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A note on sumsets of subgroups in $\mathbb{Z}_p^*$

by

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For subsets $A_1, \ldots, A_k$ of a group define $A_1 + \cdots + A_k = \{a_1 + \cdots + a_k : a_i \in A_i, 1 \le i \le k\}$. In the case that all the subsets are equal we will denote the k-fold sumset of A by $kA = \{x_1 + \cdots + x_k : x_i \in A,\ 1 \le i \le k\}$.

Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$. What is the smallest $\alpha > 0$ such that $|A| \gg p^{\alpha}$ implies that $2A$ contains $\mathbb{Z}_p^*$?

Conjecture 0.1. *Let $|A| > p^{1/2+\epsilon}$, $\epsilon > 0$. Then $2A$ contains $\mathbb{Z}_p^*$.*

It is relatively simple, using exponential sum bounds, to show that if $|A| > p^{3/4}$ then $2A \supseteq \mathbb{Z}_p^*$. Surprisingly, no improvement in the exponent has been made. An alternative approach would be to consider this conjecture from an inverse perspective. Let $|A| > p^{1/2+\epsilon}$; what is the smallest k_0 such that $k_0 A$ contains $\mathbb{Z}_p^*$? A direct application of classical counting methods using standard exponential sum bounds does not seem to yield any answer to this question. For example, using the fact that $\max_{\lambda \neq 0} |\sum_{x \in A} e_p(x\lambda)| \le \sqrt{p}$ one may show that if $|A| > p^{1/2+1/(2k)}$ then kA contains $\mathbb{Z}_p^*$.

Using combinatorial methods Glibichuk [1] gave the first answer to this question showing that $8A \supseteq \mathbb{Z}_p^*$ for $|A| \ge 2p^{1/2}$. Using an improved exponential sum bound, Schoen and Shkredov [3, Theorem 2.6] showed that $7A \supseteq \mathbb{Z}_p^*$ for $|A| > p^{1/2}$. There was subsequent improvement to this result by Shkredov and Vyugin [7] followed by Schoen and Shkredov [4]. Recently, Shkredov [5] has shown that $6A \supseteq \mathbb{Z}_p^*$ if $|A| > p^{55/112+\epsilon} = p^{.491\ldots+\epsilon}$.

In this paper we elaborate on the methods in the above-mentioned papers to show that $6A \supseteq \mathbb{Z}_p^*$ if $|A| > p^{11/23+\epsilon} = p^{.478\ldots+\epsilon}$. In addition, we extend a result of Shkredov [5] to show that $|2A| \gg |A|^{8/5-\epsilon}$ for $|A| \ll p^{5/9}$.

1. Statement of main results. Let A and B be subsets of $\mathbb{Z}_p$. Given a set A we will denote the indicator function of A by $A(\cdot)$. Define the convolution of A and B by $(A * B)(z) = \sum_{x+y=z} A(x)B(y) = |A \cap (z - B)|$.

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The *additive energy* between A and B is given by

$$\begin{aligned} E(A,B) &= |\{(x,y,z,w) \in A \times B \times A \times B : x+y=z+w\}| \\ &= \sum_z (A*B)^2(z) = \sum_z |A \cap (z-B)|^2 \\ &= \sum_z (A*-A)(z)(B*-B)(z) = \sum_z |A_z|\,|B_z|; \end{aligned}$$

here and throughout, we let $C_z = C \cap (C+z)$ for any subset C of $\mathbb{Z}_p$. In the case that $A=B$ we will write $E(A)=E(A,A)$. Similarly, we will denote the *rth additive energy* of a subset A by $E_r(A) = \sum_s |A_s|^r$.

One may also consider the additive energy in the frequency domain. Taking an exponential sum expansion, we obtain

$$E(A,B) = p^{-1} \sum_s \Big| \sum_{x\in A} e_p(sx) \Big|^2 \Big| \sum_{y \in B} e_p(sy) \Big|^2,$$

where $e_p(x) = e^{2\pi i x/p}$. For a subset A of $\mathbb{Z}_p$ we define

$$\Phi_A = \max_{\lambda \neq 0} \Big| \sum_{x \in A} e_p(\lambda x) \Big|.$$

Heath-Brown and Konyagin employed Stepanov's method in order to give a bound on the additive energy of multiplicative subgroups of $\mathbb{Z}_p^*$.

THEOREM 1.1 ([2]). *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$E(A) \ll |A|^{5/2}.$$

In [5] Shkredov gave the following combinatorial lemma.

