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On the least common multiple of Lucas subsequences

by

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1. Introduction. Matiyasevich and Guy [15] proved an interesting formula:

$$\lim_{n\to\infty} \frac{\log F_1 \cdots F_n}{\log \operatorname{lcm}(F_1, \ldots, F_n)} = \frac{\pi^2}{6}$$

valid for the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Since the least common multiple grows due to the contributions of powers of *primitive prime divisors*, that is, prime factors appearing in F_n but not in F_m for any $m < n$, the point of the proof is to describe effectively the contribution of these powers. Inspired by this formula, several generalizations have been discussed in [1, 2, 3, 13] for other sequences $(b_n)_{n\geq 0}$ of integers. A clue of these results is the *strong divisibility* condition:

$$(b_n, b_m) = |b_{\gcd(m,n)}|. \tag{S}$$

The above property ensures that the primitive divisors of b_n are essentially given by the inclusion-exclusion formula

$$\prod_{d|n} b_{n/d}^{\mu(d)},$$

and allows us to control the growth of the least common multiple. This is why strong divisibility and primitive divisors attracted the attention of many researchers [4, 8, 7, 14, 17]. Especially, a lot of effort was spent on the primitive divisors of elliptic divisibility sequences [6, 10, 11, 22].

There are few known results of the above type for general sequences without the assumption (S). In this paper, we give several results on subsequences of Lucas–Lehmer sequences, or *Lucas subsequences* for short, which do not satisfy (S). Let $(u_n)_{n\geq 0}$ be the non-degenerate binary linear sequence given by the recurrence $u_{n+2} = Au_{n+1} + Bu_n$ for all $n \geq 0$, where $u_0 = 0,\ u_1 \neq 0$,

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A and B are fixed non-zero integers. By non-degenerate we mean that the equation $x^2 - Ax - B = 0$ has two non-zero roots $\alpha,\ \beta$ such that α/β is not a root of 1. In this case, the Binet formula holds:

$$u_n = u_1\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \quad \text{for all } n \geq 0. \tag{1}$$

We assume that $|\alpha| \geq |\beta|$ and put $\kappa = (\log \gcd(A^2, B))/(2\log|\alpha|)$. We compute several cases of $(a_n)_{n\geq 0}$. We adopt the convention that $\operatorname{lcm}[s \in \mathcal{S}]$ means the least common multiple of the *non-zero* elements s of $\mathcal{S}$.

THEOREM 1. *If $a_n = |f(n)|$ for all $n \geq 1$, where $f(X) \in \mathbb{Z}[X]$ has at least two distinct roots, then*

$$\frac{\log|\prod_{1\leq k\leq n,\, a_k\neq 0} u_{a_k}|}{\log \operatorname{lcm}[u_{a_1}, \ldots, u_{a_n}]} = \frac{1}{1-\kappa} + O\left(\frac{1}{\log n}\right). \tag{2}$$

THEOREM 2. *When $f(X) = C(aX + b)^m \in \mathbb{Z}[X]$ with $a > 0$ and b coprime, then*

$$\frac{\log|\prod_{1\leq k\leq n,\, a_k\neq 0} u_{a_k}|}{\log \operatorname{lcm}[u_{a_1}, \ldots, u_{a_n}]} = \frac{\zeta(m+1)}{1-\kappa} \prod_{p|a}\left(1 - \frac{1}{p^{m+1}}\right) + O\left(\frac{1}{\log n}\right).$$

We also treat cases in which $(a_n)_{n\geq 0}$ is some arithmetic function of n, such as the Euler function $\phi(n)$ and the sum of divisors function $\sigma(n)$ (see Theorem 3), as well as the case when $(a_n)_{n\geq 0}$ is a non-degenerate binary recurrent sequence (see Theorem 4).

Note that when $b = 0$, u_{a_n} satisfies (S) and we recover the main term of [2]. The error term becomes worse because of the generality of our method. The factor $1/(1-\kappa)$ simply comes from the common divisor of all u_{a_n} and is not so important. The main terms of the two theorems give a sharp contrast. We observe some dichotomy: whenever there are distinct factors, the least common multiple and the product of subsequences become essentially the same.

Throughout the paper, we use the Landau symbols O and o and the Vinogradov symbols $\gg, \ll$ with their usual meaning. We recall that $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent and mean that $|A| \leq cB$ holds with some positive constant c, while $A = o(B)$ means that $A/B \to 0$. We also use $c_1, c_2, \ldots$ for positive computable constants. All constants which appear depend on our sequences $(u_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$.

2. Generalities. Clearly, $|\alpha| > 1$. By Baker's method, we have

$$|u_m| = |\alpha|^m |u_1|\,|\alpha - \beta|^{-1}|1 - (\beta/\alpha)^m| = \exp\big(m\log|\alpha| + O(\log(m+1))\big).$$

Evaluating this relation at $m = a_k$ for $k = 1, \dots, n$, taking logarithms and summing we get

$$\log|u_{a_1} \cdots u_{a_n}| = \log|\alpha| \sum_{k=1}^{n} a_k + O\Big(\sum_{k=1}^{n} \log(a_k + 1)\Big). \tag{3}$$

So, in applications, we shall need some information about

$$A_1(n) = \sum_{k=1}^{n} a_k \quad \text{and} \quad E_1(n) = \sum_{k=1}^{n} \log(a_k + 1). \tag{4}$$

To deal with the least common multiple, we start, as many authors do, by putting $T = \gcd(A^2, B)$, $v_n = T^{-n/2} u_n$, $A_1 = A/\sqrt{T}$, and $B_1 = B/T$. Then

$$v_n = \frac{u_1}{\sqrt{T}} \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1},$$

where $\alpha_1 = \alpha/\sqrt{T}$, $\beta_1 = \beta/\sqrt{T}$. Here, A_1^2 and B_1 are coprime integers and α_1, β_1 are the two roots of the equation $x^2 - A_1^2 x - B_1 = 0$. Put

$$w_n = \begin{cases} \dfrac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} & \text{if } n \equiv 1 \pmod 2, \\ \dfrac{\alpha_1^n - \beta_1^n}{\alpha_1^2 - \beta_1^2} & \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{5}$$

for the Lehmer numbers of the roots α_1, β_1. Then

$$u_n = \begin{cases} u_1 T^{(n-1)/2} w_n & \text{if } n \equiv 1 \pmod 2, \\ A u_1 T^{n/2-1} w_n & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{6}$$

Let $\mathcal{S}$ be the set of all primes dividing $AT u_1$, and for a prime p and a non-zero integer m let $\mu_p(m)$ be the exponent of p in the factorization of m. Since A_1^2 and B_1 are coprime, from linear forms in p-adic logarithms, we have $\mu_p(w_n) < c_p \log n$, where c_p is some constant depending on p. We put

$$\operatorname{lcm}[u_{a_1}, \dots, u_{a_n}] =: M_1 M_2, \tag{7}$$

where M_1 is the contribution to the above least common multiple of the primes from $\mathcal{S}$ and M_2 is the remaining cofactor. The above comments show that

$$\begin{aligned} \log M_1 &= \frac{\log T}{2} \max\{a_k\}_{1 \le k \le n} + O(E_1(n)), \\ \log M_2 &= \log \operatorname{lcm}[w_{a_1}, \dots, w_{a_n}] + O(E_1(n)). \end{aligned} \tag{8}$$

