ACTA ARITHMETICA
103.4 (2002)

On a theorem of Erdős and Fuchs

by

GÁBOR HORVÁTH (Budapest)

Let $k \geq 2$ be a fixed integer, let $A^{(j)} = \{a_1^{(j)}, a_2^{(j)}, \ldots\}$ $(j = 1, \ldots, k)$ be nondecreasing infinite sequences of nonnegative integers, and let

$$r_k(n) = |\{(i_1, \ldots, i_k) : a_{i_1}^{(1)} + a_{i_2}^{(2)} + \ldots + a_{i_k}^{(k)} \leq n,\ a_{i_j}^{(j)} \in A^{(j)}\ (j = 1, \ldots, k)\}|,$$

and $c > 0$.

Erdős and Fuchs [1] showed that if $k = 2$ and $A^{(1)} \equiv A^{(2)}$, then

$$r_2(n) = cn + o(n^{1/4}(\log n)^{-1/2}) \tag{1}$$

cannot hold.

Sárközy [3] extended this theorem to two sequences which are "near" in a certain sense. He proved that if

$$a_i^{(2)} - a_i^{(1)} = o((a_i^{(1)})^{1/2}(\log a_i^{(1)})^{-1}), \tag{2}$$

then (1) cannot hold. (A simple example shows that a condition of type (2) is necessary: Let $A^{(j)} = \{\sum_l \varepsilon_l 2^{lk+j} : \varepsilon_l = 0 \text{ or } 1\}$ for $j = 1, \ldots, k$. Then $r_k(n) = n + 1$, thus $r_k(n) - n = O(1)$.)

In [2] I extended this result to the case $k > 2$ and, among other things, I showed that if we assume

$$a_i^{(j)} - a_i^{(l)} = o((\min(a_i^{(j)}, a_i^{(l)}))^{1/2}(\log \min(a_i^{(j)}, a_i^{(l)}))^{-1-1/(k-1)}) \tag{3}$$

for *all* $1 \leq j < l \leq k$, then

$$r_k(n) = cn + o(n^{1/4}(\log n)^{-1/2-3/(2(k-1))}) \tag{4}$$

cannot hold. In this paper I will show that, at the price of replacing the error term in (4) by a slightly weaker one, condition (3) can be replaced by a much weaker assumption. Namely, perhaps somewhat unexpectedly, it suffices to assume that *two* of the given sequences $A^{(j)}$ are "near":

2000 *Mathematics Subject Classification*: Primary 11B34.

Theorem. *If* $k \ge 2$, $a_i^{(1)} - a_i^{(2)} = o((a_i^{(1)})^{1/2}(\log a_i^{(1)})^{-k/2})$ *and*

$$\sum_{a_i^{(j)} \le N} 1 \ll \sum_{a_i^{(1)} \le N} 1 \ll \sum_{a_i^{(j)} \le N} 1 \quad \text{for } j = 3, \dots, k,$$

then

$$r_k(n) = cn + o(n^{1/4}(\log n)^{1-3k/4}) \tag{5}$$

cannot hold.

Proof. Suppose that (5) holds. Let $v(n) = r_k(n) - cn$ and $F_j(z) = \sum_{i=1}^{\infty} z^{a_i^{(j)}}$ $(j = 1, \dots, k)$. Then for $|z| < 1$,

$$\frac{1}{1-z} F_1(z) \dots F_k(z) = \sum_{n=0}^{\infty} r_k(n) z^n = c \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} v(n) z^n$$
$$= c \frac{z}{(1-z)^2} + \sum_{n=0}^{\infty} v(n) z^n,$$

hence

$$F_1(z) \dots F_k(z) = \frac{cz}{1-z} + (1-z) \sum_{n=0}^{\infty} v(n) z^n. \tag{6}$$