LEMMA 1.2 ([5, equation (1)]). *Let A be a finite subset of an abelian group. Then*

$$\sum_s \frac{|A_s|^2}{|A+A_s|} \ll |A|^{-2} E_3(A).$$

Schoen and Shkredov ([3]) gave an estimate for $E_3(A)$.

LEMMA 1.3 ([3, Lemma 3.3]). *Let A be a multiplicative subgroup A of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$E_3(A) \ll |A|^3 \log(|A|).$$

Combining Lemmas 1.2 and 1.3 and noting that $|A+A_s| \le |(2A)_s|$ gives the following lemma.

LEMMA 1.4. *Let A be a multiplicative subgroup A of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$\sum_s \frac{|A_s|^2}{|(2A)_s|} \ll |A| \log(|A|).$$

Shkredov used this inequality in [5] to give the following estimate on the additive energy.

THEOREM 1.5 ([5, Theorem 30]). *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ such that $|A| \ll p^{2/3}$. If $E(A) \ll |A|^{3/2}\sqrt{p}\log(|A|)$ then*

$$E(A) \ll |A|^{4/3}|2A|^{2/3} \log(|A|).$$

In addition, using different methods he proved an energy estimate independent of the size of the sumset.

THEOREM 1.6 ([5, Theorem 34]). *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ such that $|A| \ll p^{2/3}$. Then*

$$E(A) \ll \max\{|A|^{22/9} \log(|A|), |A|^3 p^{-1/3} \log^{4/3}(|A|)\}.$$

Combining Theorems 1.5 and 1.6 and applying the trivial estimate $|2A| \geq |A|^4 E^{-1}(A)$ gives the following sumset estimate.

THEOREM 1.7. *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$|2A| \gg \begin{cases} |A|^{8/5} \log^{-3/5}(|A|) & \text{if } |A| \ll p^{9/17}, \\ |A|^{14/9} \log^{-1}(|A|) & \text{if } |A| \ll p^{3/5} \log^{3/5}(|A|), \\ |A| p^{1/3} \log^{-4/3}(|A|) & \text{if } |A| \gg p^{3/5} \log^{3/5}(|A|). \end{cases}$$

Here we give the following energy estimate.

THEOREM 1.8. *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$E(A) \ll \max\{|A|^{4/3}|2A|^{2/3} \log^{1/2}(|A|), |A|\,|2A|^2 p^{-1} \log(|A|)\}.$$

This allows us to improve Shkredov's sumset result in some ranges.

THEOREM 1.9. *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ with $|A| \ll p^{2/3}$. Then*

$$|2A| \gg \begin{cases} |A|^{8/5} \log^{-3/10}(|A|) & \text{if } |A| \ll p^{5/9} \log^{-1/18}(|A|), \\ |A| p^{1/3} \log^{-1/3}(|A|) & \text{if } |A| \gg p^{5/9} \log^{-1/18}(|A|). \end{cases}$$

Using the Plancherel identity or orthogonality one can very quickly prove that $\Phi_A \ll \sqrt{p}$ for a multiplicative subgroup A with $|A| \gg p^{1/2}$. This is only non-trivial when $|A| > p^{1/2}$. Shparlinski [6] improved this result in some ranges with the bound $\Phi_A \ll |A|^{7/12} p^{1/6}$ for $p^{2/5} \ll |A| \ll p^{4/7}$. Heath-Brown and Konyagin [2] used the energy estimate of Theorem 1.1 to obtain the following improvement.

THEOREM 1.10. *Let A be a multiplicative subgroup. Then*

$$\Phi_A \ll \begin{cases} \sqrt{p} & \text{if } p^{2/3} \ll |A| \le p, \\ p^{1/4}|A|^{-1/4}E^{1/4}(A) \ll p^{1/4}|A|^{3/8} & \text{if } p^{1/2} \ll |A| \ll p^{2/3}, \\ p^{1/8}E^{1/4}(A) \ll p^{1/8}|A|^{5/8} & \text{if } p^{1/3} \ll |A| \ll p^{1/2}. \end{cases}$$

Using Shkredov's energy estimate, one may improve this result in some ranges when the sumset is small. Let $|A| \ll p^{1/2}$; then

$$\Phi_A \ll p^{1/8}|A|^{1/3}|2A|^{1/6}\log^{1/4}(|A|).$$

Using the same methods employed to prove Lemma 1.3 one may obtain $E_{3/2}(A) \ll |A|^{9/4}$. If the sumset is small we are able to significantly improve this bound.

LEMMA 1.11. *Let A be a multiplicative subgroup with $|A| \ll p^{1/2}$. Then*

$$E_{3/2}(A) \ll |A|^{1/2}|2A|\log^{7/4}|A|.$$

This lemma allows us to obtain the following exponential sum bound which is an improvement of the result of Shkredov as long as $|2A| \ll |A|^{7/4-\epsilon}$.

