proof. Put $z = \pi i/\alpha$ and $r_2/N = r$ in (3.18). ∎

COROLLARY 3.22. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{k^{4M-1} \cos(2\pi rk/c') \sin(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = 0.$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.20. Replace n by $-2M$ for any positive integer M, and use

$$(3.19) \qquad \frac{1}{e^{x-ix} - 1} - \frac{1}{e^{x+ix} - 1} = \frac{i \sin(x)}{\cosh(x) - \cos(x)}. \quad ∎$$

COROLLARY 3.23. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi rk/c') \sin(2\pi k/c')}{k^{4M+1}(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$

$$= -\frac{(2\pi)^{4M+1}}{2} \sum_{j=1}^{c'} \sum_{k=0}^{2M} \frac{B_{2k+1}(j/c') \bar{B}_{4M+1-2k}(j/c' + \{r/c'\})}{(2k+1)!(4M+1-2k)!} (-1)^{k+1}$$

$$- \frac{(2\pi)^{4M+1}}{2(4M+1)!} \bar{B}_{4M+1}\left(\frac{r}{c'}\right).$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.20. Replace n by $2M$ for any positive integer M. For $n > 0$, applying Corollary 3.47 of [8], we find that

$$\psi(r/c', 0, 1+4n) - \psi(-r/c', 0, 1+4n) = 2i \sum_{k=1}^{\infty} \frac{\sin(2\pi rk/c')}{k^{4n+1}}$$

$$= \frac{i(2\pi)^{4n+1}}{(4n+1)!} \bar{B}_{4n+1}\left(\frac{r}{c'}\right).$$

Employing (3.19) and considering the imaginary parts, we arrive at the desired results. ∎

COROLLARY 3.24. *For any positive integer* M,

$$\sum_{j=1}^{c'} \sum_{k=0}^{2M-1} \frac{B_{2k}(j/c') \bar{B}_{4M-1-2k}(j/c' + \{r/c'\})}{(2k)!(4M-1-2k)!} (-1)^k = 0.$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.20. Replace n by $2M - 1$. Consider the imaginary parts. ∎

COROLLARY 3.25.

$$\alpha \sum_{k=1}^{\infty} \frac{k \cos(2\pi rk/c')}{e^{\frac{2k}{c'}(\alpha-i\pi)} - 1} + \beta \sum_{k=1}^{\infty} \frac{k \cos(2\pi rk/c')}{e^{\frac{2k}{c'}(\beta+i\pi)} - 1}$$
$$= \frac{\beta + \alpha}{4\pi^2}(\zeta(2, \{r/c'\}) + \zeta(2, 1 - \{r/c'\})) - c'/4.$$

Proof. Put $n = -1$ in Theorem 3.20. ∎

COROLLARY 3.26.

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi rk/c')\sin(2\pi k/c')}{k(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$
$$= \pi \sum_{j=1}^{c'} B_1(j/c')\bar{B}_1(j/c' + \{r/c'\}) - \pi\bar{B}_1(r/c').$$

Proof. Let $\alpha = \beta = \pi$ and $n = 0$ in Theorem 3.20. Use (3.19). ∎

COROLLARY 3.27. *For $c' \neq 1$,*

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi k/c')\sin(2\pi k/c')}{k(\cosh(2\pi k/c') - \cos(2\pi k/c'))} = \frac{(c'-2)(c'+2)\pi}{6c'}.$$

Proof. Let $r = 1$ in Corollary 3.26. ∎

COROLLARY 3.28.

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi rk/c')\sin(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = \sum_{k=1}^{\infty} \frac{\sin(2\pi rk/c')\cos(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')}$$
$$- \sum_{k=1}^{\infty} \frac{\sin(2\pi rk/c')e^{-2\pi k/c'}}{\cosh(2\pi k/c') - \cos(2\pi k/c')} + \frac{1}{2}\cot(\pi r/c').$$

Proof. Let $n = -1$ in Theorem 3.21 and consider the imaginary parts. ∎

THEOREM 3.29. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Suppose that r is an integer and c' is a positive integer with $c' \mid r$. For any integer n,*

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}(e^{2k(\alpha-i\pi)/c'} - 1)} = (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}(e^{2k(\beta+i\pi)/c'} - 1)}$$
$$- 2^{2n} \sum_{j=1}^{c'} \sum_{k=0}^{2n+2} \frac{B_k(j/c')\bar{B}_{2n+2-k}(j/c')}{k!(2n+2-k)!} \alpha^{n-k+1}(-\pi i)^k + \frac{1}{2} J_4'(n)$$

where

$$J_4'(n) = \begin{cases} ((-\beta)^{-n} - \alpha^{-n})\frac{B_{-2n}}{2n} & \text{if } n < 0, \\ \frac{1}{2}\log\alpha - \frac{1}{2}\log\beta + \frac{\pi}{2}i & \text{if } n = 0, \\ ((-\beta)^{-n} - \alpha^{-n})\zeta(1+2n) & \text{if } n > 0. \end{cases}$$

Proof. Put $z = \pi i/\alpha$ in (3.17). For $n < 0$, use (2.5), i.e., $\zeta(1+2n) = B_{-2n}/(2n)$. ∎

COROLLARY 3.30. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{k^{4M-1}\sin(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = 0.$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.29 and replace n by $-2M$ for any positive integer M. Equate the real parts. ∎

COROLLARY 3.31. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi k/c')}{k^{4M+1}\cosh(2\pi k/c') - \cos(2\pi k/c')}$$
$$= -\frac{(2\pi)^{4M+1}}{2}\sum_{j=1}^{c'}\sum_{k=0}^{2M} \frac{B_{2k+1}(j/c')\bar{B}_{4M+1-2k}(j/c')}{(2k+1)!(4M+1-2k)!}(-1)^k.$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.29 and replace n by $2M$ for any positive integer M. Equate the real parts. ∎

COROLLARY 3.32. *For any positive integer* M,

$$\sum_{j=1}^{c'}\sum_{k=0}^{2M+1} \frac{B_{2k}(j/c')\bar{B}_{4M+2-2k}(j/c')}{(2k)!(4n+2-2k)!}(-1)^k = 0.$$

Proof. Under the conditions in the proof of Corollary 3.31, equate the imaginary parts. ∎

COROLLARY 3.33.

$$\alpha\sum_{k=1}^{\infty} \frac{k}{e^{2k(\alpha-i\pi)/c'} - 1} + \beta\sum_{k=1}^{\infty} \frac{k}{e^{2k(\beta+i\pi)/c'} - 1} = -\frac{c'}{4} + \frac{1}{24}(\beta+\alpha).$$