Next, we use cyclotomy to write

$$w_n = \prod_{d \mid n} \Phi_d(\alpha_1, \beta_1), \tag{9}$$

where we put

$$\Phi_m(\alpha_1,\beta_1)=\prod_{\substack{1\le k\le m\\ \gcd(k,m)=1}}(\alpha_1-e^{2\pi ik/m}\beta_1)\quad\text{for all } m\ge 3, \tag{10}$$

and $\Phi_1(\alpha_1,\beta_1)=\Phi_2(\alpha_1,\beta_1)=1$. It is well-known that $\Phi_m(\alpha_1,\beta_1)$ is an integer which captures the primitive prime factors of the term w_m. More precisely, if we write $\Psi_m(\alpha_1,\beta_1)$ for the largest divisor of $\Phi_m(\alpha_1,\beta_1)$ consisting of primes which do not divide $\Phi_\ell(\alpha_1,\beta_1)$ for any $1\le\ell\le m$, then

$$\Phi_m(\alpha_1,\beta_1)=\delta_m\Psi_m(\alpha_1,\beta_1), \tag{11}$$

where δ_m is a divisor of m (see [19, Lemmas 6–8]). By Baker's method again, we have

$$\begin{aligned}|\Phi_m(\alpha_1,\beta_1)|&=\prod_{d\mid m}|\alpha_1^d-\beta_1^d|^{\mu(m/d)}\\ &=\prod_{d\mid m}|\alpha_1|^{d\mu(m/d)}|1-(\beta_1/\alpha_1)^d|^{\mu(m/d)}\\ &=\exp\bigl(\log|\alpha_1|\,\phi(m)+O(\tau(m)\log(m+1))\bigr).\end{aligned} \tag{12}$$

We evaluate the above relation at $m=a_k$ for $k=1,\dots,n$ and use the fact that

$$\log\prod_{k=1}^{n}\delta_{a_k}=O\Bigl(\sum_{k=1}^{n}\log(a_k+1)\Bigr)=O(E_1(n)), \tag{13}$$

to conclude that if we put

$$\mathcal{D}_n=\{d: d\mid a_k \text{ for some } 1\le k\le n\}, \tag{14}$$

then from (9)–(13) we obtain

$$\begin{aligned}\log\operatorname{lcm}[w_{a_1},\dots,w_{a_n}]&=\log\prod_{d\in\mathcal{D}_n}|\Psi_d(\alpha_1,\beta_1)|+O\Bigl(\log\prod_{k=1}^{n}\delta_{a_k}\Bigr)\\ &=\log|\alpha_1|\sum_{d\in\mathcal{D}_n}\phi(d)+O(E_1(n))\\ &\quad+O\Bigl(\sum_{d\in\mathcal{D}_n}\tau(d)\log(d+1)\Bigr)\\ &=\log|\alpha_1|\,A_2(n)+O(E_2(n)),\end{aligned} \tag{15}$$

where we write

$$A_2(n)=\sum_{d\in\mathcal{D}_n}\phi(d)\quad\text{and}\quad E_2(n)=\sum_{k=1}^{n}\tau(a_k)^2\log(a_k+1). \tag{16}$$

The last error term in (15) comes from the fact that every a_k for $k=1,\dots,n$

contributes at most $\tau(a_k)$ members $d \in \mathcal{D}_n$ and for each of them we have

$$\tau(d)\log(d+1) \le \tau(a_k)\log(a_k+1).$$

All this has been obtained without any arithmetic condition on the sequence $(a_n)_{n\ge 1}$. Let us see some examples.

3. Examples

3.1. The case of the sequences $a_n = \phi(n)$ and $a_n = \sigma(n)$. Both sequences have almost linear growth, that is, $a_n \le n^{1+o(1)}$ as $n \to \infty$, therefore both inequalities

$$E_1(n) \le n^{1+o(1)} \quad \text{and} \quad E_2(n) \le n^{1+o(1)}$$

hold as n tends to infinity. Further,

$$A_1(n) = c_{\mathbf{a}} n^2 + O(n\log n),$$

with $c_{\mathbf{a}} = 3/\pi^2$ or $\pi^2/12$ according to whether $a_n = \phi(n)$ or $a_n = \sigma(n)$. As for $\mathcal{D}_n$, we cut it into two parts:

$$\mathcal{D}_{1,n} = \{d \in \mathcal{D}_n : 1 \le d \le n/(\log n)^{1/4}\}.$$

Here we use the trivial estimate

$$\sum_{d\in\mathcal{D}_{1,n}} \phi(d) \le \sum_{d\le n/(\log n)^{1/4}} d = O\Big(\frac{n^2}{(\log n)^{1/2}}\Big).$$

Put $\mathcal{D}_{2,n} = \mathcal{D}_n \setminus \mathcal{D}_{1,n}$. If $d \in \mathcal{D}_{1,n}$, then $d = \phi(u)/v$, where $u \le n$ and $v \le (\log n)^{1/4}$ in the case of $a_k = \phi(k)$. When $a_k = \sigma(k)$, we have $d = \sigma(u)/v$ for some $u \le n$, where $v \le c_1(\log n)^{1/4}\log\log n$ for some constant c_1. Here, we use the fact that $\sigma(u) \le c_1 u \log\log u$ for all $u \ge 3$ with some constant c_1. Each of the sets $\{\phi(u) \le n\}$ and $\{\sigma(u) \le c_1 n\log\log n\}$ has $O(n/(\log n)^{1-\varepsilon})$ elements (see [5] or [9, Theorems 1 and 14]), where $\varepsilon > 0$ can be taken to be as small as we wish and will be fixed later. Thus,

$$\#\mathcal{D}_{2,n} = O\Big(\frac{n\log\log n}{(\log n)^{3/4-\varepsilon}}\Big) = O\Big(\frac{n}{(\log n)^{1/2}}\Big)$$

provided that we choose $\varepsilon = 1/10$. Hence,

$$\sum_{d\in\mathcal{D}_{2,n}} \phi(d) \le n\#\mathcal{D}_{2,n} = O\Big(\frac{n^2}{(\log n)^{1/2}}\Big),$$

and we get the estimate

$$\frac{\log|u_{a_1}\cdots u_{a_n}|}{\log \operatorname{lcm}[u_{a_1},\dots,u_{a_n}]} \gg \sqrt{\log n}.$$

In particular,

$$\log \operatorname{lcm}[u_{a_1},\dots,u_{a_n}] = o(\log|u_{a_1}\cdots u_{a_n}|) \quad \text{as } n\to\infty,$$

a phenomenon that does not happen for the sequences dealt with in [2].

We record this as the following result.

THEOREM 3. *If $a_n = \phi(n)$ for all $n \geq 1$, then*

$$\log \operatorname{lcm}[u_{a_1}, \ldots, u_{a_n}] = o(\log |u_{a_1} \cdots u_{a_n}|) \quad \text{as } n \to \infty.$$

The same conclusion holds when $a_n = \sigma(n)$ for all $n \geq 1$.

