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On the diophantine equation $ax^2 + b^m = 4y^n$

by

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Let a, b, x, y, m, n be positive integers. Many special cases of the diophantine equation

$$ax^2 + b^m = 4y^n,$$

where $(ax, by) = 1$, a, b are square-free integers, $y > 1$, m is odd and n is an odd prime, have been considered in the last few years (see [1, 3, 7–11]). Le Maohua [4–6] studied this equation in full generality and proved that it has only a finite number of solutions (a, b, x, y, m, n) with $n > 5$.

Following almost the same method as Le Maohua but using a recent result of Bilu, Hanrot and Voutier [2], we are able to prove:

Theorem. *The diophantine equation*

$$ax^2 + b^{2k+1} = 4y^n, \tag{1}$$

where a, b, x, y, k, n *are positive integers such that* $(ax, b) = 1$, a, b *are square-free integers*, $k \geq 0$, n *is an odd prime*, $(n, h) = 1$ *where* h *is the class number of the field* $\mathbb{Q}(\sqrt{-ab})$ *and* $y > 1$, *has no solutions in* (a, b, x, y, k, n) *when* $n > 13$ *and has exactly six solutions for* $7 \leq n \leq 13$, *given by*

$$\begin{aligned}(a, b, k, n, y) = {} & (1, 7, 0, 13, 2), (1, 7, 1, 7, 2), (1, 19, 0, 7, 5),\\ & (3, 5, 0, 7, 2), (5, 7, 1, 7, 3), (13, 3, 0, 7, 4).\end{aligned}$$

Further if $a = 1$, $n = 5$, *then* (1) *has exactly* 2 *solutions given by* $k = 0$ *and* $(b, y) = (7, 2)$, $(11, 3)$.

We are grateful to Professor Bilu for his valuable suggestions and also for providing us with a copy of [1]. He also informed us that similar results using similar approach have been obtained by Bugeaud [3]. Our paper was submitted independently although later than that of Bugeaud.

We start by giving some important definitions.

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Definitions. A *Lehmer pair* is a pair (α, β) of algebraic integers such that $(\alpha+\beta)^2$ and $\alpha\beta$ are non-zero co-prime rational integers and α/β is not a root of unity. For a Lehmer pair (α, β) one defines the corresponding *sequence of Lehmer numbers* by

$$\widetilde{u}_n = \widetilde{u}_n(\alpha, \beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n \text{ is odd,} \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{if } n \text{ is even.} \end{cases}$$

A prime number p is a *primitive divisor* of $\widetilde{u}_n(\alpha, \beta)$ if p divides $\widetilde{u}_n$ but does not divide $(\alpha^2 - \beta^2)^2 \widetilde{u}_1 \widetilde{u}_2 \ldots \widetilde{u}_{n-1}$.

In [2] it has been shown that

Lemma 1. *For every integer $n > 30$, $\widetilde{u}_n(\alpha, \beta)$ has a primitive divisor.*

Also in [12], for $6 < n \le 30$, all Lehmer pairs (α, β) such that $\widetilde{u}_n(\alpha, \beta)$ has no primitive divisors have been listed.

We also need the following classical lemma, going back to the work of Ljunggren [8, 9] (and perhaps earlier), and in the present form proved in [1].

Lemma 2. *Let a, b, x, y be positive integers, with a and b square-free and $\gcd(ax, by) = 1$. Assume that $ax^2 + by^2 = 4c^n$, where c and n are positive integers such that $\gcd(n, 6h(-ab)) = 1$. Then there exist integers x_1 and y_1 such that*

$$\frac{x\sqrt{a} + y\sqrt{-b}}{2} = \left(\frac{x_1\sqrt{a} + y_1\sqrt{-b}}{2}\right)^n.$$

Proof of the Theorem. Let (a, b, x, y, k, n) be a solution of (1). Then applying Lemma 2 with $c = y$, $y = b^m$, we can find integers c, d such that

$$\frac{x\sqrt{a} + b^k\sqrt{-b}}{2} = \left(\frac{c\sqrt{a} + d\sqrt{-b}}{2}\right)^n, \tag{2}$$

where c, d are rational integers such that $4y = ac^2 + bd^2$ and $(ac, bd) = 1$. Let

$$\alpha = \frac{c\sqrt{a} + d\sqrt{-b}}{2}, \qquad \beta = \frac{c\sqrt{a} - d\sqrt{-b}}{2}.$$

Then from equation (2), we get

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \pm\frac{b^k}{d}. \tag{3}$$

It is easy to verify that (α, β) is a Lehmer pair and so from (3) it follows that all the prime divisors of the corresponding nth Lehmer number $\widetilde{u}_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ must divide b. Since $(\alpha^2 - \beta^2)^2 = -abc^2d^2$, these prime divisors also divide $(\alpha^2 - \beta^2)^2$. Hence the nth Lehmer number $\widetilde{u}_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ has no primitive divisors. Since n is a prime, by Lemma 1 and Table 2 in [2] when $n > 13$ there is no Lehmer number $\widetilde{u}_n(\alpha, \beta)$ which has no primitive divisors and so no solution of equation (1) when $n > 13$. For $7 \le n \le 13$, in [2] all Lehmer pairs (α, β) for which $\widetilde{u}_n(\alpha, \beta)$ have no primitive divisors have been listed; we consider each value of n separately.