Proof. Put $n = -1$ in Theorem 3.29 and use $\zeta(-1) = -1/12$. ∎

COROLLARY 3.34.

$$\sum_{k=1}^{\infty} \frac{k\cos(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = \sum_{k=1}^{\infty} \frac{ke^{-2\pi k/c'}}{\cosh(2\pi k/c') - \cos(2\pi k/c')} - \frac{c'}{4\pi} + \frac{1}{12}.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.33. Use

$$(3.20) \quad \frac{1}{e^{x-ix}-1} + \frac{1}{e^{x+ix}-1} = \frac{\cos(x)}{\cosh(x) - \cos(x)} - \frac{e^{-x}}{\cosh(x) - \cos(x)}. \quad ∎$$

COROLLARY 3.35.

$$\sum_{k=1}^{\infty} \frac{1}{k(e^{2k(\alpha-i\pi)/c'} - 1)} - \sum_{k=1}^{\infty} \frac{1}{k(e^{2k(\beta+i\pi)/c'} - 1)}$$
$$= \frac{1}{12c'}(\beta - \alpha) + \frac{1}{4}\log\frac{\alpha}{\beta} + \frac{(c'-1)(c'-2)\pi i}{12c'}.$$

Proof. Put $n = 0$ in Theorem 3.29. ∎

COROLLARY 3.36.

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi k/c')}{k(\cosh(2\pi k/c') - \cos(2\pi k/c'))} = \frac{(c'-1)(c'-2)\pi}{12c'}.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.35 and apply (3.19). ∎

COROLLARY 3.37. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{k^{4M+1}\cos(2\pi k/c')}{\cosh(2\pi k/c') - \cos(2\pi k/c')} = \sum_{k=1}^{\infty} \frac{k^{4M+1}e^{-2\pi k/c'}}{\cosh(2\pi k/c') - \cos(2\pi k/c')} + \frac{B_{4M+2}}{4M+2}.$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.29 and replace n by $-2M - 1$ for any positive integer M. Use (3.20). ∎

COROLLARY 3.38. *For any positive integer* M,

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi k/c')}{k^{4M-1}(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$
$$= \sum_{k=1}^{\infty} \frac{e^{-2\pi k/c'}}{k^{4M-1}(\cosh(2\pi k/c') - \cos(2\pi k/c'))}$$
$$- \frac{(2\pi)^{4M-1}}{2}\sum_{j=1}^{c'}\sum_{k=0}^{2M} \frac{B_{2k}(j/c')\bar{B}_{4M-2k}(j/c')}{(2k)!(4M-2k)!}(-1)^k - \zeta(4M-1).$$

Proof. Let $\alpha = \beta = \pi$ in Theorem 3.29 and replace n by $2M - 1$. Equate the real parts. ∎

COROLLARY 3.39.

$$\sum_{j=1}^{c'}\sum_{k=0}^{2M-1} \frac{B_{2k+1}(j/c')\bar{B}_{4M-1-2k}(j/c')}{(2k+1)!(4M-1-2k)!}(-1)^k = 0.$$

Proof. Under the conditions in the proof of Corollary 3.38, equate the imaginary parts. ∎

4. Case $c' = 2$. In this section, we assume $c' = 2$. Then using the results in Section 3 of [8], we obtain a number of infinite series identities, many of which are new.

4.1. $N \nmid r_1$ **and** $N \nmid (r_1 + r_2)$**.** Put $c' = 2$ in Theorem 3.1. It is easy to see that

$$(4.1) \quad \sum_{k=1}^{\infty} \frac{\cosh((2(r + r') - 2[r] - 1)\pi ik/2 + (1 - 2\{r\})\alpha k/2)}{k^{2n+1} \sinh((\alpha - i\pi)k/2)}$$

$$= 2^{-2n-1} \sum_{k=1}^{\infty} \frac{\cosh((r + r')2\pi ik + (1 - 2\{r\})\alpha k)}{k^{2n+1} \sinh(\alpha k)}$$

$$+ (-1)^{[r]} \sum_{k=0}^{\infty} \frac{\sinh((2(r + r')\pi i + (1 - 2\{r\})\alpha)(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\alpha(2k + 1)/2)},$$

and

$$(4.2) \quad \sum_{k=1}^{\infty} \frac{\cosh((-2r + 2[r + r'] + 1)\pi ik/2 + (1 - 2\{r + r'\})\beta k/2)}{k^{2n+1} \sinh((\beta + i\pi)k/2)}$$

$$= 2^{-2n-1} \sum_{k=1}^{\infty} \frac{\cosh(-2\pi irk + (1 - 2\{r + r'\})\beta k)}{k^{2n+1} \sinh(\beta k)}$$

$$+ (-1)^{[r+r']} \sum_{k=0}^{\infty} \frac{\sinh((-2\pi ir + (1 - 2\{r + r'\})\beta)(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\beta(2k + 1)/2)}.$$

THEOREM 4.1. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Suppose that* r, r' *are real numbers such that* $r, r + r'$ *are not integers. Then, for any integer* n,

$$(-1)^{[r]}\alpha^{-n} \sum_{k=0}^{\infty} \frac{\sinh((2(r + r')\pi i + (1 - 2\{r\})\alpha)(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\alpha(2k + 1)/2)}$$

$$= (-1)^{[r+r']}(-\beta)^{-n} \sum_{k=0}^{\infty} \frac{\sinh((-2\pi ir + (1 - 2\{r + r'\})\beta)(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\beta(2k + 1)/2)}$$

$$- 2^{2n+1} \sum_{j=1}^{2} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j - \{r+r'\}}{2}\right) \bar{B}_{2n+2-k}\left(\frac{j + \varrho_N/N}{2}\right)}{k!(2n + 2 - k)!} (-\pi i)^k \alpha^{n-k+1}$$

$$+ \sum_{k=0}^{2n+2} \frac{\bar{B}_k(r + r') \bar{B}_{2n+2-k}(\varrho_N/N)}{k!(2n + 2 - k)!} (\pi i)^k \alpha^{n-k+1}.$$

Proof. Put (4.1) and (4.2) in the first equation in Theorem 3.1 and use Theorem 3.1 in [8]. ∎

Similarly we obtain the following theorem.

