On a theorem of Erdős and Fuchs

by

GÁBOR HORVÁTH (Budapest)

Let \( k \geq 2 \) be a fixed integer, let \( A^{(j)} = \{a^{(j)}_1, a^{(j)}_2, \ldots \} \) \((j = 1, \ldots, k)\) be nondecreasing infinite sequences of nonnegative integers, and let
\[
 r_k(n) = |\{(i_1, \ldots, i_k) : a^{(1)}_{i_1} + a^{(2)}_{i_2} + \ldots + a^{(k)}_{i_k} \leq n, \ a^{(j)}_{i_j} \in A^{(j)} \ (j = 1, \ldots, k)\}|,
\]
and \( c > 0 \).

Erdős and Fuchs [1] showed that if \( k = 2 \) and \( A^{(1)} \equiv A^{(2)} \), then
\[
 (1) \quad r_2(n) = cn + o(n^{1/4}(\log n)^{-1/2})
\]
cannot hold.

Sárközy [3] extended this theorem to two sequences which are “near” in a certain sense. He proved that if
\[
 (2) \quad a^{(2)}_i - a^{(1)}_i = o((a^{(1)}_i)^{1/2}(\log a^{(1)}_i)^{-1}),
\]
then (1) cannot hold. (A simple example shows that a condition of type (2) is necessary: Let \( A^{(j)} = \{\sum_i \varepsilon_i 2^{i+j} : \varepsilon_i = 0 \text{ or } 1\} \) for \( j = 1, \ldots, k \). Then \( r_k(n) = n + 1 \), thus \( r_k(n) - n = O(1) \).)

In [2] I extended this result to the case \( k > 2 \) and, among other things, I showed that if we assume
\[
 (3) \quad a^{(j)}_i - a^{(l)}_i = o((\min(a^{(j)}_i, a^{(l)}_i))^{1/2}(\log \min(a^{(j)}_i, a^{(l)}_i))^{-1/(k-1)})
\]
for all \( 1 \leq j < l \leq k \), then
\[
 (4) \quad r_k(n) = cn + o(n^{1/4}(\log n)^{-1/2-3/(2(k-1))})
\]
cannot hold. In this paper I will show that, at the price of replacing the error term in (4) by a slightly weaker one, condition (3) can be replaced by a much weaker assumption. Namely, perhaps somewhat unexpectedly, it suffices to assume that \( \text{two} \) of the given sequences \( A^{(j)} \) are “near”:

2000 Mathematics Subject Classification: Primary 11B34.
Theorem. If \( k \geq 2 \), \( a_i^{(1)} - a_i^{(2)} = o((a_i^{(1)})^{1/2}(\log a_i^{(1)})^{-k/2}) \) and

\[
\sum_{a_i^{(j)} \leq N} 1 \ll \sum_{a_i^{(1)} \leq N} 1 \ll \sum_{a_i^{(j)} \leq N} 1 \quad \text{for } j = 3, \ldots, k,
\]

then

\[
r_k(n) = cn + o(n^{1/4}(\log n)^{1-3k/4})
\]

cannot hold.

Proof. Suppose that (5) holds. Let \( v(n) = r_k(n) - cn \) and \( F_j(z) = \sum_{i=1}^{\infty} z^{a_i^{(j)}} \) (\( j = 1, \ldots, k \)). Then for \( |z| < 1 \),

\[
\frac{1}{1-z} F_1(z) \ldots F_k(z) = \sum_{n=0}^{\infty} r_k(n) z^n = c \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} v(n) z^n
\]

\[
= c \frac{z}{(1-z)^2} + \sum_{n=0}^{\infty} v(n) z^n,
\]

hence

\[
F_1(z) \ldots F_k(z) = \frac{cz}{1-z} + (1-z) \sum_{n=0}^{\infty} v(n) z^n.
\]