3.2. The case of the sequences $a_n = |b_n|$ with $(b_n)_{n\geq 1}$ binary recurrent. Since we are working very generally, we shall assume that

$$b_{n+2} = Cb_{n+1} + Db_n,$$

where C and D are non-zero integers such that the equation $\lambda^2 - C\lambda - D = 0$ has two distinct roots γ, δ with γ/δ not a root of 1. Then

$$b_n = \eta\gamma^n + \zeta\delta^n,$$

with some non-zero algebraic numbers η, ζ in $\mathbb{K} = \mathbb{Q}(\gamma)$. We assume that $|\gamma| \geq |\delta|$. Thus,

$$A_1(n) = \sum_{k=1}^{n} |b_n|.$$

We also assume that we work only with the numbers $k = 1, \ldots, n$ such that $b_k \neq 0$. It is easy to see that if there exists such a k with $b_k = 0$, then it is unique. Indeed, if not, then say $b_{k_1} = b_{k_2} = 0$ for integers $k_1 < k_2$. Regarding these two equations as a degenerate homogeneous linear system in the unknowns η, ζ whose coefficient matrix is

$$\begin{pmatrix} \gamma^{k_1} & \delta^{k_1} \\ \gamma^{k_2} & \delta^{k_2} \end{pmatrix},$$

we find that $(\gamma/\delta)^{k_2-k_1} = 1$, which is not allowed because γ/δ is not a root of unity. By Baker's bound,

$$A_1(n) \geq |b_n| = \exp(n \log |\gamma| + O(\log n)). \tag{17}$$

This gives us the main term for $\log |u_{a_1} \cdots a_{a_n}|$. It remains to study $\log \operatorname{lcm}[u_{a_1}, \ldots, u_{a_n}]$. Clearly,

$$E_1(n) = \exp(o(n)) \quad \text{and} \quad E_2(n) = \exp(o(n)) \quad \text{as } n \to \infty.$$

To get $A_2(n)$, we put $T_1 = \gcd(C^2, D)$, $\gamma_1 = \gamma^2/T_1$, $\delta_1 = \delta^2/T_1$ and

$$b_n = T_1^{\lfloor n/2 \rfloor} z_n,$$

where

$$z_n = \eta_1\gamma_1^{\lfloor n/2 \rfloor} + \zeta_1\delta_1^{\lfloor n/2 \rfloor} \quad \text{with} \quad (\eta_1, \zeta_1) = \begin{cases} (\eta, \zeta) & \text{if } n \equiv 0 \pmod 2, \\ (\eta\gamma, \zeta\delta) & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Let $\mathcal{T}$ be the finite set of primes sitting above some prime ideal π from $\mathcal{O}_{\mathbb{K}}$ which appears with non-zero exponent in the factorization of one of the

principal fractional ideals generated by $\gamma, \delta, \eta, \zeta, \gamma - \delta$ in $\mathbb{K}$. We split $\mathcal{D}_n$ into three subsets as follows. We take

$$\mathcal{D}_{1,n} = \{d \in \mathcal{D}_n : d \le |\gamma|^{n/2}\}.$$

Since $d \,|\, a_k$ for some $k = 1, \dots, n$ and since each a_k has $a_k^{o(1)} = \exp(o(n))$ divisors as $n \to \infty$, we get

$$\sum_{d \in \mathcal{D}_{1,n}} \phi(d) = O\big(n|\gamma|^{n/2} \exp(o(n))\big) \le |\gamma|^{(1/2+o(1))n} \quad \text{as } n \to \infty. \tag{18}$$

Next we take

$$\mathcal{D}_{2,n} = \{d \in \mathcal{D}_n \setminus \mathcal{D}_{1,n} : d \,|\, a_i \text{ and } d \,|\, a_j \text{ for some } i < j \in \{1, \dots, n\}\}.$$

Since $d > |\gamma|^{n/2}$ and $a_k = O(|\gamma|^k)$ for both $k = i$ and j, it follows that $i \ge n/2 + O(1)$, therefore

$$j - i \le n/2 + O(1).$$

Now write $d = d_1 d_2$, where d_1 is the contribution to d from primes coming from $\mathcal{T}$ and d_2 is the contribution from the remaining primes. Since γ_1 and δ_1 are coprime, it follows, again by the theory of linear forms in p-adic logarithms, that $\mu_p(c_m) < c(p) \log(m+1)$ for all primes p, with some constant c_p depending on p. This shows that

$$\log d_1 = \frac{\log T_1}{2} n + O(\log(n+1)).$$

As for d_2, we have $d_2 \,|\, z_i$ and $d_2 \,|\, z_j$. Since η and δ are invertible modulo d_2, we get

$$\left(\frac{\gamma}{\delta}\right)^i \equiv -\frac{\zeta}{\eta} \pmod{z_2} \quad \text{and} \quad \left(\frac{\gamma}{\delta}\right)^j \equiv -\frac{\zeta}{\eta} \pmod{z_2},$$

from which we deduce that

$$\left(\frac{\gamma}{\zeta}\right)^{j-i} \equiv 1 \pmod{z_2}.$$

Thus, z_2 divides the sth term of the Lucas sequence $(\gamma^s - \delta^s)/(\gamma - \delta)$ with $s = j - i \le n/2 + O(1)$. Each such term has $\exp(o(n))$ divisors as $n \to \infty$, and there are only $O(n)$ possibilities for s. Hence,

$$\sum_{d \in \mathcal{D}_{2,n}} \phi(d) \le n|\gamma|^{n/2} \exp(o(n)) = |\gamma|^{(1/2+o(1))n} \quad \text{as } n \to \infty. \tag{19}$$

Finally, look at numbers $d \in \mathcal{D}_{3,n} = \mathcal{D}_n \setminus (\mathcal{D}_{1,n} \cup \mathcal{D}_{2,n})$. Each of these numbers divides a unique $a_k = k_d$ and they are all $> |\gamma|^{n/2}$. Further, each number $d > |\gamma|^{n/2}$ which divides a_k for some k is either in $\mathcal{D}_{3,n}$ or in $\mathcal{D}_{2,n}$. Using the formula

$$m = \sum_{d \mid m} \phi(d)$$

and adding into our sums also all the divisors $d \le |\gamma|^{n/2}$ of all the numbers a_k for $k \in \{1, \dots, n\}$ (at most n values for k, at most $\exp(o(n))$ as $n \to \infty$ values for d for each k, and none exceeding $|\gamma|^{n/2}$), we easily get

$$\sum_{d \in \mathcal{D}_{3,n}} \phi(d) = \sum_{k=1}^{n} a_k + O\big(n|\gamma|^{n/2+o(n)} \exp(o(n))\big) = A_1(n) + O(|\gamma|^{n/2+o(n)}). \tag{20}$$

Putting (18)–(20) together and using also (17), we get

$$A_2(n) = \sum_{k=1}^{3} \sum_{d \in \mathcal{D}_{k,n}} \phi(d) = A_1(n) + O(|\gamma|^{n/2+o(n)}) = (1 + o(1))A_1(n),$$

which leads to quite the opposite conclusion to the one in the previous case, namely

$$\log \operatorname{lcm}[u_{a_1}, \dots, u_{a_n}] = (1 + o(1)) \log |u_{a_1} \cdots u_{a_n}| \quad \text{as } n \to \infty.$$

Further, note that the expression for $A_1(n)$ can be simplified when $|\gamma| > |\delta|$ (that is, when both γ and δ are real), since then

$$|a_n| = |\eta|\,|\gamma|^n + O(|\delta|^n) \quad \text{for all } n \ge 1,$$

therefore

$$A_1(n) = \frac{|\eta\gamma|}{|\gamma| - 1} |\gamma|^n + O(|\gamma|^{c_2 n}),$$

where c_2 is any constant satisfying $\log |\delta| / \log |\gamma| < c_2 < 1$.

We record the following result.