THEOREM 4.2. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Suppose that* r, r' *are real numbers such that* $r, r + r'$ *are not integers. Then, for any integer* n,

$$(-1)^{[r]}\alpha^{-n-1/2}\sum_{k=0}^{\infty}\frac{\cosh((2(r+r')\pi i+(1-2\{r\})\alpha)(2k+1)/2)}{(2k+1)^{2n+2}\cosh(\alpha(2k+1)/2)}$$

$$=(-1)^{[r+r']}(-\beta)^{-n-1/2}\sum_{k=0}^{\infty}\frac{\cosh((-2\pi ir+(1-2\{r+r'\})\beta)(2k+1)/2)}{(2k+1)^{2n+2}\cosh(\beta(2k+1)/2)}$$

$$-2^{2n+2}\sum_{j=1}^{2}\sum_{k=0}^{2n+3}\frac{B_k\big(\frac{j-\{r+r'\}}{2}\big)\bar{B}_{2n+3-k}\big(\frac{j+\varrho_N/N}{2}\big)}{k!(2n+3-k)!}(-\pi i)^k\alpha^{n-k+3/2}$$

$$+\sum_{k=0}^{2n+3}\frac{\bar{B}_k(r+r')\bar{B}_{2n+3-k}(\varrho_N/N)}{k!(2n+3-k)!}(\pi i)^k\alpha^{n-k+3/2}.$$

COROLLARY 4.3. *For any positive integer M,*

$$(-1)^{[r]}\alpha^{M}\sum_{k=0}^{\infty}\frac{\sinh((2(r+r')\pi i+(1-2\{r\})\alpha)(2k+1)/2)}{(2k+1)^{1-2M}\cosh(\alpha(2k+1)/2)}$$

$$=(-1)^{[r+r']}(-\beta)^{M}\sum_{k=0}^{\infty}\frac{\sinh((-2\pi ir+(1-2\{r+r'\})\beta)(2k+1)/2)}{(2k+1)^{1-2M}\cosh(\beta(2k+1)/2)}.$$

Proof. Put $n=-M$ in Theorem 4.1. ∎

COROLLARY 4.4.

$$(-1)^{[r]}\sum_{k=0}^{\infty}\frac{\sinh((2(r+r')\pi i+(1-2\{r\})\alpha)(2k+1)/2)}{(2k+1)\cosh(\alpha(2k+1)/2)}$$

$$=(-1)^{[r+r']}\sum_{k=0}^{\infty}\frac{\sinh((-2\pi ir+(1-2\{r+r'\})\beta)(2k+1)/2)}{(2k-1)\cosh(\beta(2k+1)/2)}+A,$$

where

$$A:=\begin{cases}\pi i/4 & \text{if }[\varrho_N/N]\text{ is odd},\\ -\pi i/4 & \text{if }[\varrho_N/N]\text{ is even}.\end{cases}$$

Proof. Put $n=0$ in Theorem 4.1. ∎

COROLLARY 4.5. *For any positive integer M,*

$$(-1)^{[r]}\alpha^{M-1/2}\sum_{k=0}^{\infty}\frac{\cosh((2(r+r')\pi i+(1-2\{r\})\alpha)(2k+1)/2)}{(2k+1)^{2-2M}\cosh(\alpha(2k+1)/2)}$$

$$=(-1)^{[r+r']}(-\beta)^{M-1/2}\sum_{k=0}^{\infty}\frac{\cosh((-2\pi ir+(1-2\{r+r'\})\beta)(2k+1)/2)}{(2k+1)^{2-2M}\cosh(\beta(2k+1)/2)}.$$

Proof. Let $n=-M$ in Theorem 4.2. If $M>1$, then Corollary 4.5 is obvious. If $M=1$, then we can check that the sums over the Bernoulli polynomials vanish. ∎

Let us assume that r is a real number with $0 < r < 1$ and $r' = 0$. We see that

$$\varrho_N/N = -2[Nr] - r.$$

Thus

(4.3) $$[\varrho_N/N] = -2[-Nr] - 1, \quad \{\varrho_N/N\} = 1 - r.$$

THEOREM 4.6. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be a real number with* $0 < r < 1$. *Then, for any integer* n,

$$\alpha^{-n} \sum_{k=0}^{\infty} \frac{\sinh((2\pi i r + \alpha(1 - 2r))(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\alpha(2k + 1)/2)}$$

$$= (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{\sinh((-2\pi i r + \beta(1 - 2r))(2k + 1)/2)}{(2k + 1)^{2n+1} \cosh(\beta(2k + 1)/2)}$$

$$- 2^{2n+1} \sum_{j=1}^{2} \sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j-r}{2}\right) \bar{B}_{2n+2-k}\left(\frac{j-r}{2}\right)}{k!(2n + 2 - k)!} (-\pi i)^k \alpha^{n-k+1}$$

$$+ \sum_{k=0}^{2n+2} \frac{B_k(r) B_{2n+2-k}(r)}{k!(2n + 2 - k)!} (-\pi i)^k \alpha^{n-k+1}.$$

Proof. Let $r' = 0$ and $0 < r < 1$ in Theorem 4.1 and apply (4.3). ∎

THEOREM 4.7. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be a real number with* $0 < r < 1$. *Then, for any integer* n,

$$\alpha^{-n-1/2} \sum_{k=0}^{\infty} \frac{\cosh((2\pi i r + \alpha(1 - 2r))(2k + 1)/2)}{(2k + 1)^{2n+2} \cosh(\alpha(2k + 1)/2)}$$

$$= (-\beta)^{-n-1/2} \sum_{k=0}^{\infty} \frac{\cosh((-2\pi i r + \beta(1 - 2r))(2k + 1)/2)}{(2k + 1)^{2n+2} \cosh(\beta(2k + 1)/2)}$$

$$- 2^{2n+2} \sum_{j=1}^{2} \sum_{k=0}^{2n+3} \frac{B_k\left(\frac{j-r}{2}\right) \bar{B}_{2n+3-k}\left(\frac{j-r}{2}\right)}{k!(2n + 3 - k)!} (-\pi i)^k \alpha^{n-k+3/2}$$

$$- \sum_{k=0}^{2n+3} \frac{B_k(r) \bar{B}_{2n+3-k}(r)}{k!(2n + 3 - k)!} (-\pi i)^k \alpha^{n-k+3/2}.$$

Theorems 4.6 and 4.7 can be divided into two parts, i.e., the real parts and the imaginary parts. We use

$$\sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y,$$
$$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$$

for any real numbers x, y.