Let ε be a fixed small positive number, N a large positive integer, $m(n) = [\varepsilon n^{1/2}(\log n)^{-k/2}]$, $m = m(N)$, $z = re(\alpha)$, where $r = 1 - 1/N$ and $e(\alpha) = e^{2\pi i \alpha}$ (for real α). Let

$$J = \int_0^1 |F_1(z) \dots F_k(z)| \left| \frac{1-z^m}{1-z} \right|^2 d\alpha,$$
$$J_1 = c \int_0^1 |1-z|^{-1} \left| \frac{1-z^m}{1-z} \right|^2 d\alpha, \tag{7}$$
$$J_2 = \int_0^1 \left| (1-z) \sum_{n=0}^{\infty} v(n) z^n \right| \left| \frac{1-z^m}{1-z} \right|^2 d\alpha.$$

Then, by (6),

$$J \le J_1 + J_2. \tag{8}$$

We first estimate J. By (7),

$$J \ge \left| \int_0^1 F_1(z) \overline{F_2(z)} F_3(z) \dots F_k(z) \left| \frac{1-z^m}{1-z} \right|^2 d\alpha \right|$$
$$= \left| \int_0^1 \Big(F_1(z) \overline{F_2(z)} \sum_{t=0}^{m-1} r^t e(-t\alpha) \Big) \Big(F_3(z) \dots F_k(z) \sum_{t=0}^{m-1} r^t e(t\alpha) \Big) d\alpha \right|.$$

Let

$$\sum_{b=-\infty}^{\infty} g_b e(b\alpha) = F_1(z)\,\overline{F_2(z)} \sum_{t=0}^{m-1} r^t e(-t\alpha),$$

$$\sum_{i=0}^{\infty} h_i e(i\alpha) = F_3(z)\ldots F_k(z) \sum_{t=0}^{m-1} r^t e(t\alpha)$$

(so that all the coefficients g_b, h_i are nonnegative). Then

$$J \geq \Big|\int_0^1 \sum_{b=-\infty}^{\infty} g_b e(b\alpha) \sum_{i=0}^{\infty} h_i e(i\alpha)\, d\alpha\Big| = \sum_{b+i=0} g_b h_i \geq \sum_{m/4\leq i\leq m/2} g_{-i} h_i. \tag{9}$$

If $m/4 \leq i \leq m/2$, then

$$\begin{aligned} h_i &= \sum_{\substack{a_{i_3}^{(3)}+\ldots+a_{i_k}^{(k)}+t=i\\ 0\leq t\leq m-1}} r^{a_{i_3}^{(3)}+\ldots+a_{i_k}^{(k)}+t} \\ &\geq r^N \sum_{\substack{a_{i_3}^{(3)}+\ldots+a_{i_k}^{(k)}+t=i\\ 0\leq t\leq m/2}} 1 \gg \sum_{a_{i_3}^{(3)}+\ldots+a_{i_k}^{(k)}\leq m/4} 1 \end{aligned}$$

since $r^N = (1-1/N)^N \to 1/e$.

For $k > 2$, since

$$\sum_{a_{i_j}^{(j)}\leq m/(4(k-2))} 1 \gg \sum_{a_{i_1}^{(1)}\leq m/(4(k-2))} 1 \qquad (j = 3, \ldots, k),$$

it follows that for $m/4 \leq i \leq m/2$,

$$\begin{aligned} h_i &\gg \sum_{a_{i_3}^{(3)}+\ldots+a_{i_k}^{(k)}\leq m/4} 1 \geq \Big(\sum_{a_{i_3}^{(3)}\leq m/(4(k-2))} 1\Big)\ldots\Big(\sum_{a_{i_k}^{(k)}\leq m/(4(k-2))} 1\Big) \\ &\gg \Big(\sum_{a_{i_1}^{(1)}\leq m/(4(k-2))} 1\Big)^{k-2}, \end{aligned}$$

and thus, by (9),

$$\begin{aligned} J &\gg \sum_{m/4\leq i\leq m/2} g_{-i}\Big(\sum_{a_{i_1}^{(1)}\leq m/(4(k-2))} 1\Big)^{k-2} \\ &= \Big(\sum_{a_{i_1}^{(1)}\leq m/(4(k-2))} 1\Big)^{k-2} \sum_{m/4\leq i\leq m/2} g_{-i}. \end{aligned} \tag{10}$$

Since $m = m(N) = [\varepsilon N^{1/2}(\log N)^{-k/2}]$ is eventually nondecreasing, and