THEOREM 4. *If $a_n = |b_n|$, where $(b_n)_{n \ge 1}$ is a non-degenerate binary recurrence, then*

$$\log \operatorname{lcm}[u_{a_1}, \dots, u_{a_n}] = (1 + o(1)) \log |u_{a_1} \cdots u_{a_n}| \quad \text{as } n \to \infty.$$

3.3. The case of the Lucas sequence of the second kind. Jones and Kiss [12] studied the least common multiple of the sequence u_{mn}/u_n for $m > 0$. For completeness, we study the case for $m = 2$ directly by our method which will give us a good comparison. Thus $(u_n)_{n \ge 1}$ is replaced by $(L_n)_{n \ge 1}$ given by $L_0 = 2,\ L_1 = A$. In this case, the analog of formula (1) is

$$L_n = \alpha^n + \beta^n.$$

By Baker's method, we have again

$$|L_m| = \exp\big(m \log |\alpha| + O(\log(m + 1))\big),$$

so formula (3) holds for this case also:

$$\log |L_{a_1} \cdots L_{a_n}| = \log |\alpha|\, A_1(n) + O(E_1(n)). \tag{21}$$

It remains to estimate the least common multiple. The analogue of formula (6) is

$$L_n = \begin{cases} T^{(n-1)/2} A w_{2n}/w_n & \text{if } n \equiv 1 \pmod 2, \\ A u_1 T^{n/2} w_{2n}/w_n & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{22}$$

We now see that the analogues of formulas (7) and (8) are

$$\operatorname{lcm}[L_{a_1}, \dots, L_{a_n}] =: M_1 M_2, \tag{23}$$

where again M_1 is the contribution to the above least common multiple of the primes from $\mathcal{S}$ and M_2 is the contribution of the remaining primes, then we have

$$\begin{aligned} \log M_1 &= \frac{\log T}{2} \max\{a_k\}_{1 \le k \le n} + O(E_1(n)), \\ \log M_2 &= \log \operatorname{lcm}[w_{2a_1}/w_{a_1}, \dots, w_{2a_n}/w_{a_n}] + O(E_1(n)). \end{aligned} \tag{24}$$

Now observe that by cyclotomicity, we have

$$\frac{w_{2m}}{w_m} = \delta_{2m} \delta_m^{-1} \prod_{\substack{d \mid 2m \\ d \nmid m}} \Psi_d(\alpha_1, \beta_1),$$

and now the previous argument shows that if we put

$$\mathcal{D}'_n = \{d : d \mid 2a_k \text{ but } d \nmid a_k \text{ for some } k \in \{1, \dots, n\}\},$$

then in fact

$$\log \operatorname{lcm}[w_{2a_1}/w_{a_1}, \dots, w_{2a_n}/w_{a_n}] = \log |\alpha_1| \, A_3(n) + O(E_2(n)),$$

where

$$A_3(n) = \sum_{d \in \mathcal{D}'_n} \phi(d).$$

As a concluding example, take $a_k = k$. Then

$$A_1(n) = \sum_{k \le n} k = n^2/2 + O(n).$$

Clearly,

$$E_1(n) \le \sum_{k \le n} \log(k+1) = O(n \log n).$$

Next

$$\begin{aligned} \log M_1 &= \frac{T}{2} n + O(E_1(n)) = O(n \log n), \\ \log M_2 &= \log |\alpha_1| \, A_3(n) + O(E_2(n)), \end{aligned}$$

where

$$A_3(n) = \sum_{d \in \mathcal{D}'_n} \phi(d) \quad \text{with} \quad \mathcal{D}'_n = \{2, 4, \dots, 2n\}.$$

Observe that $\mathcal{D}'_n$ is the set of even numbers less than or equal to $2n$. So,

$$A_3(n) = \sum_{\substack{d \equiv 0 \pmod 2 \\ d \le 2n}} \phi(d) = \sum_{d \le 2n} \phi(d) - \sum_{1 \le k \le n} \phi(2k-1) := S_1 + S_2.$$

Clearly,

$$S_1 = \frac{(2n)^2}{2\zeta(2)} + O(n \log n) = \frac{2n^2}{\zeta(2)} + O(n \log n).$$

It is well-known that if $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree h with leading coefficient a_h, then

$$\sum_{k \le n} \phi(f(k)) = c_f a_h (h+1)^{-1} n^{h+1} + O(n^h \log n)$$

with

$$c_f = \sum_{k=1}^{\infty} \frac{\mu(k)\rho_f(k)}{k^2},$$

where $\rho_f(n)$ is the number of solutions $x \pmod k$ of the congruence $f(x) \equiv 0 \pmod k$ (see [18]). For the particular case of the polynomial $f(x) = 2x-1$, we have $\rho_f(k) = 1$ if k is odd and $\rho_f(k) = 0$ if k is even, so

$$c_f = \sum_{k \equiv 1 \pmod 2} \frac{\mu(k)}{k^2} = \prod_{p \ge 3} \left(1 - \frac{1}{p^2}\right) = \frac{4}{3\zeta(2)},$$

so since $h = 1,\ a_h = 2$, we have

$$S_2 = \frac{4n^2}{3\zeta(2)} + O(n \log n),$$

leading to

$$A_3(n) = \left(2 - \frac{4}{3}\right) \frac{n^2}{\zeta(2)} + O(n \log n) = \frac{2n^2}{3\zeta(2)} + O(n \log n).$$

Unfortunately, given that our method is so general, the error terms are not very good, and are worse than the ones obtained in [1] and [2], for example. That is, for our particular case, we have

$$E_2(n) \le \sum_{d \le 2n} \tau(d)^2 \log(d+1) = O(n(\log n)^5),$$

so that

$$\log \operatorname{lcm}[L_1, \ldots, L_n] = \log M_1 + \log M_2 = \left(\frac{2 \log |\alpha_1|}{3\zeta(2)}\right) n^2 + O(n(\log n)^5).$$

We find that the analogue of (2) for the Lucas sequence of the second kind is

$$\frac{\log|L_1\cdots L_n|}{\log \operatorname{lcm}[L_1,\dots,L_n]} = \frac{(\log|\alpha|)/2}{(2\log|\alpha_1|)/(3\zeta(2))} + O\bigg(\frac{(\log n)^5}{n}\bigg) = \frac{3\zeta(2)}{4(1-\kappa)} + O\bigg(\frac{(\log n)^5}{n}\bigg).$$

We record this as follows.

THEOREM 5. *We have*

$$\frac{\log|L_1\cdots L_n|}{\log \operatorname{lcm}[L_1,\dots,L_n]} = \frac{3\zeta(2)}{4(1-\kappa)} + O\bigg(\frac{(\log n)^5}{n}\bigg).$$

Here, the error term is slightly worse than in [12] because of our general approach.

3.4. The case when $a_k = |f(k)|$ with a polynomial $f(X) \in \mathbb{Z}[X]$. In this section, we treat the case when $a_k = |f(k)|$ with $f(X) \in \mathbb{Z}[X]$. Say

$$f(X) = C_0X^m + C_1X^{m-1} + \cdots + C_m \in \mathbb{Z}[X]$$

has degree $m \ge 1$. We assume that $C_0 > 0$. As in the previous cases, we only work with numbers k such that $f(k) \ne 0$. Clearly, the equation $f(k) = 0$ has at most m solutions k. We have

$$A_1(n) = \sum_{1\le k\le n} |f(k)| = \frac{C_0}{m+1}n^{m+1} + O(n^m),$$
$$E_1(n) = \sum_{1\le k\le n} \log(|f(k)|+1) = O(n\log n),$$

so, by (3), we have

$$\log\Big|\prod_{\substack{1\le k\le n\\ a_k\ne 0}} u_{a_k}\Big| = \frac{C_0\log|\alpha|}{m+1}n^{m+1} + O(n^m\log n). \tag{25}$$

To get $A_2(n)$, first we put $C = \gcd(C_0,\dots,C_m)$ and write $f(X) = Cg(X)$. Further, putting $\alpha_0 = \alpha^C$, $\beta_0 = \beta^C$ and

$$v_k = \frac{\alpha_0^k - \beta_0^k}{\alpha_0 - \beta_0} \quad \text{for } k \ge 0,$$

we have

$$u_{a_k} = \frac{\alpha^{f(k)} - \beta^{f(k)}}{\alpha - \beta} = \frac{\alpha_0^{g(k)} - \beta_0^{g(k)}}{\alpha_0 - \beta_0}u_C = v_{g(k)}u_C.$$