LEMMA 1.12. *Let A be a multiplicative subgroup with $|A| \ll p^{1/2}$. Then*

$$\Phi_A \ll p^{1/8}|A|^{-1/8}|2A|^{1/4}E^{1/8}(|A|)\log^{7/16}(|A|).$$

In particular, applying Theorem 1.8 we have

$$\Phi_A \ll p^{1/8}|A|^{1/24}|2A|^{1/3}\log^{1/2}(|A|).$$

With Lemma 1.12 in tow, we may now prove our main result.

THEOREM 1.13. *Let A be a multiplicative subgroup of $\mathbb{Z}_p^*$ with $|A| \gg p^{11/23}\log^{12/23}(|A|)$. Then*

$$6A \supseteq \mathbb{Z}_p^*.$$

Proof. Fix a in $\mathbb{Z}_p^*$. We may assume that $|A| \ll p^{1/2}$ as the result is already known in the range $|A| \gg p^{1/2}$.

Let N be the number of solutions to the equation

$$x_1 + x_2 + y_1 + y_2 = ay_3$$

with $x_1, x_2 \in 2A$ and $y_1, y_2, y_3 \in A$. Taking an exponential sum expansion, we obtain

$$N = \frac{|2A|^2|A|^3}{p} + \frac{1}{p}\sum_{\lambda\neq 0}\Big(\sum_{x\in 2A} e_p(\lambda x)\Big)^2\Big(\sum_{y\in A} e_p(\lambda y)\Big)^2\Big(\sum_{z\in A} e_p(-\lambda z a)\Big),$$

which by the Plancherel identity implies $N > 0$ as long as $|2A|\,|A|^3 > p\Phi_A^3$. Applying Lemma 1.12 gives the condition

$$|2A|\,|A|^3 \gg p^{11/8}|2A|\,|A|^{1/8}\log^{3/2}(|A|),$$

which in turn leads to

$$|A| \gg p^{11/23} \log^{12/23}(|A|). \blacksquare$$

2. A few preliminary lemmas. We begin with a lemma of Shkredov and Vyugin [7, Corollary 5.1] which is a generalization of a result of Heath-Brown and Konyagin [2]. We say that a subset $S \neq \{0\}$ is *A-invariant* if $SA = \{sa : s \in S,\ a \in A\} = S$, that is, S is a union of cosets of A and possibly $\{0\}$.

Lemma 2.1 (Shkredov and Vyugin [7, Corollary 5.1]). *Let A be a multiplicative subgroup of $\mathbb{Z}_p$ and let S_1, S_2, S_3 be A-invariant sets such that $|S_1 \setminus \{0\}|\,|S_2 \setminus \{0\}|\,|S_3 \setminus \{0\}| \ll \min\{|A|^5, p^3|A|^{-1}\}$. Then*

$$\sum_{z\in S_3} (S_1 * S_2)(z) \ll |A|^{-1/3}(|S_1|\,|S_2|\,|S_3|)^{2/3}.$$

Remark. The above lemma has been modified slightly from its original form in order to allow S_1, S_2, S_3 to contain the zero element. One may check that the additional terms in $\sum_{z\in S_3}(S_1 * S_2)(z)$ allowing S_1, S_2 to contain the zero element only affect the implied constant.

We can now give a variant of a result of Shkredov [5].

Lemma 2.2 ([5, Corollary 18]). *Let $k \gg 1$ and S_1, S_2 be A-invariant sets and let M be any A-invariant subset of the set $\{z : (S_1 * S_2)(z) \geq k\}$. If $|S_1|\,|S_2|\,|M|\,|A| \ll \min\{|A|^6, p^3\}$ then*

$$\sum_{z\in M} (S_1 * S_2)^r(z) \ll |S_1|^2|S_2|^2|A|^{-1}k^{r-3} \quad \textit{for } 1 \leq r < 3,$$

$$\sum_{z\in M} (S_1 * S_2)^3(z) \ll |S_1|^2|S_2|^2|A|^{-1} \log(|S_1|^2|S_2|^2|A|^{-2}k^{-3}).$$