THEOREM 4.8. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be a real number with* $0 < r < 1$. *Then, for any integer* n,

$$\alpha^{-n} \sum_{k=0}^{\infty} \frac{\sinh(\alpha(1-2r)(2k+1)/2)\cos(\pi r(2k+1))}{(2k+1)^{2n+1}\cosh(\alpha(2k+1)/2)}$$

$$= (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{\sinh(\beta(1-2r)(2k+1)/2)\cos(\pi r(2k+1))}{(2k+1)^{2n+1}\cosh(\beta(2k+1)/2)}$$

$$- 2^{2n+1} \sum_{j=1}^{2} \sum_{k=0}^{n+1} \frac{B_{2k}\left(\frac{j-r}{2}\right)\bar{B}_{2n+2-2k}\left(\frac{j-r}{2}\right)}{(2k)!(2n+2-2k)!} \alpha^{n-k+1}(-\beta)^k$$

$$+ \sum_{k=0}^{n+1} \frac{B_{2k}(r)B_{2n+2-2k}(r)}{(2k)!(2n+2-2k)!} \alpha^{n-k+1}(-\beta)^k.$$

Proof. Equate the real parts in Theorem 4.6. ∎

THEOREM 4.9. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be a real number with* $0 < r < 1$. *Then, for any integer* n,

$$\alpha^{-n} \sum_{k=0}^{\infty} \frac{\cosh(\alpha(1-2r)(2k+1)/2)\sin(\pi r(2k+1))}{(2k+1)^{2n+1}\cosh(\alpha(2k+1)/2)}$$

$$= -(-\beta)^{-n} \sum_{k=0}^{\infty} \frac{\cosh(\beta(1-2r)(2k+1)/2)\sin(\pi r(2k+1))}{(2k+1)^{2n+1}\cosh(\beta(2k+1)/2)}$$

$$+ 2^{2n+1}\pi \sum_{j=1}^{2} \sum_{k=0}^{n} \frac{B_{2k+1}\left(\frac{j-r}{2}\right)\bar{B}_{2n+1-2k}\left(\frac{j-r}{2}\right)}{(2k+1)!(2n+1-2k)!} \alpha^{n-k}(-\beta)^k$$

$$- \pi \sum_{k=0}^{n} \frac{B_{2k+1}(r)B_{2n+1-2k}(r)}{(2k+1)!(2n+1-2k)!} \alpha^{n-k}(-\beta)^k.$$

Proof. Equate the imaginary parts in Theorem 4.6. ∎

THEOREM 4.10. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be a real number with* $0 < r < 1$. *Then, for any integer* n,

$$\alpha^{-n-1/2} \sum_{k=0}^{\infty} \frac{\cosh(\alpha(1-2r)(2k+1)/2)\cos(\pi r(2k+1))}{(2k+1)^{2n+2}\cosh(\alpha(2k+1)/2)}$$

$$= (-1)^{n+1}\beta^{-n-1/2} \sum_{k=0}^{\infty} \frac{\sinh(\beta(1-2r)(2k+1)/2)\sin(\pi r(2k+1))}{(2k+1)^{2n+2}\cosh(\beta(2k+1)/2)}$$

$$- 2^{2n+2} \sum_{j=1}^{2} \sum_{k=0}^{n+1} \frac{B_{2k}\left(\frac{j-r}{2}\right)\bar{B}_{2n+3-2k}\left(\frac{j-r}{2}\right)}{(2k)!(2n+3-2k)!} \alpha^{n-k+3/2}(-\beta)^k$$

$$- \sum_{k=0}^{n+1} \frac{B_{2k}(r)\bar{B}_{2n+3-2k}(r)}{(2k)!(2n+3-2k)!} \alpha^{n-k+3/2}(-\beta)^k.$$

Proof. Equate the real parts in Theorem 4.7. ∎

We see that the imaginary parts in Theorem 4.7 give the same theorem as Theorem 4.10.

COROLLARY 4.11. *For any positive integer M,*

$$\alpha^{2M} \sum_{k=0}^{\infty} \frac{(2k+1)^{4M-1} \sinh(\alpha(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{\cosh(\alpha(2k+1)/2)}$$
$$= \beta^{2M} \sum_{k=0}^{\infty} \frac{(2k+1)^{4M-1} \sinh(\beta(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{\cosh(\beta(2k+1)/2)}.$$

Proof. Put $n = -2M$ in Theorem 4.8. ∎

COROLLARY 4.12. *For any positive integer M,*

$$\alpha^{2M-1} \sum_{k=0}^{\infty} \frac{(2k+1)^{2M-3} \cosh(\alpha(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{\cosh(\alpha(2k+1)/2)}$$
$$= \beta^{2M-1} \sum_{k=0}^{\infty} \frac{(2k+1)^{2M-3} \cosh(\beta(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{\cosh(\beta(2k+1)/2)}.$$

Proof. Put $n = -2M + 1$ in Theorem 4.9. ∎

COROLLARY 4.13. *For any positive integer M,*

$$\alpha^{2M-3/2} \sum_{k=0}^{\infty} \frac{(2k+1)^{4M-4} \cosh(\alpha(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{\cosh(\alpha(2k+1)/2)}$$
$$= \beta^{2M-3/2} \sum_{k=0}^{\infty} \frac{(2k+1)^{4M-4} \sinh(\beta(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{\cosh(\beta(2k+1)/2)}.$$

Proof. Put $n = -2M + 1$ in Theorem 4.10. The sums of Bernoulli polynomials vanish when $n = -1$. ∎

Corollaries 4.11–4.13 are symmetric with respect to α and β.

COROLLARY 4.14. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{(2k+1)^{4M-3} \sinh(\pi(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{\cosh(\pi(2k+1)/2)} = 0.$$

Proof. Replace $-n$ by $2M - 1$ and let $\alpha = \beta = \pi$ in Theorem 4.8. ∎

COROLLARY 4.15. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{(2k+1)^{4M-1} \cosh(\pi(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{\cosh(\pi(2k+1)/2)} = 0.$$

Proof. Replace $-n$ by $2M$ and let $\alpha = \beta = \pi$ in Theorem 4.9. ∎

COROLLARY 4.16. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{(2k+1)^{4M-2} \cosh(\pi(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{\cosh(\pi(2k+1)/2)} = 0.$$

Proof. Replace $-n$ by $2M$ and let $\alpha = \beta = \pi$ in Theorem 4.10. ∎

COROLLARY 4.17.

$$\sum_{k=0}^{\infty} \frac{\sinh(\alpha(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{(2k+1) \cosh(\alpha(2k+1)/2)}$$

$$= \sum_{k=0}^{\infty} \frac{\sinh(\beta(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{(2k+1) \cosh(\beta(2k+1)/2)}.$$

Proof. Put $n = 0$ in Theorem 4.8. ∎

COROLLARY 4.18.