Thus, instead of working with the sequences $\{u_n\}_{n\ge1}$ and $a_k = |f(k)|$ for $1 \le k \le n$, we can work with $\{u_Cv_n\}_{n\ge1}$ and $b_k = |g(k)|$ for $1 \le k \le n$. The

characteristic equation for the sequence $\{u_C v_n\}_{n\ge 1}$ is

$$X^2 - A_0 X - B_0 = 0,$$

where $A_0 = \alpha^C + \beta^C = u_{2C}/u_C$ and $B_0 = -(\alpha\beta)^C = (-1)^{C-1}B^C$. The Lehmer sequence $\{w_n\}_{n\ge 0}$ associated to $\{v_n\}_{n\ge 0}$ is given by formula (5) with the roots $\alpha_1 = \alpha_0/\sqrt{T_0},\ \beta_1 = \beta_0/\sqrt{T_0}$, where $T_0 = \gcd(A_0^2, B_0)$. The arguments from the beginning of Section 2 show that

$$\operatorname{lcm}[u_{a_1}, \dots, u_{a_n}] = M_1 M_2,$$

where

$$M_1 = \frac{\log T_0}{2} \max\{|g(k)|\}_{1\le k\le n} + O(E_1(n)),$$
$$M_2 = \log \operatorname{lcm}[w_{b_1}, \dots, w_{b_k}] + O(E_1(n)).$$

Clearly,

$$M_1 = O(n^m \log n).$$

By formula (15), we have

$$M_2 = \log|\alpha_1|\, A_2(n) + O(E_2(n)),$$

where

$$A_2(n) = \sum_{d\in\mathcal{D}_n} \phi(d) \quad \text{and} \quad E_2(n) = \sum_{k\le n} \tau(b_k)^2 \log(b_k+1),$$

and

$$\mathcal{D}_n = \{d : d \mid g(k) \text{ for some } k \in [1,n] \text{ with } g(k)\neq 0\}.$$

By a result of van der Corput (see [20]), we have

$$\sum_{\substack{1\le k\le n\\ g(k)\neq 0}} \tau(|g(k)|)^i = O(n(\log n)^{c(i)}) \tag{26}$$

for all positive integers i, where $c(i)$ is some constant depending on i and g. We put $c_1 = \max\{c(1), m\}$ and $c_2 = c(2)$. In particular, from the above estimate (26) with $i = 2$ we get

$$E_2(n) = O\Big(\log n \sum_{\substack{1\le k\le n\\ g(k)\neq 0}} \tau(|g(k)|)^2\Big) = O(n(\log n)^{c_2+1}).$$

It remains to understand $A_2(n)$. For this, we split the set $\mathcal{D}_n$ into three subsets according to whether d is small, or k is small, or both are large.

We put

$$\mathcal{D}_{1,n} = \{d \in \mathcal{D}_n : d \le n^m/(\log n)^{c_1+1}\}.$$

Then

$$\sum_{d\in\mathcal{D}_{1,n}}\phi(d)\le\frac{n^m\#\mathcal{D}_n}{(\log n)^{c_1+1}}\le\frac{n^m}{(\log n)^{c_1+1}}\sum_{\substack{1\le k\le n\\ g(k)\ne 0}}\tau(|g(k)|)=O\Big(\frac{n^{m+1}}{\log n}\Big). \tag{27}$$

Next, let

$$\mathcal{D}_{2,n}=\{d : d\,|\,g(k)\text{ for some }k\le n/(\log n)^{c_1+1}\text{ with }g(k)\ne 0\}.$$

Then

$$\sum_{d\in\mathcal{D}_{2,n}}\phi(d)\le\max\{|g(k)|\}_{k\le n/(\log n)^{c_1+1}}\#\mathcal{D}_n=O\Big(\frac{n^{m+1}}{\log n}\Big). \tag{28}$$

We now look at the numbers $d\in\mathcal{D}_n\setminus(\mathcal{D}_{1,n}\cup\mathcal{D}_{2,n})$. Since $|g(k)|\le c_3k^m$ for all $k\ge 1$ with some constant c_3, we may write $d=|g(k)|/e$, where $n/(\log n)^{c_1+1}\le k\le n$ and $1\le e\le c_3(\log n)^{c_1+1}$. Furthermore, since $C_0>0$ and $k>n/(\log n)^{c_1+1}$, it follows that for large enough n, the number $g(k)$ is positive. So, from now on we shall simply write $g(k)$ for such k instead of $|g(k)|$. Put $\mathcal{K}_n=[n/(\log n)^{c_1+1},n]$ and $\mathcal{E}_n=[1,c_3(\log n)^{c_1+1}]$.

It turns out that from here on the argument (and indeed, the answer), splits into two cases according to whether or not $g(X)$ (or $f(X)$) has at least two distinct roots.

3.4.1. *Proof of Theorem 1.* We start with a preliminary result about polynomials satisfying a certain functional equation.

LEMMA 1. *Let $f(X)\in\mathbb{C}[X]$ of degree m and let $r\ne 0,s,\eta$ be complex numbers with r not a root of unity such that*

$$f(rX+s)=\eta f(X). \tag{29}$$

Then $f(X)=(aX+b)^m$ for some complex numbers a and b such that $as=b(r-1)$.

Proof. Identifying the leading coefficient in (29), we get $\eta=r^m$. We prove the lemma by induction on m. For $m=1$, $f(X)=aX+b$, so the relation $f(rX+s)=f(X)$ gives $a(rX+s)+b=r(aX+b)$, so $as=b(r-1)$, as desired. Assume now that $m\ge 2$ and that the claim is true for polynomials of degree smaller than m, and let $f(X)$ be a polynomial of degree m such that $f(rX+s)=r^mf(X)$. Taking derivatives, we get $f'(rX+s)=r^{m-1}f'(X)$, so, by the induction hypothesis, $f'(X)=(aX+b)^{m-1}$ where $as=b(r-1)$. Thus, $f(X)=\frac{1}{m}(aX+b)^m+d$ for some number d. But then (29) becomes

$$\frac{1}{m}(a(rX+s)+b)^m+d=\frac{r^m}{m}(aX+b)^m+r^md.$$

Since $a(rX+s)+b=arX+as+b=r(aX+b)$, it follows that we must have $d=r^md$, i.e. $d(r^m-1)=0$, so $d=0$ because r is not a root of unity. We thus get $f(X)=(a_1X+b_1)^m$, where $a_1=a/m^{1/m}$, $b_1=b/m^{1/m}$. ∎

We next have the following lemma.

LEMMA 2. *There exists a constant c_4 such that for $n > n_0$ the number of solutions $(k_1, k_2, e_1, e_2) \in \mathcal{K}_n^2 \times \mathcal{E}_n^2$ with $k_1 \neq k_2$ of the equation*

$$\frac{g(k_1)}{e_1} = \frac{g(k_2)}{e_2} \tag{30}$$

is at most $(\log n)^{c_4}$.