Let \( \varepsilon \) be a fixed small positive number, \( N \) a large positive integer, \( m(n) = \lfloor \varepsilon n^{1/2}(\log n)^{-k/2} \rfloor \), \( m = m(N) \), \( z = re(\alpha) \), where \( r = 1 - 1/N \) and \( e(\alpha) = e^{2\pi i \alpha} \) (for real \( \alpha \)). Let

\[
J = \int_0^1 |F_1(z) \ldots F_k(z)| \left| \frac{1-z^m}{1-z} \right|^2 d\alpha,
\]

(7)

\[
J_1 = \int_0^1 |1-z|^{-1} \left| \frac{1-z^m}{1-z} \right|^2 d\alpha,
\]

\[
J_2 = \int_0^1 (1-z) \sum_{n=0}^{\infty} v(n) z^n \left| \frac{1-z^m}{1-z} \right|^2 d\alpha.
\]

Then, by (6),

\[
J \leq J_1 + J_2.
\]

We first estimate \( J \). By (7),

\[
J \geq \int_0^1 \left| F_1(z) F_2(z) F_3(z) \ldots F_k(z) \left| \frac{1-z^m}{1-z} \right|^2 \right| d\alpha
\]

\[
= \int_0^1 \left( F_1(z) F_2(z) \sum_{t=0}^{m-1} r^t e(-t\alpha) \right) \left( F_3(z) \ldots F_k(z) \sum_{t=0}^{m-1} r^t e(t\alpha) \right) d\alpha.
\]
Let
\[ \sum_{b=-\infty}^{\infty} g_b e(b\alpha) = F_1(z) \frac{F_2(z)}{F_2(z)} \sum_{t=0}^{m-1} r^t e(-t\alpha), \]
\[ \sum_{i=0}^{\infty} h_i e(i\alpha) = F_3(z) \ldots F_k(z) \sum_{t=0}^{m-1} r^t e(t\alpha) \]
(so that all the coefficients \( g_b, h_i \) are nonnegative). Then

\[ J \geq \left| \int_{b=-\infty}^{1} \sum_{i=0}^{\infty} g_b e(b\alpha) \sum_{i=0}^{\infty} h_i e(i\alpha) \, d\alpha \right| = \sum_{b+i=0}^{m/4 \leq i \leq m/2} g_b h_i \geq \sum_{m/4 \leq i \leq m/2} g_{-i} h_i. \]  

If \( m/4 \leq i \leq m/2 \), then
\[ h_i = \sum_{0 \leq t \leq m-1} r^{a_{i_3}^{(3)} + \ldots + a_{i_k}^{(k)} + t} \geq r^N \sum_{0 \leq t \leq m/2} 1 \Rightarrow \sum_{a_{i_3}^{(3)} + \ldots + a_{i_k}^{(k)} \leq m/4} 1 \]

since \( r^N = (1 - 1/N)^N \to 1/e. \)

For \( k > 2 \), since
\[ \sum_{a_{i_j}^{(j)} \leq m/(4(k-2))} 1 \Rightarrow \sum_{a_{i_1}^{(1)} \leq m/(4(k-2))} 1 \quad (j = 3, \ldots, k), \]

it follows that for \( m/4 \leq i \leq m/2 \),
\[ h_i \Rightarrow \sum_{a_{i_3}^{(3)} + \ldots + a_{i_k}^{(k)} \leq m/4} 1 \geq \left( \sum_{a_{i_3}^{(3)} \leq m/(4(k-2))} 1 \right) \ldots \left( \sum_{a_{i_k}^{(k)} \leq m/(4(k-2))} 1 \right) \]
\[ \Rightarrow \left( \sum_{a_{i_1}^{(1)} \leq m/(4(k-2))} 1 \right)^k \]

and thus, by (9),

\[ J \geq \sum_{m/4 \leq i \leq m/2} g_{-i} \left( \sum_{a_{i_1}^{(1)} \leq m/(4(k-2))} 1 \right)^{k-2} \]
\[ = \left( \sum_{a_{i_1}^{(1)} \leq m/(4(k-2))} 1 \right)^{k-2} \sum_{m/4 \leq i \leq m/2} g_{-i}. \]