$$\sum_{k=0}^{\infty} \frac{\cosh(\pi(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{(2k+1) \cosh(\pi(2k+1)/2)} = \frac{\pi}{8}.$$

Proof. Put $n = 0$ and let $\alpha = \beta = \pi$ in Theorem 4.9. ∎

COROLLARY 4.19. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{\sinh(\pi(1-2r)(2k+1)/2) \cos(\pi r(2k+1))}{(2k+1)^{4M-1} \cosh(\pi(2k+1)/2)}$$

$$= -\frac{(2\pi)^{4M-1}}{2} \sum_{j=1}^{2} \sum_{k=0}^{2M} \frac{B_{2k}\left(\frac{j-r}{2}\right) \bar{B}_{4M-2k}\left(\frac{j-r}{2}\right)}{(2k)!(4M-2k)!} (-1)^k$$

$$+ \frac{\pi^{4M-1}}{2} \sum_{k=0}^{2M} \frac{B_{2k}(r) B_{4M-2k}(r)}{(2k)!(4M-2k)!} (-1)^k.$$

Proof. Put $n = 2M - 1$ for any positive integer M and let $\alpha = \beta = \pi$ in Theorem 4.8. ∎

COROLLARY 4.20. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{\cosh(\pi(1-2r)(2k+1)/2) \sin(\pi r(2k+1))}{(2k+1)^{4M+1} \cosh(\pi(2k+1)/2)}$$

$$= \frac{(2\pi)^{4M+1}}{2} \sum_{j=1}^{2} \sum_{k=0}^{2M} \frac{B_{2k+1}\left(\frac{j-r}{2}\right) \bar{B}_{4M+1-2k}\left(\frac{j-r}{2}\right)}{(2k+1)!(4M+1-2k)!} (-1)^k$$

$$- \frac{\pi^{4M+1}}{2} \sum_{k=0}^{2M} \frac{B_{2k+1}(r) B_{4M+1-2k}(r)}{(2k+1)!(4M+1-2k)!} (-1)^k.$$

Proof. Put $n = 2M$ for any positive integer M and let $\alpha = \beta = \pi$ in Theorem 4.9. ∎

REMARK 4.21. If we put $r = 1/2$, then Theorem 4.9 yields Proposition 3.21 in [8], which is found in Ramanujan's Notebooks [11] and established by other authors [8, p. 13].

4.2. $N \mid r_1$ and $N \nmid (r_1 + r_2)$. In this case, we see that

$$\frac{\varrho_N}{N} = -2\left[-r_2 + \frac{r_2}{2N}\right] - 2N\left[\frac{r_2}{N}\right] + \left[\frac{r_2}{N}\right].$$

Thus $\{\varrho_N/N\} = 0$. First, put $c' = 2$ in Theorem 3.12. It is easy to see that

$$(4.4) \quad \sum_{k=1}^{\infty} \frac{\cos(\pi r k)}{k^{2n+1}(e^{(\alpha-i\pi)k} - 1)}$$

$$= \sum_{k=1}^{\infty} \frac{\cos(2\pi r k)}{(2k)^{2n+1}(e^{2\alpha k} - 1)} - \sum_{k=0}^{\infty} \frac{\cos(\pi r(2k+1))}{(2k+1)^{2n+1}(e^{\alpha(2k+1)} + 1)},$$

and

$$(4.5) \quad \sum_{k=1}^{\infty} \frac{\cosh((2[r]+1)\pi i k/2 + (1-2\{r\})\beta k/2)}{k^{2n+1}\sinh((\beta+i\pi)k/2)}$$

$$= \sum_{k=1}^{\infty} \frac{\cosh((1-2\{r\})\beta k)}{(2k)^{2n+1}\sinh(\beta k)} + (-1)^{[r]}\sum_{k=0}^{\infty} \frac{\sinh((1-2\{r\})\beta(2k+1)/2)}{(2k+1)^{2n+1}\cosh(\beta(2k+1)/2)}.$$

We divide the two sums of Bernoulli polynomials in Theorem 3.12 into the real and imaginary parts, namely,

$$(4.6) \quad \sum_{j=1}^{2}\sum_{k=0}^{2n+2} \frac{B_k\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+2-k}\left(\frac{j+[r]}{2}\right)}{k!(2n+2-k)!}\,\alpha^{n-k+1}(-\pi i)^k$$

$$= \sum_{j=1}^{2}\sum_{k=0}^{n+1} \frac{B_{2k}\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+2-2k}\left(\frac{j+[r]}{2}\right)}{(2k)!(2n+2-2k)!}\,\alpha^{n-k+1}(-\beta)^k$$

$$- \pi i\sum_{j=1}^{2}\sum_{k=0}^{n} \frac{B_{2k+1}\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+1-2k}\left(\frac{j+[r]}{2}\right)}{(2k+1)!(2n+1-2k)!}\,\alpha^{n-k}(-\beta)^k.$$

From the imaginary parts of (4.6),

$$(4.7) \quad \sum_{j=1}^{2}\sum_{k=0}^{n} \frac{B_{2k+1}\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+1-2k}\left(\frac{j+[r]}{2}\right)}{(2k+1)!(2n+1-2k)!}\,\alpha^{n-k}(-\beta)^k$$

$$= -\frac{(-\beta)^n}{2(2n+1)!}\,B_{2n+1}(f_r),$$

where

$$f_r = \begin{cases} 1 - \{r\}/2 & \text{if } [r] \text{ is even,} \\ 1/2 - \{r\}/2 & \text{if } [r] \text{ is odd.} \end{cases}$$

In the case of $n = 0$, we find that

(4.8) $\frac{1}{2}\log(1 - e^{-2\pi i r}) - \log(1 - e^{-\pi i r}) = \frac{1}{2}\log|\cot(\pi r/2)| + \pi i \delta/4,$

where

$$\delta = \begin{cases} -1 & \text{if } [r] \equiv 0 \pmod{4}, \\ -3 & \text{if } [r] \equiv 1 \pmod{4}, \\ 3 & \text{if } [r] \equiv 2 \pmod{4}, \\ 1 & \text{if } [r] \equiv 3 \pmod{4}. \end{cases}$$

For $n > 0$,

(4.9) $\psi(-r/2, 0, 2n + 1) = 2^{-2n-1}\psi(-r, 0, 2n + 1)$
$$+ 2^{-2n-1}e^{-\pi i r}\psi(-r, 1/2, 2n + 1).$$