Proof. Observe first that if $e_1 = e_2$, then $g(k_1) = g(k_2)$. However, for large n, $g'(k)$ is positive for all $k > n/(\log n)^{c_1+1}$, and in particular $g(k)$ is increasing for $k \in \mathcal{K}_n$, so the above equation implies $k_1 = k_2$, which is not allowed. Thus, for large n, any solution (k_1, k_2, e_1, e_2) will have $e_1 \neq e_2$. Write

$$g(X) = C_0'X^m + C_1'X^{m-1} + \cdots + C_m', \quad \text{where} \quad C_i' = C_i/C \quad (i = 0, \ldots, m).$$

Observe that

$$C_0'^{m-1} m^m g(X) = (C_0' mX + C_1')^m + h(X),$$

where $h(X) \in \mathbb{Z}[X]$ is of degree at most $m-2$. Thus, from (30) we get

$$\begin{aligned} &C_0'^{m-1} m^m (e_2 g(k_1) - e_1 g(k_2)) \\ &\quad = e_2 (C_0' mk_1 + C_1')^m - e_1 (C_0' mk_2 + C_1')^m + e_2 h(k_1) - e_1 h(k_2) = 0, \end{aligned}$$

therefore if we put $\ell(X) = C_0' mX + C_1'$ and $\ell_i = \ell(k_i)$ for $i = 1, 2$, then

$$|e_2 \ell_1^m - e_1 \ell_2^m| = O(e_1 k_2^{m-2} + e_2 k_1^{m-2}) = O(n^{m-2} (\log n)^{c_1+1}). \tag{31}$$

The left-hand side above equals

$$\prod_{\zeta^m = 1} |e_1^{1/m} \ell_1 - \zeta e_2^{1/m} \ell_2|, \tag{32}$$

where $e_1^{1/m}$ and $e_2^{1/m}$ stand for the real positive roots of order m of e_1 and e_2 respectively. If ζ is a complex non-real root of unity of order m, then

$$|e_1^{1/m} \ell_1 - \zeta e_2^{1/m} \ell_2| \geq |\mathrm{Im}(\zeta)| e_2^{1/m} \ell_2 \gg \frac{n}{(\log n)^{c_1+1}}, \tag{33}$$

and a similar inequality holds when $\zeta = -1$ and m is even. Thus, using inequality (33) to bound from below all factors of the product (32) except for the one corresponding to $\zeta = 1$, and comparing the inequality obtained in this way with (31), we get

$$|e_1^{1/m} \ell_1 - e_2^{1/m} \ell_2| \ll \frac{(\log n)^{c_5}}{n},$$

where $c_5 = mc_1 + m$. In particular,

$$\left| \alpha(e_1, e_2) - \frac{\ell_2}{\ell_1} \right| \ll \frac{(\log n)^{c_5}}{\ell_1^2},$$

where $\alpha(e_1, e_2) = (e_1/e_2)^{1/m}$. Write $\delta = \gcd(\ell_1, \ell_2)$, $\ell_1 = \delta m_1$, $\ell_2 = \delta m_2$. We then get

$$\left|\alpha(e_1, e_2) - \frac{m_2}{m_1}\right| < \frac{c_6(\log n)^{c_5}}{\delta^2 m_1^2}, \tag{34}$$

where c_6 is some positive constant.

Suppose first that $\delta^2 < 2c_6(\log n)^{c_5}$. Then δ can take only $O((\log n)^{c_5/2})$ positive integer values. By a result of Worley [21], inequality (34) implies that

$$\frac{m_1}{m_2} = \frac{ap_k + bp_{k-1}}{aq_k + bq_{k-1}} \quad \text{or} \quad \frac{ap_{k+1} + bp_{k-1}}{aq_{k+1} + bq_{k-1}}$$

for some integers $k \geq 1$, $a \geq 1$ and b with $a|b| < 2c_6(\log n)^{c_5}$, where $\{p_k/q_k\}_{k\geq 0}$ is the kth convergent to $\alpha(e_1, e_2)$. Since $\max\{m_1, m_2\} \leq n$, we have $k = O(\log n)$ uniformly in e_1 and e_2. Since there are $O((\log n)^{2c_1+2})$ choices for the pair (e_1, e_2); next, for the number $\alpha(e_1, e_2)$, $O((\log n)^{c_5/2})$ choices for δ; and then $O((\log n)^{2c_5+1})$ choices for the triple (a, b, k), we get a totality of $O((\log n)^{2c_1+2.5c_5+3})$ choices for (ℓ_1, ℓ_2), hence, for (k_1, k_2), in this instance.

Assume next that $\delta^2 > 2c_6(\log n)^{c_5}$. We then have

$$\left|\alpha(e_1, e_2) - \frac{m_2}{m_1}\right| < \frac{1}{2m_1^2}.$$

Either $\alpha(e_1, e_2) = m_2/m_1$ is rational, so the expression on the left above is 0, or $\alpha(e_1, e_2)$ is irrational and $m_2/m_1 = p_k/q_k$ is a convergent to $\alpha(e_1, e_2)$ by a criterion of Legendre. Here, as before, $k = O(\log n)$. Fix e_1, e_2, m_1, m_2. Then

$$\frac{m_1}{m_2} = \frac{\ell_1}{\ell_2} = \frac{C_0' m k_1 + C_1'}{C_0' m k_2 + C_1'},$$

so

$$k_2 = rk_1 + s, \quad \text{where} \quad r = \frac{m_2}{m_1} \quad \text{and} \quad s = \frac{C_1'(m_2 - m_1)}{C_0' m m_1}.$$

Note that $r \neq 1$, because if not, then $m_1 = m_2 = 1$, so $k_2 = k_1$, which is not allowed. Since r is also positive, it follows that r is not a root of unity. Going back to (30), we get

$$\frac{g(rk_1 + s)}{g(k_1)} = \eta \quad \text{with} \quad \eta = \frac{e_2}{e_1}.$$

Since r, s, η are fixed, the above is a polynomial relation in k_1, so it has at most m roots, unless the rational function $g(rX + s)/g(X)$ is constant η, which is not the case by Lemma 1 and the fact that $g(X)$ has at least two distinct zeros. Thus, when e_1, e_2, m_1, m_2 are fixed, there are at most m possibilities for k_1, and then k_2 is uniquely determined. This shows that

the number of solutions of equation (30) in this case is $O((\log n)^{2c_1+3})$. The lemma now follows with $c_4 = 2c_1 + 2.5c_5 + 4 = (2.5m+2)c_1 + 2.5m + 4$. ∎

For each $d \in \mathcal{D}_n \setminus (\mathcal{D}_{1,n} \cup \mathcal{D}_{2,n})$ let $r(d)$ be the number of representations of d in the form $d = g(k)/e$ for some $k \in \mathcal{K}_n$ and $e \in \mathcal{E}_n$. Lemma 2 shows that if we put

$$\mathcal{D}_{3,n} = \{d \in \mathcal{D}_n \setminus (\mathcal{D}_{1,n} \cup \mathcal{D}_{2,n}) : r(d) > 1\},$$

then

$$\#\mathcal{D}_{3,n} = O((\log n)^{c_4}). \tag{35}$$

We now use the relation

$$m = \sum_{d|m} \phi(d)$$

with $m = g(k)$ in the following way:

$$g(k) = \sum_{\substack{e|g(k)\\ e\in\mathcal{E}_n}} \phi\left(\frac{g(k)}{e}\right) + O\left(g(k) \sum_{\substack{e|g(k)\\ e>c_3(\log n)^{c_1+1}}} \frac{1}{e}\right), \tag{36}$$

which we rewrite as

$$g(k) = \sum_{\substack{e|g(k)\\ e\in\mathcal{E}_n}} \phi\left(\frac{g(k)}{e}\right) + O\left(n^m \sum_{\substack{e|g(k)\\ e>c_3(\log n)^{c_1+1}}} \frac{1}{e}\right). \tag{37}$$

We sum up the above relation for all $k \in \mathcal{K}_n$ getting

$$\begin{aligned}\sum_{k\in\mathcal{K}_n} g(k) = {} & \sum_{d\in\mathcal{D}_n\setminus(\mathcal{D}_{1,n}\cup\mathcal{D}_{2,n})} \phi(d) + O(n^m(\#\mathcal{D}_{3,n})^2) \\ & + O\left(n^m \sum_{e>c_3(\log n)^{c_1+1}} \frac{1}{e} \sum_{\substack{k\in\mathcal{K}_n\\ g(k)\equiv 0 \,(\mathrm{mod}\, e)}} 1\right).\end{aligned} \tag{38}$$