If $n = 13$, then $\alpha = (1 + \sqrt{-7})/2$, which correspondingly gives $k = 0$, $a = 1$, $b = 7$, $c = 1$, $d = 1$ and consequently $y = (ac^2 + bd^2)/4 = 2$, $x = 181$ is the only solution of the equation $ax^2 + b^{2k+1} = 4y^{13}$.

If $n = 11$, then there is no Lehmer number which has no primitive divisors and so no solution of equation (1).

If $n = 7$, then

$$\alpha = (1 + \sqrt{-7})/2,\ (1 + \sqrt{-19})/2,\ (\sqrt{3} + \sqrt{-5})/2,\ (\sqrt{5} + \sqrt{-7})/2,$$
$$(\sqrt{13} + \sqrt{-3})/2,\ (\sqrt{14} + \sqrt{-22})/2.$$

The first five values of α give us:

- $y = 2$ as a solution of $x^2 + 7^3 = 4y^7$ $(x = 13)$,
- $y = 5$ as a solution of $x^2 + 19 = 4y^7$ $(x = 559)$,
- $y = 2$ as a solution of $3x^2 + 5 = 4y^7$ $(x = 13)$,
- $y = 3$ as a solution of $5x^2 + 7^3 = 4y^7$ $(x = 41)$,
- $y = 4$ as a solution of $13x^2 + 3 = 4y^7$ $(x = 71)$.

If $n = 5$ and $a = 1$, then the values of α for which $\widetilde{u}_5(\alpha, \beta)$ has no primitive divisors as given in [2] are:

$$\alpha = (1 + \sqrt{-5})/2,\ (1 + \sqrt{-7})/2,\ (1 + \sqrt{-11})/2,\ (1 + \sqrt{-15})/2,$$

which gives us:

- $y = 2$ as a solution of $x^2 + 7 = 4y^5$ $(x = 11)$,
- $y = 3$ as a solution of $x^2 + 11 = 4y^5$ $(x = 31)$.

No solutions are found when $\alpha = (1 + \sqrt{-5})/2$, $(1 + \sqrt{-15})/2$. ∎

References

[1] Y. Bilu, *On Le's and Bugeaud's papers about the equation* $ax^2 + b^{2m-1} = 4c^p$, preprint.

[2] Y. Bilu, G. Hanrot and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers* (with an appendix by M. Mignotte), J. Reine Angew. Math. 539 (2001), 75–122.

[3] Y. Bugeaud, *On some exponential diophantine equations*, Monatsh. Math. 132 (2001), 93–97.

[4] M. H. Le, *On the diophantine equations* $d_1x^2 + 2^{2m}d_2 = y^n$ *and* $d_1x^2 + d_2 = 4y^n$, Proc. Amer. Math. Soc. 118 (1993), 67–70.

[5] —, *On the diophantine equation* $D_1x^2 + D_2^m = 4y^n$, Monatsh. Math. 120 (1995), 121–125.

[6] M. H. Le, *A note on the number of solutions of the generalized Ramanujan–Nagell equation* $D_1x^2 + D_2 = 4p^n$, J. Number Theory 62 (1997), 100–106.
[7] —, *On the diophantine equation* $x^2 + D = 4p^n$, ibid. 41 (1992), 87–97.
[8] W. Ljunggren, *On the diophantine equation* $x^2 + D = 4y^q$, Monatsh. Math. 75 (1971), 136–143.
[9] —, *New theorems concerning the diophantine equation* $x^2 + D = 4y^q$, Acta Arith. 21 (1972), 183–191.
[10] B. Persson, *On a diophantine equation in two unknowns*, Ark. Mat. 1 (1949), 45–57.
[11] B. Stolt, *Die Anzahl von Lösungen gewisser diophantischer Gleichungen*, Arch. Math. (Basel) 8 (1957), 393–400.
[12] P. M. Voutier, *Primitive divisors of Lucas and Lehmer sequences*, Math. Comp. 64 (1995), 869–888.

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