Since \( m = m(N) = [\varepsilon N^{1/2}(\log N)^{-k/2}] \) is eventually nondecreasing, and
\[ a_{i_1}^{(1)} - a_{i_1}^{(2)} = o((a_{i_1}^{(1)})^{1/2}(\log a_{i_1}^{(1)})^{-k/2}), \]

it follows that if \( a_{i_1}^{(1)} \leq N \), then

\[ |a_{i_1}^{(1)} - a_{i_1}^{(2)}| \leq m(a_{i_1}^{(1)})/4 \leq m(N)/4 = m/4 \]

for all sufficiently large \( a_{i_1}^{(1)} \).

Hence, for all sufficiently large \( N \), if \( a_{i_1}^{(1)} \leq N \), then \( |a_{i_1}^{(1)} - a_{i_1}^{(2)}| \leq m/4 \). If

\[ a_{i_1}^{(1)} \leq N - m, \]

then \( a_{i_1}^{(2)} \leq a_{i_1}^{(1)} + |a_{i_1}^{(2)} - a_{i_1}^{(1)}| \leq N - m + m/4 < N \) and

\[
0 = m/4 - m/4 \leq i - |a_{i_1}^{(2)} - a_{i_1}^{(1)}| \leq i + a_{i_1}^{(1)} - a_{i_1}^{(2)} \leq i + |a_{i_1}^{(2)} - a_{i_1}^{(1)}| \\
\leq m/2 + m/4 < m - 1,
\]

thus

\[
g_{-i} = \sum_{0 \leq t \leq m-1} r^{a_{i_1}^{(1)} + a_{i_1}^{(2)} + t} \\
\geq \sum_{0 \leq t \leq m-1} r^{a_{i_1}^{(1)} + a_{i_1}^{(2)} + t} \geq r^{3N} \sum_{a_{i_1}^{(1)} \leq N-m} 1 \gg \sum_{a_{i_1}^{(1)} \leq N-m} 1.
\]

Hence, by (10) and (11),

\[
J \gg m\left( \sum_{a_{i_1}^{(1)} \leq m/(4(k-2))} 1 \right)^{k-2} \sum_{a_{i_1}^{(1)} \leq N-m} 1.
\]

Since \( a_{i}^{(2)} - a_{i}^{(1)} = a_{i}^{(1)} (a_{i}^{(2)}/a_{i}^{(1)} - 1) \) and \( a_{i}^{(2)} - a_{i}^{(1)} = o(m(a_{i}^{(1)})) \), so that

\[ a_{i}^{(2)}/a_{i}^{(1)} = 1 + o(m(a_{i}^{(1)})/a_{i}^{(1)}) = 1 + o(1), \]

it follows that

\[
a_{i}^{(2)} - a_{i}^{(1)} \\
= o(m(a_{i}^{(1)})) = o((a_{i}^{(1)})^{1/2}(\log a_{i}^{(1)})^{-k/2}) \\
= o((a_{i}^{(2)})^{1/2}(\log a_{i}^{(2)})^{-k/2})(a_{i}^{(1)}/a_{i}^{(1)} - 1)^{1/2}((\log a_{i}^{(2)})(\log a_{i}^{(1)})^{-1})^{k/2} \\
= o((a_{i}^{(2)})^{1/2}(\log a_{i}^{(2)})^{-k/2}) = o(m(a_{i}^{(2)})).
\]

As \( m \) is eventually nondecreasing, it follows that if \( a_{i}^{(2)} \leq N \), then \( |a_{i}^{(1)} - a_{i}^{(2)}| \leq m(a_{i}^{(2)})/4 \leq m(N)/4 = m/4 \) for all sufficiently large \( a_{i}^{(2)} \). Hence, for all sufficiently large \( N \), if \( a_{i}^{(2)} \leq N \), then \( |a_{i}^{(1)} - a_{i}^{(2)}| \leq m/4 \). Furthermore,