The term on the left in (38) is obviously

$$\sum_{k\in\mathcal{K}_n} (C_0' k^m + O(k^{m-1})) = \frac{C_0' n^{m+1}}{m+1} + O\left(\frac{n^{m+1}}{\log n}\right).$$

The first term on the right in (38) is

$$A_2(n) - \sum_{d\in\mathcal{D}_{1,n}\cup\mathcal{D}_{2,n}} \phi(d) = A_2(n) + O\left(\frac{n^{m+1}}{\log n}\right),$$

by estimates (27) and (28). The second term on the right in (38) is of order $O(n^m(\log n)^{2c_4})$ by (35). For the last term on the right in (38), we use the

fact that

$$\sum_{\substack{k\in\mathcal{K}_n\\ g(k)\equiv 0 \,(\mathrm{mod}\, e)}} 1 = \rho_g(e)\left\lfloor \frac{\#\mathcal{K}_n}{e}\right\rfloor + O(\rho_g(e)) \ll \begin{cases} n\rho_g(e)/e & \text{if } e\le n,\\ \rho_g(e) & \text{if } e>n,\end{cases}$$

where ρ_g has the same meaning as in Section 3.3. Consequently, the last term on the right in (38) is of order

$$n^{m+1}\sum_{c_3(\log n)^{c_1+1}<e\le n}\frac{\rho_g(e)}{e^2} + n^m \sum_{\substack{n<e\\ e\mid g(k)\text{ for some } k\in\mathcal{K}_n}}\frac{\rho_g(e)}{e} =: S_1+S_2.$$

From the Ore–Nagell theorem (see [16]), we have $\rho_g(e)\ll m^{\omega(e)}$. Thus,

$$\begin{aligned} S_1 &= \frac{n^{m+1}}{(\log n)^{c_1+1}}\sum_{e\le n}\frac{\rho_g(e)}{e} \ll \frac{n^{m+1}}{(\log n)^{c_1+1}}\sum_{e\le n}\frac{m^{\omega(e)}}{e}\\ &= \frac{n^{m+1}}{(\log n)^{c_1+1}}\prod_{p\le n}\left(1+\frac{m}{p}+\frac{m}{p^2}+\cdots\right)\\ &\ll \frac{n^{m+1}}{(\log n)^{m+1}}\exp\left(\sum_{p\le n}\frac{m}{p}+O(1)\right)\\ &\ll \frac{n^{m+1}}{(\log n)^{c_1+1}}\exp(m\log\log n+O(1))\\ &\ll \frac{n^{m+1}}{(\log n)^{c_1+1-m}} = O\left(\frac{n^{m+1}}{\log n}\right). \end{aligned}$$

Here, we used the fact that $c_1\ge m$.

For S_2, we use the estimate $\omega(e)=o(\log e)$ as $e\to\infty$ to conclude that $\rho_g(e)\le m^{o(\log e)}=e^{o(1)}$ as $e\to\infty$. In particular, $\rho(e)<e^{1/2}$ for all $e>n$ and n sufficiently large. Thus,

$$\begin{aligned} S_2 &\ll n^m \sum_{\substack{n<e\\ e\mid g(k)\text{ for some } k\in\mathcal{K}_n}}\frac{1}{\sqrt{e}} \ll n^{m-1/2}\sum_{1\le k\le n}\tau(|g(k)|)\\ &\ll n^{m+1/2}(\log n)^{c_1+1} = O\left(\frac{n^{m+1}}{\log n}\right). \end{aligned}$$

So, the last term on the right in (38) is $S_1+S_2=O(n^{m+1}/\log n)$. From (38), we now get

$$A_2(n)=\frac{C_0' n^{m+1}}{m+1}+O\left(\frac{n^{m+1}}{\log n}\right).$$

Hence

$$\log \operatorname{lcm}[u_{a_1}, \dots, u_{a_n}] = \frac{C_0' \log |\alpha_1|}{m+1} n^{m+1} + O\left(\frac{n^{m+1}}{\log n}\right).$$

Since $\alpha_1 = \alpha_0/\sqrt{T_0} = \alpha^C/\sqrt{T_0}$, and

$$\log \Big| \prod_{\substack{1 \le k \le n \\ a_k \ne 0}} u_{a_k} \Big| = \frac{\log |\alpha|\, C_0}{m+1} n^{m+1} + O\left(\frac{n^{m+1}}{\log n}\right)$$

(see (25)), we get

$$\frac{\log |\prod_{1 \le k \le n,\, a_k \ne 0} u_{a_k}|}{\log \operatorname{lcm}[u_{a_1}, \dots, u_{a_n}]} = \frac{1}{1-\kappa_0} + O\left(\frac{1}{\log n}\right),$$

where

$$\kappa_0 = \frac{\gcd(A_0^2, B_0)}{2 \log |\alpha_0|} = \frac{\gcd((u_{2C}/u_C)^2, B^C)}{2 \log |\alpha|^C}.$$

It is easy to show using (5) that κ_0 does not depend on C, so in particular $\kappa_0 = \kappa$. The proof of Theorem 1 is finished.

3.4.2. *Proof of Theorem 2.* We start with the following lemma.

LEMMA 3. *We have $g(X) = (aX+b)^m$ for some coprime integers $a > 0$ and b.*

Proof. We can clearly write $g(X) = (aX+b)^m$ for some complex numbers a and b. Identifying the first two coefficients we get $C_0' = a^m$, $C_1' = ma^{m-1}b$, so $b/a = C_1'/(mC_0') \in \mathbb{Q}$. Further, $a^m = C_0' > 0$, so we may assume, up to replacing (a, b) by $(a\zeta, b\zeta)$, where ζ is some root of order m of unity, that $a = a_1 \rho^{1/m}$, where $a_1 > 0$ is an integer and $\rho > 0$ is an integer which is mth power free. Since $b/a \in \mathbb{Q}$ and $b^m = C_m'$, it follows that $b = b_1 \rho^{1/m}$ for some integer b_1. Thus, $g(X) = \rho(a_1 X + b_1)^m$, so ρ divides all the coefficients of $g(X)$, therefore $\rho = 1$. ∎

When $g(X)$ had at least two roots, we found a suitable set of large numbers $d = g(k)/e$ for which $r(d) = 1$, namely all numbers in $\mathcal{D}_n \setminus (\mathcal{D}_{1,n} \cup \mathcal{D}_{2,n})$ minus $\mathcal{D}_{3,n}$. In the present case, we replace this by the following.

LEMMA 4. *Every $d = g(k)/e \in \mathcal{D}_n \setminus (\mathcal{D}_{1,n} \cup \mathcal{D}_{2,n})$ can be uniquely represented as $d = g(k)/e$ for some e which is mth power free.*

Proof. This is trivial since if $g(k_1)/e_1 = g(k_2)/e_2$, then, by Lemma 3, we have $e_1/e_2 = ((ak_1+b)/(ak_2+b))^m$, and the number on the left is mth power free, while the number on the right is an mth power. Thus, both are equal to 1, so $e_1 = e_2$ and $k_1 = k_2$. ∎

We also need the following easy fact about multiplicative functions.