$a_{i_1}^{(1)} - a_{i_1}^{(2)} = o((a_{i_1}^{(1)})^{1/2}(\log a_{i_1}^{(1)})^{-k/2})$, it follows that if $a_{i_1}^{(1)} \le N$, then $|a_{i_1}^{(1)} - a_{i_1}^{(2)}| \le m(a_{i_1}^{(1)})/4 \le m(N)/4 = m/4$ for all sufficiently large $a_{i_1}^{(1)}$. Hence, for all sufficiently large N, if $a_{i_1}^{(1)} \le N$, then $|a_{i_1}^{(1)} - a_{i_1}^{(2)}| \le m/4$. If $a_{i_1}^{(1)} \le N - m$, then $a_{i_1}^{(2)} \le a_{i_1}^{(1)} + |a_{i_1}^{(2)} - a_{i_1}^{(1)}| \le N - m + m/4 < N$ and

$$\begin{aligned}0 = m/4 - m/4 \le i - |a_{i_1}^{(2)} - a_{i_1}^{(1)}| \le i + a_{i_1}^{(1)} - a_{i_1}^{(2)} \le i + |a_{i_1}^{(2)} - a_{i_1}^{(1)}|\\ \le m/2 + m/4 < m - 1,\end{aligned}$$

thus

$$\begin{aligned}(11)\qquad g_{-i} &= \sum_{\substack{a_{i_1}^{(1)} - a_{i_2}^{(2)} - t = -i\\ 0 \le t \le m-1}} r^{a_{i_1}^{(1)} + a_{i_2}^{(2)} + t}\\ &\ge \sum_{\substack{a_{i_1}^{(1)} - a_{i_1}^{(2)} - t = -i\\ 0 \le t \le m-1\\ a_{i_1}^{(1)}, a_{i_1}^{(2)} \le N}} r^{a_{i_1}^{(1)} + a_{i_1}^{(2)} + t} \ge r^{3N} \sum_{a_{i_1}^{(1)} \le N-m} 1 \gg \sum_{a_{i_1}^{(1)} \le N-m} 1.\end{aligned}$$

Hence, by (10) and (11),

$$(12)\qquad J \gg m\Big(\sum_{a_{i_1}^{(1)} \le m/(4(k-2))} 1\Big)^{k-2} \sum_{a_{i_1}^{(1)} \le N-m} 1.$$

Since $a_i^{(2)} - a_i^{(1)} = a_i^{(1)}(a_i^{(2)}/a_i^{(1)} - 1)$ and $a_i^{(2)} - a_i^{(1)} = o(m(a_i^{(1)}))$, so that $a_i^{(2)}/a_i^{(1)} = 1 + o(m(a_i^{(1)})/a_i^{(1)}) = 1 + o(1)$, it follows that

$$\begin{aligned}&a_i^{(2)} - a_i^{(1)}\\ &\quad = o(m(a_i^{(1)})) = o((a_i^{(1)})^{1/2}(\log a_i^{(1)})^{-k/2})\\ &\quad = o((a_i^{(2)})^{1/2}(\log a_i^{(2)})^{-k/2})(a_i^{(1)}(a_i^{(2)})^{-1})^{1/2}((\log a_i^{(2)})(\log a_i^{(1)})^{-1})^{k/2}\\ &\quad = o((a_i^{(2)})^{1/2}(\log a_i^{(2)})^{-k/2}) = o(m(a_i^{(2)})).\end{aligned}$$

As m is eventually nondecreasing, it follows that if $a_i^{(2)} \le N$, then $|a_i^{(1)} - a_i^{(2)}| \le m(a_i^{(2)})/4 \le m(N)/4 = m/4$ for all sufficiently large $a_i^{(2)}$. Hence, for all sufficiently large N, if $a_i^{(2)} \le N$, then $|a_i^{(1)} - a_i^{(2)}| \le m/4$. Furthermore,

$$\sum_{a_{i_j}^{(j)} \le N - 5m/4} 1 \ll \sum_{a_{i_1}^{(1)} \le N - 5m/4} 1 \quad \text{for } j = 3, \ldots, k,$$