\[
\sum_{a_{i}^{(j)} \leq N-5m/4} 1 \ll \sum_{a_{i}^{(1)} \leq N-5m/4} 1 \quad \text{for } j = 3, \ldots, k,
\]
and $r_k(N) \sim cN$, thus

$$N \ll r_k(N/2) \leq r_k(N - \lceil N/2 \rceil) \leq r_k(N - 5m/4)$$

$$= \sum_{a_{i_1}^{(1)} + \ldots + a_{i_k}^{(k)} \leq N - 5m/4} 1 \leq \prod_{j=1}^k \sum_{a_{i_j}^{(j)} \leq N - 5m/4} 1$$

$$\ll \left( \prod_{j=1}^k \sum_{j \neq 2} a_{i_1}^{(1)} \leq N - 5m/4 \right) \left( \sum_{a_{i_2}^{(2)} \leq N - m} 1 \right) \leq \left( \sum_{a_{i_1}^{(1)} \leq N - m} 1 \right)^k,$$

hence

$$\sum_{a_{i_1}^{(1)} \leq N - m} 1 \gg N^{1/k}.$$  (13)

By a similar argument for $k > 2$ and for all sufficiently large $N$, if $a_i^{(2)} \leq N$, then $|a_i^{(1)} - a_i^{(2)}| \leq m/(8(k - 2))$. Thus

$$m \ll r_k \left( \frac{m}{8(k - 2)} \right) \leq \prod_{j=1}^k \sum_{a_{i_j}^{(j)} \leq m/(8(k - 2))} 1$$

$$\ll \left( \prod_{j=1}^k \sum_{j \neq 2} a_{i_1}^{(1)} \leq m/(8(k - 2)) \right) \left( \sum_{a_{i_2}^{(2)} \leq m/(4(k - 2))} 1 \right) \leq \left( \sum_{a_{i_1}^{(1)} \leq m/(4(k - 2))} 1 \right)^k,$$

hence

$$\sum_{a_{i_1}^{(1)} \leq m/(4(k - 2))} 1 \gg m^{1/k}.$$  (14)

By (12)–(14),

$$J \gg mm^{(k-2)/k} N^{1/k} = m^{2-2/k} N^{1/k}.$$  (15)

We now estimate $J_1$ and $J_2$. Since

$$|1 - z|^2 = (1 - r \cos 2\pi \alpha)^2 + (r \sin 2\pi \alpha)^2 = (1 - r)^2 + 2r(1 - \cos 2\pi \alpha)$$

$$= \frac{1}{N^2} + 4r \sin^2 \pi \alpha$$

and

$$|(2/\pi) \pi \alpha| \leq |\sin \pi \alpha| \quad \text{for } |\alpha| \leq 1/2,$$
it follows that $\max(1/N^2, \alpha^2) \ll |1 - z|^2$, thus $\max(1/N, \alpha) \ll |1 - z|$. Hence

$$J_1 = c \int_0^{1/N} |1 - z|^{-1} \left| \frac{1 - z^m}{1 - z} \right|^2 d\alpha \ll m^2 \int_0^{1/N} |1 - z|^{-1} d\alpha$$

$$\ll m^2 \left( \int_0^{1/N} |1 - z|^{-1} d\alpha + \int_{1/N}^{1/2} |1 - z|^{-1} d\alpha \right)$$

$$\ll m^2 \left( \frac{1}{N} + \int_{1/N}^{1/2} \frac{1}{\alpha} d\alpha \right) \leq m^2 (1 + \log N)$$

$$\ll m^2 \log N.$$  

By Cauchy’s inequality and Parseval’s formula,

$$J_2 = \int_0^{1} \left| (1 - z) \sum_{n=0}^{\infty} v(n) z^n \right| \left| \frac{1 - z^m}{1 - z} \right|^2 d\alpha$$

$$\leq 2 \int_0^{1} \left| \sum_{n=0}^{\infty} v(n) z^n \right| \left| \frac{1 - z^m}{1 - z} \right| d\alpha$$