Proof. Let $l_i = (S_1 * S_2)(z_i)$, $z_i \neq 0$, where $l_1 \geq l_2 \geq \cdots$ are arranged in decreasing order. For each z in the coset $aA = \{aa' : a' \in A\}$, $a \in \mathbb{Z}_p$, note that $(S_1 * S_2)(z) = (S_1 * S_2)(a)$. By the coset a_iA we will mean the coset on which $l_i = (S_1 * S_2)(a_i)$. Let M be any A-invariant subset of the set $\{z : (S_1 * S_2)(z) \geq k\}$ and $M_i = \bigcup_{j=1}^{i} a_jA \subseteq M$. From Lemma 2.1 we have

$$l_i|A|i \leq \sum_{j=1}^{i} |A|l_j \leq \sum_{z\in M_i} (S_1 * S_2)(z) \ll i^{2/3}|A|^{1/3}|S_1|^{2/3}|S_2|^{2/3},$$

as long as $i|A|\,|S_1|\,|S_2| \ll |M|\,|S_1|\,|S_2| \ll \min\{|A|^5, p^3|A|^{-1}\}$. Now,

$$\sum_{z\in M}(S_1*S_2)^r(z)\le |A|\sum_{i\ll|S_1|^2|S_2|^2|A|^{-2}k^{-3}} l_i^r$$
$$\ll |A|\sum_{i\ll|S_1|^2|S_2|^2|A|^{-2}k^{-3}} (i^{-1/3}|A|^{-2/3}|S_1|^{2/3}|S_2|^{2/3})^r. \ \blacksquare$$

3. Additive energy bound: Proof of Theorem 1.8. We may assume that

$$E(A)\gg \max\{|A|^{4/3}|2A|^{2/3}\log^{1/2}(|A|), |A|\,|2A|^2p^{-1}\log(|A|)\}.$$

Combining this with the energy estimate from Theorem 1.1 we may also assume that

$$|2A|\ll \max\{|A|^{7/4}\log^{-3/4}(|A|), |A|^{3/4}p^{1/2}\log^{-1/2}(|A|)\}.$$

Write

$$E(A)=\sum_s |A_s|^2\ll \sum_{s\in M_1}|A_s|^2,$$

where $M_1=\{s:|A_s|\gg k_1:=|A|^{-2}E(A)\}$. Note that we have the trivial estimate $|M_1|\ll |A|^2k_1^{-1}=|A|^4E^{-1}(|A|)$. Now Lemma 1.4 gives

$$E(A)=\sum_s|A_s|^2\ll \frac{E(A)}{|A|\log(A)}\sum_{s\in M_2^c}\frac{|A_s|^2}{|(2A)_s|}+\sum_{s\in M_2}|A_s|^2\ll \sum_{s\in M_2}|A_s|^2,$$

where $M_2=\{s:s\in M_1,\ |(2A)_s|\gg k_2:=|A|^{-1}\log^{-1}(|A|)E(A)\}$.

By Lemma 2.1 we have $k_2|M_2|\ll |A|^{-1/3}|2A|^{4/3}|M_2|^{2/3}$, yielding $|M_2|\ll |2A|^4|A|^{-1}k_2^{-3}$ as long as $|2A|^2|M_2|\ll \min\{|A|^5, p^3|A|^{-1}\}$. In order to see that the first bound holds, one may note that $|M_2|\ll|M_1|$ combined with our assumptions on the size of energy and sumset. To show that $|2A|^2|M_2|\ll p^3|A|^{-1}$ we use an exponential sum expansion,

$$|M_2|k_2\ll \sum_{s\in M}|(2A)_s|\ll \frac{1}{p}\sum_m\Big|\sum_{x\in 2A}e_p(xm)\Big|^2\Big(\sum_{x\in M_2}e_p(xm)\Big),$$

together with the bound $\max_{m\ne 0}|\sum_{x\in M_2}e_p(xm)|\ll p^{1/2}|M_2|^{1/2}|A|^{-1/2}$, to deduce

$$|M_2|k_2\ll \max\{p^{-1}|2A|^2|M_2|, p^{1/2}|2A|\,|M_2|^{1/2}|A|^{-1/2}\}.$$

If the first of these two bounds holds then $E(A)\ll |A|\,|2A|^2p^{-1}\log(|A|)$. We may then assume that $|M_2|\ll p|2A|^2|A|^{-1}k_2^{-2}$, which implies that $|2A|^2|M_2|\ll p|2A|^4|A|\log^2(|A|)E^{-2}(A)\ll p^3|A|^{-1}$.

Therefore, for $|A|\ll p^{2/3}$, we have $|M_2|\ll |2A|^4|A|^{-1}k_2^{-3}$. Using this fact we may again reduce the number of terms:

$$E(A) = \sum_s |A_s|^2 \ll k_3^2|M_2| + \sum_{s\in M_3} |A_s|^2 \ll \sum_{s\in M_3} |A_s|^2,$$

where $M_3 = \{s : s \in M_2,\ |A_s| \gg k_3 := |2A|^{-2}|A|^{-1}\log^{-3/2}(|A|)E^2(A)\}$.