In (4.9), the last term on the right hand side is expressed as

(4.10) $2^{-2n-1}e^{-\pi i r}\psi(-r, 1/2, 2n + 1)$
$$= \sum_{k=0}^{\infty} \frac{\cos(\pi r(2k+1))}{(2k+1)^{2n+1}} - i\sum_{k=0}^{\infty} \frac{\sin(\pi r(2k+1))}{(2k+1)^{2n+1}}.$$

THEOREM 4.22. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Let r be any real number that is not an integer. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=0}^{\infty} \frac{2\cos(\pi r(2k+1))}{(2k+1)^{2n+1}(e^{\alpha(2k+1)} + 1)}$$

$$= (-1)^{[r]+1}(-\beta)^{-n}\sum_{k=0}^{\infty} \frac{\sinh(\beta(1 - 2\{r\})(2k+1)/2)}{(2k+1)^{2n+1}\cosh(\beta(2k+1)/2)}$$

$$+ 2^{2n+1}\sum_{j=1}^{2}\sum_{k=0}^{n+1} \frac{B_{2k}\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+2-2k}\left(\frac{j+[r]}{2}\right)}{(2k)!(2n+2-2k)!} \alpha^{n-k+1}(-\beta)^k$$

$$- \sum_{k=0}^{n+1} \frac{\bar{B}_{2k}(r)B_{2n+2-2k}}{(2k)!(2n+2-2k)!} \alpha^{n-k+1}(-\beta)^k + \frac{\pi i(-\beta)^n\bar{B}_{2n+1}(r)}{2(2n+1)!}$$

$$+ \frac{2^{2n}\pi i(-\beta)^n B_{2n+1}(f_r)}{(2n+1)!} + I(n),$$

where

$$I(n) := \begin{cases} 2^{2n}(-\beta)^n(-2n-1)! \, \mathscr{Z}_+(-2n, \{r/2\}) \\ \quad - 2^{-1}(-\beta)^n(-2n-1)! \, \mathscr{Z}_+(-2n, \{r\}) & \text{if } n < 0, \\ \frac{1}{2}\log|\cot(\pi r/2)| + \pi i \delta/4 & \text{if } n = 0, \\ \alpha^{-n}\left(\sum_{k=0}^{\infty} \frac{\cos(\pi r(2k+1))}{(2k+1)^{2n+1}} - i\sum_{k=0}^{\infty} \frac{\sin(\pi r(2k+1))}{(2k+1)^{2n+1}}\right) & \text{if } n > 0, \end{cases}$$

and $\mathscr{Z}_+(s,x)$, f_r, δ are defined in (2.6), (4.7), (4.8), respectively.

Proof. Apply (4.4), (4.5) and Theorem 3.42 in [8] to Theorem 3.12. Employing (4.6)–(4.10), we obtain the desired results. ∎

If $0 < r < 1$, then the real parts of Theorem 4.22 can be compared with Theorem 4.17 in [5, p. 346].

Next, put $c' = 2$ in Theorem 3.13. Similarly we obtain the following theorem.

THEOREM 4.23. *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2$. *Let* r *be any real number that is not an integer. Then, for any integer* n,

$$\alpha^{-n-1/2}\sum_{k=0}^{\infty} \frac{2i\sin(\pi r(2k+1))}{(2k+1)^{2n+2}(e^{\alpha(2k+1)}+1)}$$

$$= (-1)^{[r]+1}(-\beta)^{-n-1/2}\sum_{k=0}^{\infty} \frac{\cosh(\beta(1-2\{r\})(2k+1)/2)}{(2k+1)^{2n+2}\cosh(\beta(2k+1)/2)}$$

$$- 2^{2n+1}\pi i \sum_{j=1}^{2}\sum_{k=0}^{n+1} \frac{B_{2k+1}\left(\frac{j-\{r\}}{2}\right)\bar{B}_{2n+2-2k}\left(\frac{j+[r]}{2}\right)}{(2k+1)!(2n+2-2k)!} \alpha^{n-k+1/2}(-\beta)^k$$

$$- \pi i \sum_{k=0}^{n+1} \frac{\bar{B}_{2k+1}(r)B_{2n+2-2k}}{(2k+1)!(2n+2-2k)!} \alpha^{n-k+1/2}(-\beta)^k$$

$$+ \frac{(-1)^{n+1}\alpha^{1/2}\beta^{n+1}\bar{B}_{2n+2}(r)}{2(2n+2)!} + \frac{2^{2n+1}(-1)^n\alpha^{1/2}\beta^{n+1}B_{2n+2}(f_r)}{2(2n+2)!} + \mathcal{I}(n),$$

where

$$\mathcal{I}(n) := \begin{cases} -2^{2n+1}(-\beta)^{n+1/2}(-2n-2)! \, \mathscr{Z}_-(-2n-1, \{r/2\}) \\ \quad + 2^{-1}(-\beta)^{n+1/2}(-2n-2)! \, \mathscr{Z}_-(-2n-1, \{r\}) & \text{if } n < -1, \\ \frac{(-1)^{[r]}i}{2}\alpha^{1/2}\csc(\pi\{r\}) & \text{if } n = -1, \\ \alpha^{-n-1/2}\left(\sum_{k=0}^{\infty} \frac{\cos(\pi r(2k+1))}{(2k+1)^{2n+2}} - i\sum_{k=0}^{\infty} \frac{\sin(\pi r(2k+1))}{(2k+1)^{2n+2}}\right) & \text{if } n \geq 0, \end{cases}$$

and $\mathscr{Z}_-(s,x)$, f_r are defined in (2.6), (4.7), respectively.

If $0 < r < 1$, then the imaginary parts of Theorem 4.23 can be compared with Theorem 4.17 in [5, p. 347].

COROLLARY 4.24.
$$\sum_{k=0}^{\infty} \frac{2\cos(\pi r(2k+1))}{(2k+1)(e^{\alpha(2k+1)}-1)} = (-1)^{[r]+1}\sum_{k=0}^{\infty} \frac{\sinh((1-2\{r\})\beta(2k+1)/2)}{(2k+1)\cosh(\beta(2k+1)/2)}$$
$$+ \frac{1}{2}\log\left|\cot\left(\frac{\pi r}{2}\right)\right|.$$

Proof. Let $n = 0$ in Theorem 4.22. ∎

COROLLARY 4.25.
$$\alpha^{1/2}\sum_{k=0}^{\infty} \frac{2\sin(\pi r(2k+1))}{e^{\alpha(2k+1)}+1} = (-1)^{[r]+1}\beta^{1/2}\sum_{k=0}^{\infty} \frac{\cosh(\beta(1-2\{r\})(2k+1)/2)}{\cosh(\beta(2k+1)/2)}$$
$$+ \frac{1}{2}\beta^{1/2}(1-\{r\}) + \frac{(-1)^{[r]}}{2}\alpha^{1/2}\csc(\pi\{r\}).$$

Proof. Let $n = -1$ in Theorem 4.23. ∎

This last corollary is similar to Corollary 4.20 in [5, p. 349].