and $r_k(N) \sim cN$, thus

$$\begin{aligned} N &\ll r_k(N/2) \le r_k(N-[N/2]) \le r_k(N-5m/4) \\ &= \sum_{a_{i_1}^{(1)}+\ldots+a_{i_k}^{(k)} \le N-5m/4} 1 \le \prod_{j=1}^{k} \sum_{a_{i_j}^{(j)} \le N-5m/4} 1 \\ &\ll \Big(\prod_{\substack{j=1\\ j\ne 2}}^{k} \sum_{a_{i_1}^{(1)} \le N-5m/4} 1\Big)\Big(\sum_{a_{i_2}^{(1)} \le N-m} 1\Big) \le \Big(\sum_{a_{i_1}^{(1)} \le N-m} 1\Big)^k, \end{aligned}$$

hence

$$\sum_{a_{i_1}^{(1)} \le N-m} 1 \gg N^{1/k}. \tag{13}$$

By a similar argument for $k > 2$ and for all sufficiently large N, if $a_i^{(2)} \le N$, then $|a_i^{(1)} - a_i^{(2)}| \le m/(8(k-2))$. Thus

$$\begin{aligned} m &\ll r_k\Big(\frac{m}{8(k-2)}\Big) \le \prod_{j=1}^{k} \sum_{a_{i_j}^{(j)} \le m/(8(k-2))} 1 \\ &\ll \Big(\prod_{\substack{j=1\\ j\ne 2}}^{k} \sum_{a_{i_1}^{(1)} \le m/(8(k-2))} 1\Big)\Big(\sum_{a_{i_2}^{(1)} \le m/(4(k-2))} 1\Big) \le \Big(\sum_{a_{i_1}^{(1)} \le m/(4(k-2))} 1\Big)^k, \end{aligned}$$

hence

$$\sum_{a_{i_1}^{(1)} \le m/(4(k-2))} 1 \gg m^{1/k}. \tag{14}$$

By (12)–(14),

$$J \gg m m^{(k-2)/k} N^{1/k} = m^{2-2/k} N^{1/k}. \tag{15}$$

We now estimate J_1 and J_2. Since

$$\begin{aligned} |1-z|^2 &= (1-r\cos 2\pi\alpha)^2 + (r\sin 2\pi\alpha)^2 = (1-r)^2 + 2r(1-\cos 2\pi\alpha) \\ &= \frac{1}{N^2} + 4r\sin^2 \pi\alpha \end{aligned}$$

and

$$|(2/\pi)\pi\alpha| \le |\sin \pi\alpha| \quad \text{ for } |\alpha| \le 1/2,$$

it follows that $\max(1/N^2, \alpha^2) \ll |1-z|^2$, thus $\max(1/N, \alpha) \ll |1-z|$. Hence

$$J_1 = c\int_0^1 |1-z|^{-1}\left|\frac{1-z^m}{1-z}\right|^2 d\alpha \ll m^2\int_0^1 |1-z|^{-1}\,d\alpha \tag{16}$$
$$\ll m^2\Big(\int_0^{1/N} |1-z|^{-1}\,d\alpha + \int_{1/N}^{1/2} |1-z|^{-1}\,d\alpha\Big)$$
$$\ll m^2\bigg(\frac{1}{N}N + \int_{1/N}^{1/2}\frac{1}{\alpha}\,d\alpha\bigg) \le m^2(1+\log N)$$
$$\ll m^2\log N.$$

By Cauchy's inequality and Parseval's formula,

$$J_2 = \int_0^1\Big|(1-z)\sum_{n=0}^{\infty} v(n)z^n\Big|\left|\frac{1-z^m}{1-z}\right|^2 d\alpha \tag{17}$$
$$\le 2\int_0^1\Big|\sum_{n=0}^{\infty} v(n)z^n\Big|\left|\frac{1-z^m}{1-z}\right| d\alpha$$
$$\ll \Big(\int_0^1\Big|\sum_{n=0}^{\infty} v(n)z^n\Big|^2 d\alpha\Big)^{1/2}\Big(\int_0^1\left|\frac{1-z^m}{1-z}\right|^2 d\alpha\Big)^{1/2}$$
$$\le \Big(\sum_{n=0}^{\infty}|v(n)|^2 r^{2n}\Big)^{1/2} m^{1/2}.$$