Finally, applying Lemma 2.2 we have

$$E(A) \ll |A|^4|2A|^2\log^{3/2}(|A|)E^{-2}(|A|),$$

as long as $|A|^2|M_3| \ll |2A|^2|M_2| \ll \min\{|A|^5, p^3|A|^{-1}\}$.

4. $E_{3/2}(A)$**: Proof of Lemma 1.11.** Let $l_i = |A_{z_i}|$, $z_i \neq 0$, where $l_1 \geq l_2 \geq \cdots$ are arranged in decreasing order. For each z in the coset $aA = \{aa' : a' \in A\}$, $a \in \mathbb{Z}_p$, note that $|A_z| = |A_a|$. By the coset a_iA we will mean the coset on which $l_i = |A_{a_i}|$. Let M be any A-invariant subset of the set $\{z : |A_z| \geq k\}$, and $M_i = \bigcup_{j=1}^{i} a_jA \subseteq M$. Set $k = |2A|^2|A|^{-3}$.

We have

$$l_i|A|i \leq \sum_{j=1}^{i} |A|l_j \leq \sum_{z\in M_i} |A_z|.$$

Now

$$\sum_{z\in M_i} |A_z| = \sum_{z\in M_i} \frac{|A_z|}{|(2A)_z|^{1/2}}|(2A)_z|^{1/2} \leq \Big(\sum_z \frac{|A_z|^2}{|2A_z|}\Big)^{1/2}\Big(\sum_{z\in M_i}|2A_z|\Big)^{1/2}.$$

Therefore, by Lemma 1.4,

$$l_i^2|A|^2i^2 \ll |A|\log(|A|)\sum_{z\in M_i}|2A_z|.$$

Noting that $|M_i| \ll |A|^2k^{-1}$ we have $|M_i|\,|2A|^2 \ll |A|^5$. Hence we can apply Lemma 2.1 to get

$$l_i^2|A|^2i^2 \ll |2A|^{4/3}i^{2/3}|A|^{4/3}\log|A|.$$

This implies

$$l_i \ll |2A|^{2/3}i^{-2/3}|A|^{-1/3}\log^{1/2}|A|$$

for $i \ll |A-A|\,|A|^{-1} \leq |A|$. Finally

$$\begin{aligned}\sum_z |A_z|^{3/2} &\ll k^{1/2}|A|^2 + |A|\sum_{i\ll|A|}|l_i|^{3/2}\\ &\ll k^{1/2}|A|^2 + |A|^{1/2}|2A|\log^{7/4}(|A|),\end{aligned}$$

giving the desired result.

5. Exponential sum bound: Proof of Lemma 1.12. We begin by expanding the sum below and performing a basic substitution:

$$|A|\Big|\sum_{x\in A} e_p(\lambda x)\Big|^2 = \sum_{y\in A}\Big|\sum_{x\in A} e_p(\lambda yx)\Big|^2$$
$$= \sum_{x_1,x_2\in A}\sum_{y\in A} e_p(\lambda y(x_1-x_2)) = \sum_s |A_s| \sum_{y\in A} e_p(\lambda ys).$$

Now we may take absolute values and estimate from above:

$$|A|\Phi_A^2 \le \sum_s |A_s|\Big|\sum_{y\in A} e_p(\lambda ys)\Big|.$$

Applying Hölder's inequality we have

$$|A|\Phi_A^2 \ll \Big(\sum_s |A_s|^{4/3}\Big)^{3/4}\Big(\sum_s\Big|\sum_{y\in A} e_p(\lambda ys)\Big|^4\Big)^{1/4},$$

which by the Plancherel identity gives

$$|A|\Phi_A^2 \ll \Big(\sum_s |A_s|^{4/3}\Big)^{3/4} p^{1/4}E^{1/4}(A). \tag{5.1}$$

Another application of Hölder's inequality shows that

$$\sum_s |A_s|^{4/3} = \sum_s |A_s|\,|A_s|^{1/3} \ll \Big(\sum_s |A_s|^{3/2}\Big)^{2/3}|A|^{2/3},$$

and by Lemma 1.11,

$$\sum_s |A_s|^{4/3} \ll |A|^{2/3}(|A|^{1/2}|2A|\log^{7/4}(|A|))^{2/3} \ll |A|\,|2A|^{2/3}\log^{7/6}(|A|).$$

Putting this estimate into (5.1) gives the stated result.

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