LEMMA 5. *We have*

$$n^m \prod_{p|n}\left(1-\frac{1}{p^m}\right) = \sum_{\substack{e|n^m \\ e \ m\text{th power free}}} \phi\left(\frac{n^m}{e}\right).$$

Proof. Both functions above are multiplicative, the one on the left for obvious reasons, while the one on the right because it is the convolution of the multiplicative function $n \mapsto \phi(n^m)$ with the characteristic function of the set of mth power free numbers. If $n = p^\alpha$ for some prime p and integer exponent $\alpha \geq 1$, then the formula becomes

$$\begin{aligned} p^{(\alpha-1)m}(p^m-1) &= \sum_{f=0}^{m-1} \phi(p^{\alpha m-f}) \\ &= \sum_{f=0}^{m-1} (p-1)p^{m\alpha-f-1} \\ &= (p-1)p^{(\alpha-1)m}(1+p+\cdots+p^{m-1}) \\ &= (p-1)p^{(\alpha-1)m}\left(\frac{p^m-1}{p-1}\right) \\ &= p^{(\alpha-1)m}(p^m-1), \end{aligned}$$

which is what we wanted. ∎

We now continue our argument. Instead of (36) which leads immediately to (37), we use Lemma 5 to deduce that the relation analogous to (37) in this case is

$$\sum_{\substack{e|ak+b \\ \mu(e)^2=1}} \mu(e)\left(\frac{ak+b}{e}\right)^m = \sum_{\substack{e|g(k) \\ e<c_3(\log n)^{c_1+1} \\ e \ m\text{th power free}}} \phi\left(\frac{(ak+b)^m}{e}\right) + O\left(n^m \sum_{\substack{e|g(k) \\ e>c_3(\log n)^{c_1+1}}} \frac{1}{e}\right).$$

We now sum up the above relation over all $k \in \mathcal{K}_n$ getting

$$\sum_{k\in\mathcal{K}_n} \sum_{\substack{e|ak+b \\ \mu(e)^2=1}} \mu(e)\left(\frac{ak+b}{e}\right)^m = \sum_{k\in\mathcal{K}_n} \sum_{\substack{e|g(k) \\ e<c_3(\log n)^{c_1+1} \\ e \ m\text{th power free}}} \phi\left(\frac{(ak+b)^m}{e}\right) + O\left(n^m \sum_{k\in\mathcal{K}_n} \sum_{\substack{e|g(k) \\ e>c_3(\log n)^{c_1+1}}} \frac{1}{e}\right). \tag{39}$$

The issue of overcounting elements in $\mathcal{D}_n$ no longer appears by Lemma 4, so the right-hand side of (39) above is equal to $A_2(n) + O(n^{m+1}/\log n)$. Note that if $e \mid ak+b$ for some $k \in \mathcal{K}_n$, then e and a are coprime and $e \le an+b$. We change the order of summation on the left hand side of (39):

$$
\begin{aligned}
(40)\qquad & \sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m} \sum_{\substack{k\in\mathcal{K}_n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} (a^m k^m + O(n^{m-1})) \\
&= a^m \sum_{\substack{1\le e\le an+b\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m} \sum_{\substack{k\in\mathcal{K}_n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m + O\Bigg(n^{m-1}\#\mathcal{K}_n \sum_{e\le an+b} \frac{1}{e}\Bigg) \\
&= a^m \sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m} \Bigg(\sum_{\substack{1\le k\le n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m - \sum_{\substack{1\le k\le n/(\log n)^{c_1+1}\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m \Bigg) \\
&\quad + O(n^m \log n) \\
&= a^m \sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m} \sum_{\substack{1\le k\le n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m + O(n^m \log n) \\
&\quad + O\Bigg(\frac{n^{m+1}}{(\log n)^{(m+1)(c_1+1)}} \sum_{e\le an+|b|} \frac{1}{e}\Bigg) \\
&= a^m \sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m} \sum_{\substack{1\le k\le n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m + O\Bigg(\frac{n^{m+1}}{\log n}\Bigg).
\end{aligned}
$$

For the inner sum, we use Abel's summation formula together with the fact that the counting function of the set of $k \le n$ such that $ak+b \equiv 0 \pmod e$ is $n/e + O(1)$. We get

$$
\begin{aligned}
\sum_{\substack{1\le k\le n\\ ak+b\equiv 0 \,(\mathrm{mod}\, e)}} k^m &= \Bigg(\frac{n}{e} + O(1)\Bigg) n^m - m \int_1^n \Bigg(\frac{t}{e} + O(1)\Bigg) t^{m-1}\, dt \\
&= \frac{n^{m+1}}{e} + O(n^m) - m \int_1^n \frac{t^m}{e}\, dt + O\Bigg(\int_1^n t^{m-1}\, dt\Bigg) \\
&= \frac{n^{m+1}}{e} - \Bigg(\frac{m t^{m+1}}{m+1}\Bigg|_{t=1}^{t=n}\Bigg) + O(n^m) = \frac{n^{m+1}}{(m+1)e} + O(n^m).
\end{aligned}
$$

Inserting this into (40), we get

$$
\begin{aligned}
A_2(n) &= a^m \sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^m}\left(\frac{n^{m+1}}{(m+1)e}+O(n^m)\right)+O\left(\frac{n^{m+1}}{\log n}\right)\\
&= \frac{a^m n^{m+1}}{m+1}\sum_{\substack{1\le e\le an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^{m+1}}+O\left(n^m\sum_{e\le an+b}\frac{1}{e}+\frac{n^{m+1}}{\log n}\right)\\
&= \frac{a^m n^{m+1}}{m+1}\left(\sum_{\substack{e\ge 1\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^{m+1}}-\sum_{\substack{e>an+b\\(e,a)=1\\ \mu(e)^2=1}} \frac{\mu(e)}{e^{m+1}}\right)+O\left(\frac{n^{m+1}}{\log n}\right)\\
&= \frac{a^m n^{m+1}}{m+1}\prod_{p\nmid a}\left(1-\frac{1}{p^{m+1}}\right)+O\left(n^{m+1}\sum_{e>an+b}\frac{1}{e^2}+\frac{n^{m+1}}{\log n}\right)\\
&= \left(\frac{a^m\zeta(m+1)^{-1}}{m+1}\prod_{p\mid a}\left(1-\frac{1}{p^{m+1}}\right)^{-1}\right)n^{m+1}+O\left(\frac{n^{m+1}}{\log n}\right)\\
&= \left(\frac{C_0'\zeta(m+1)^{-1}}{m+1}\prod_{p\mid a}\left(1-\frac{1}{p^{m+1}}\right)^{-1}\right)n^{m+1}+O\left(\frac{n^{m+1}}{\log n}\right).
\end{aligned}
$$

So, we conclude that

$$
\log\Big|\prod_{\substack{1\le k\le n\\ a_k\ne 0}} u_{a_k}\Big| = \frac{\log|\alpha|\,C_0}{m+1}n^{m+1}+O\left(\frac{n^{m+1}}{\log n}\right),
$$

while

$$
\begin{aligned}
\log \operatorname{lcm}[u_{a_1},\dots,u_{a_n}] &= \left(\frac{\log|\alpha_1|\,C_0'}{(m+1)\zeta(m+1)}\prod_{p\mid a}\left(1-\frac{1}{p^{m+1}}\right)^{-1}\right)n^{m+1}\\
&\quad +O\left(\frac{n^{m+1}}{\log n}\right),
\end{aligned}
$$

This leads to

$$
\frac{\log\left|\prod_{1\le k\le n,\,a_k\ne 0} u_{a_k}\right|}{\log \operatorname{lcm}[u_{a_1},\dots,u_{a_n}]} = \frac{\zeta(m+1)}{1-\kappa}\prod_{p\mid a}\left(1-\frac{1}{p^{m+1}}\right)+O\left(\frac{1}{\log n}\right).
$$

Thus, we obtained Theorem 2.

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