By our assumption, $v(n) = o(n^{1/4}(\log n)^{1-3k/4})$, therefore for every $\eta > 0$, there exists a natural number K (≥ 2) such that for all $n \ge K$, $|v(n)| \le \eta n^{1/4}(\log n)^{1-3k/4}$ and $n^{1/4}(\log n)^{1-3k/4}$ is nondecreasing. Then for all $N \ge K$,

$$\sum_{n=0}^{\infty}|v(n)|^2 r^{2n} \le \sum_{n=0}^{K-1}|v(n)|^2 + \eta^2\sum_{n=K}^{\infty} n^{1/2}(\log n)^{2-3k/2} r^{2n}$$
$$\le \sum_{n=0}^{K-1}|v(n)|^2 + \eta^2 N N^{1/2}(\log N)^{2-3k/2}$$
$$+\,\eta^2\sum_{j=0}^{\infty}\sum_{n=2^jN+1}^{2^{j+1}N} n^{1/2}(\log n)^{2-3k/2} r^n.$$

Since

$$\sum_{j=0}^{\infty}\sum_{n=2^jN+1}^{2^{j+1}N} n^{1/2}(\log n)^{2-3k/2}r^n$$
$$\le \sum_{j=0}^{\infty} 2^jN(2^{j+1}N)^{1/2}(\log(2^{j+1}N))^{2-3k/2}r^{2^jN}$$
$$\le N^{3/2}(\log N)^{2-3k/2}\sum_{j=0}^{\infty}2^{j+j/2+1/2}e^{-2^j} = C_0N^{3/2}(\log N)^{2-3k/2},$$

it follows that

$$\sum_{n=0}^{\infty}|v(n)|^2r^{2n} \le \sum_{n=0}^{K-1}|v(n)|^2 + \eta^2N^{3/2}(\log N)^{2-3k/2}(1+C_0)$$
$$< \eta N^{3/2}(\log N)^{2-3k/2}$$

for $\eta < (2(1+C_0))^{-1}$ and for $N > N_0(\eta)$. Thus

$$\sum_{n=0}^{\infty}|v(n)|^2r^{2n} = o(N^{3/2}(\log N)^{2-3k/2}). \tag{18}$$

By (17) and (18),

$$J_2 = o(N^{3/4}(\log N)^{1-3k/4}m^{1/2}). \tag{19}$$

By (8), (15), (16), and (19),

$$m^{2-2/k}N^{1/k} \ll m^2\log N + o(m^{1/2}N^{3/4}(\log N)^{1-3k/4}). \tag{20}$$

Since $m = [\varepsilon N^{1/2}(\log N)^{-k/2}]$, (20) yields

$$\left(\frac{\varepsilon}{2}N^{1/2}(\log N)^{-k/2}\right)^{2-2/k}N^{1/k}$$
$$\ll \varepsilon^2N(\log N)^{-k}\log N + o(\varepsilon^{1/2}N^{1/4}(\log N)^{-k/4}N^{3/4}(\log N)^{1-3k/4})$$

for all sufficiently large N, hence $\varepsilon^{3/2-2/k} \ll \varepsilon^{3/2} + o(1)$. Thus $\varepsilon^{-2/k} \ll 1$; but this cannot hold for sufficiently small ε. This completes the proof of the theorem.

Acknowledgements. The author would like to thank Professors Imre Z. Ruzsa and András Sárközy for helpful suggestions.

References

[1] P. Erdős and W. H. J. Fuchs, *On a problem of additive number theory*, J. London Math. Soc. 31 (1956), 67–73.

[2] G. Horváth, *On a generalization of theorem of Erdős and Fuchs*, to appear.

[3] A. Sárközy, *On a theorem of Erdős and Fuchs*, Acta Arith. 37 (1980), 333–338.

Department of Algebra and Number Theory
Eötvös Loránd University
Pázmány Péter sétány 1/c
H-1117, Budapest, Hungary
E-mail: horvathg@cs.elte.hu

Received on 16.3.2000 (3778)