PROPOSITION 4.26. *Let r be any real number that is not an integer. Then, for any positive integer M,*
$$\sum_{k=0}^{\infty} \frac{\sin(\pi r(2k+1))}{(2k+1)^{2M-1}} = \frac{(-1)^{M-1}\pi^{2M-1}}{2(2M-1)!}(\bar{B}_{2M-1}(r) + 2^{2M-1}B_{2M-1}(f_r)),$$
$$\sum_{k=0}^{\infty} \frac{\cos(\pi r(2k+1))}{(2k+1)^{2M}} = \frac{(-1)^{M-1}\pi^{2M}}{2(2M)!}(\bar{B}_{2M}(r) - 2^{2M-1}B_{2M}(f_r)),$$
where f_r is defined in (4.7).

Proof. Let $n = M-1$ and equate the imaginary parts and the real parts in Theorems 4.22 and 4.23, respectively. ∎

Proposition 4.26 should be compared with Corollary 3.47 in [8] or Theorem 3.2 in [4, p. 166].

PROPOSITION 4.27. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\alpha^{-n-1/2}\sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)^{2n+2}(e^{\alpha(2k+1)}+1)}$$
$$= (-1)^n\beta^{-n-1/2}\sum_{k=0}^{\infty} \frac{\operatorname{sech}(\beta(2k+1)/2)}{(2k+1)^{2n+2}}$$
$$- \pi\sum_{k=0}^{n+1} \frac{(2^{2k}-2^{2n+2})B_{2k+1}(1/4)B_{2n+2-2k}}{(2k+1)!(2n+2-2k)!}\alpha^{n-k+1/2}(-\beta)^k + \mathcal{I}_1(n),$$

where

$$\mathcal{I}_1(n) := \begin{cases} 2^{-4n-3}\alpha^{-n-1/2}B_{-2n-1}(1/4)/(2n+1) & \text{if } n < -1, \\ \frac{1}{2}\alpha^{1/2} & \text{if } n = -1, \\ -4^{-2n-2}\alpha^{-n-1/2}(\zeta(2n+2,1/4) - \zeta(2n+2,3/4)) & \text{if } n \geq 0. \end{cases}$$

Proof. Put $r = 1/2$ in Theorem 4.23. For $n < -1$, we see [8]

$$\zeta(-2n-1,1/4) - \zeta(-2n-1,3/4) = \frac{(-1)^n 2^{-6n-4}\pi^{-2n-1}B_{-2n-1}(1/4)}{(-2n-1)!}.$$

For the sums of Bernoulli polynomials, use [1]

$$B_n(1/2) = -(1 - 2^{1-n})B_n, \quad n \geq 0. \quad \blacksquare$$

Proposition 4.27 is already given by Berndt [5, p. 348], in a different form with Euler polynomials $E_n(x)$, $0 \leq n < \infty$, under the assumption $h_1 = h_2 = 1/2$, where $E_n(x)$ is defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi).$$

COROLLARY 4.28.

$$\alpha^{1/2}\sum_{k=0}^{\infty}\frac{2(-1)^k}{e^{\alpha(2k+1)} + 1} - \frac{1}{2}\alpha^{1/2} = -\beta^{1/2}\sum_{k=0}^{\infty}\operatorname{sech}(\beta(2k+1)/2).$$

Proof. Let $n = -1$ in Proposition 4.27. $\blacksquare$

Berndt has also given a proof of this last corollary.

EXAMPLE 4.29. We obtain a class of interesting infinite series identities from Theorem 4.22 or 4.23 for special values of r. For example, let $r = 1/4$ in Theorem 4.22. Then, using

$$\cos(\pi(2k+1)/4) = \begin{cases} 1/\sqrt{2} & \text{if } k \equiv 0, 3 \pmod 4, \\ -1/\sqrt{2} & \text{if } k \equiv 1, 2 \pmod 4, \end{cases}$$

we have, for any positive integer M,

$$\sqrt{2}\,\alpha^M\sum_{k\equiv 0,3\,(\mathrm{mod}\,4)}\frac{(2k+1)^{2M-1}}{e^{\alpha(2k+1)} - 1} - \sqrt{2}\,\alpha^M\sum_{k\equiv 1,2\,(\mathrm{mod}\,4)}\frac{(2k+1)^{2M-1}}{e^{\alpha(2k+1)} - 1}$$

$$= -(-\beta)^M\sum_{k=0}^{\infty}\frac{(2k+1)^{2M-1}\sinh(\beta(2k+1)/4)}{\cosh(\beta(2k+1)/2)}$$

$$- (2^{6M-2} - 2^{4M-2})|B_{2M}|\pi^{2M}(-\beta)^{-M} + 2^{-2M}(2M)!(-\beta)^{-M}\mathcal{Z}_+(2M,1/8),$$

$$\sum_{k\equiv 0,3\,(\mathrm{mod}\,4)}\frac{1}{(2k+1)(e^{\alpha(2k+1)} - 1)} - \sum_{k\equiv 1,2\,(\mathrm{mod}\,4)}\frac{1}{(2k+1)(e^{\alpha(2k+1)} - 1)}$$

$$= -\frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}\frac{\sinh(\beta(2k+1)/4)}{(2k+1)\cosh(\beta(2k+1)/2)} + \frac{1}{2}\log(1 + \sqrt{2}),$$

and

$$\alpha^{-M} \sum_{k \equiv 0,3 \,(\mathrm{mod}\, 4)} \frac{(2k+1)^{-2M-1}}{e^{\alpha(2k+1)} - 1} - \alpha^{-M} \sum_{k \equiv 1,2 \,(\mathrm{mod}\, 4)} \frac{(2k+1)^{-2M-1}}{e^{\alpha(2k+1)} - 1}$$

$$= -\frac{(-\beta)^{-M}}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{\sinh(\beta(2k+1)/4)}{(2k+1)^{2M+1} \cosh(\beta(2k+1)/2)}$$

$$+ \frac{2^{2M+1}}{\sqrt{2}} \sum_{k=0}^{M+1} \mathcal{B}_{M,2k}(3/8, 1/2) \alpha^{M-k+1} (-\beta)^k$$

$$+ \frac{2^{2M+1}}{\sqrt{2}} \sum_{k=0}^{M+1} \mathcal{B}_{M,2k}(1/8, 0) \alpha^{M-k+1} (-\beta)^k$$

$$- \frac{1}{\sqrt{2}} \sum_{k=0}^{M+1} \mathcal{B}_{M,2k}(1/4, 0) \alpha^{M-k+1} (-\beta)^k$$

$$+ 2^{-2M-2} \alpha^{-M} (\mathcal{Z}_+(2M, 1/8) - \mathcal{Z}_+(2M, 3/8)),$$

where

$$\mathcal{B}_{n,k}(x, y) := \frac{B_k(x) B_{2n+2-k}(y)}{k!(2n+2-k)!} \quad \text{and} \quad \mathcal{B}_{n,k}(x) := \mathcal{B}_{n,k}(x, x).$$