$$\ll \left( \int_0^{1} \left| \sum_{n=0}^{\infty} v(n) z^n \right|^2 d\alpha \right)^{1/2} \left( \int_0^{1} \left| \frac{1 - z^m}{1 - z} \right|^2 d\alpha \right)^{1/2}$$

$$\leq \left( \sum_{n=0}^{\infty} |v(n)|^2 r^{2n} \right)^{1/2} m^{1/2}.$$  

By our assumption, $v(n) = o(n^{1/4} (\log n)^{1-3k/4})$, therefore for every $\eta > 0$, there exists a natural number $K (\geq 2)$ such that for all $n \geq K$, $|v(n)| \leq \eta n^{1/4} (\log n)^{1-3k/4}$ and $n^{1/4} (\log n)^{1-3k/4}$ is nondecreasing. Then for all $N \geq K$,

$$\sum_{n=0}^{\infty} |v(n)|^2 r^{2n} \leq \sum_{n=0}^{K-1} |v(n)|^2 + \eta^2 \sum_{n=K}^{\infty} n^{1/2} (\log n)^{2-3k/2} r^{2n}$$

$$\leq \sum_{n=0}^{K-1} |v(n)|^2 + \eta^2 N N^{1/2} (\log N)^{2-3k/2}$$

$$+ \eta^2 \sum_{j=0}^{\infty} \sum_{n=2^{j+1} N+1}^{2^{j+1} N} n^{1/2} (\log n)^{2-3k/2} r^n.$$
Since
\[\sum_{j=0}^{\infty} \sum_{n=2^j N+1}^{2^{j+1} N} n^{1/2} (\log n)^{2-3k/2} r^n \leq \sum_{j=0}^{\infty} 2^j N (2^{j+1} N)^{1/2} (\log(2^{j+1} N))^{2-3k/2} r^j N \leq N^{3/2} (\log N)^{2-3k/2} \sum_{j=0}^{\infty} 2^{j+j/2+1/2} e^{-2j} = C_0 N^{3/2} (\log N)^{2-3k/2},\]
it follows that
\[\sum_{n=0}^{\infty} |v(n)|^2 r^{2n} \leq \sum_{n=0}^{K-1} |v(n)|^2 + \eta^2 N^{3/2} (\log N)^{2-3k/2} (1 + C_0) < \eta N^{3/2} (\log N)^{2-3k/2}\]
for \(\eta < (2(1 + C_0))^{-1}\) and for \(N > N_0(\eta)\). Thus
\[\sum_{n=0}^{\infty} |v(n)|^2 r^{2n} = o(N^{3/2} (\log N)^{2-3k/2}). \tag{18}\]

By (17) and (18),
\[J_2 = o(N^{3/4} (\log N)^{1-3k/4} m^{1/2}). \tag{19}\]

By (8), (15), (16), and (19),
\[m^{2-2/k} N^{1/k} \ll m^2 \log N + o(m^{1/2} N^{3/4} (\log N)^{1-3k/4}). \tag{20}\]

Since \(m = [\varepsilon N^{1/2} (\log N)^{-k/2}]\), (20) yields
\[
\left( \frac{\varepsilon}{2} N^{1/2} (\log N)^{-k/2} \right)^{2-2/k} N^{1/k} \ll \varepsilon^2 N (\log N)^{-k} \log N + o(\varepsilon^{1/2} N^{1/4} (\log N)^{-k/4} N^{3/4} (\log N)^{1-3k/4})
\]
for all sufficiently large \(N\), hence \(\varepsilon^{3/2-2/k} \ll \varepsilon^{3/2} + o(1)\). Thus \(\varepsilon^{-2/k} \ll 1\); but this cannot hold for sufficiently small \(\varepsilon\). This completes the proof of the theorem.

**Acknowledgements.** The author would like to thank Professors Imre Z. Ruzsa and András Sárközy for helpful suggestions.

**References**


Department of Algebra and Number Theory  
Eötvös Loránd University  
Pázmány Péter sétány 1/c  
H-1117, Budapest, Hungary  
E-mail: horvathg@cs.elte.hu

Received on 16.3.2000