4.3. $N \,|\, r_1$ and $N \,|\, (r_1 + r_2)$. Suppose that r is even in Theorem 3.29. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}(e^{(\alpha-i\pi)k} - 1)} = \sum_{k=1}^{\infty} \frac{2^{-2n-1}}{k^{2n+1}(e^{2\alpha k} - 1)}$$

$$- \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}(e^{\alpha(2k+1)} + 1)},$$

using Theorem 3.72 in [8], we find that

$$(4.11) \quad \alpha^{-n} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}(e^{\alpha(2k+1)} + 1)}$$

$$= (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}(e^{\beta(2k+1)} + 1)}$$

$$+ 2^{2n} \sum_{j=1}^{2} \sum_{k=0}^{2n+2} \frac{B_k(j/2) \bar{B}_{2n+2-k}(j/2)}{k!(2n+2-k)!} (-\pi i)^k \alpha^{n-k+1}$$

$$- \frac{1}{2} \sum_{k=0}^{n+1} \frac{B_{2k} B_{2n+2-2k}}{(2k)!(2n+2-2k)!} \alpha^{n-k+1} (-\beta)^k + 2^{-2n-2} K(n) - \frac{1}{2} J_4'(n),$$

where
$$K(n) := \begin{cases} ((-\beta)^{-n} - \alpha^{-n})\frac{B_{-2n}}{2n} & \text{if } n < 0, \\ \frac{1}{2}\log\alpha - \frac{1}{2}\log\beta & \text{if } n = 0, \\ ((-\beta)^{-n} - \alpha^{-n})\zeta(1+2n) & \text{if } n > 0, \end{cases}$$

and $J_4'(n)$ is defined in Theorem 3.29.

THEOREM 4.30. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+1}(e^{\alpha(2k+1)}+1)}$$

$$= (-\beta)^{-n}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+1}(e^{\beta(2k+1)}+1)}$$

$$+ \sum_{k=0}^{n+1}\frac{(2^{2n+1}-2^{2k-1})(1-2^{-2k})B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!}\alpha^{n-k+1}(-\beta)^k + I_1(n),$$

where
$$I_1(n) := \begin{cases} (2^{-2n-2}-2^{-1})((-\beta)^{-n}-\alpha^{-n})\frac{B_{-2n}}{2n} & \text{if } n < 0, \\ (2^{-2}-2^{-1})(\frac{1}{2}\log\alpha - \frac{1}{2}\log\beta) & \text{if } n = 0, \\ (2^{-2n-2}-2^{-1})((-\beta)^{-n}-\alpha^{-n})\zeta(2n+1) & \text{if } n > 0. \end{cases}$$

Proof. In (4.11), the imaginary parts of the sums of Bernoulli polynomials can be expressed as

$$-\pi i\sum_{j=1}^{2}\sum_{k=0}^{n}\frac{B_{2k+1}(j/2)\bar{B}_{2n+1-2k}(j/2)}{(2k+1)!(2n+1-2k)!}\alpha^{n-k}(-\beta)^k = \begin{cases} \pi i/4 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Use the definition of $J_4'(n)$ in Theorem 3.29 and apply the formula (see [1])
$$B_n(1/2) = -(1-2^{1-n})B_n, \quad n \geq 0.$$

Then the desired results follow. ∎

If $n \neq 0$, then Theorem 4.30 has been first stated by Malukar [9]. Berndt [5] has also proved Theorem 4.30 for $n \neq 0$. In the case of $n < 0$, Theorem 4.30 has been proved by Rao and Ayyar [12] and Nanjundiah [10].

COROLLARY 4.31.
$$\alpha\sum_{k=0}^{\infty}\frac{2k+1}{e^{\alpha(2k+1)}+1} = -\beta\sum_{k=0}^{\infty}\frac{2k+1}{e^{\beta(2k+1)}+1} + \frac{1}{24}(\beta+\alpha).$$

Proof. Let $n = -1$ in Theorem 4.30. Use $\zeta(-1) = -1/12$. ∎

COROLLARY 4.32.
$$\sum_{k=0}^{\infty}\frac{2k+1}{e^{\pi(2k+1)}+1} = \frac{1}{24}.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 4.31. ∎

COROLLARY 4.33. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{(2k+1)^{4M+1}}{e^{\pi(2k+1)}+1} = \frac{B_{4M+2}}{8M+4}(2^{4M+1}-1).$$

Proof. Put $n = -2M-1$ for any positive integer M and let $\alpha = \beta = \pi$ in Theorem 4.30. ∎

COROLLARY 4.34.

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(e^{\alpha(2k+1)}+1)} + \frac{1}{8}\log\alpha = \sum_{k=0}^{\infty} \frac{1}{(2k+1)(e^{\beta(2k+1)}+1)} + \frac{1}{8}\log\beta.$$

Proof. Put $n = 0$ in Theorem 4.30. ∎

COROLLARY 4.35. *For any positive integer M,*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{4M-1}(e^{\pi(2k+1)}+1)}$$
$$= \pi^{4M-1} \sum_{k=0}^{2M} \frac{(2^{4M-2}-2^{2k-2})(1-2^{-2k})B_{2k}B_{4M-2k}}{(2k)!(4M-2k)!}(-1)^k$$
$$- (2^{-4M}-2^{-1})\zeta(4M-1).$$

Proof. Put $n = 2M-1$ for any positive integer and let $\alpha = \beta = \pi$ in Theorem 4.30. ∎

Corollary 4.33 has already been found by Glaisher [6], Sandham [13] and Berndt [5]. Corollary 4.34 has also been established by Nanjundiah [10] and Berndt [5]. Berndt has also proved Corollary 4.35.

REMARK 4.36. The conclusion of Theorem 3.29 for r odd coincides with Theorem 4.30. Proposition 4.27 is the case $c' = 4$ and r odd of Theorem 3.21.

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