ACTA ARITHMETICA
157.1 (2013)

Multiplicative functions over short segments

by

Olivier Bordellès (Aiguilhe)

1. Introduction and result. In what follows, $[t]$ is the integer part of t and $\varepsilon > 0$ is an arbitrary small real number which does not need to be the same at each occurrence.

In contrast to the long-sum case, there are few articles dealing with short sums of multiplicative functions in the literature. By "short sums" we mean sums of the shape

$$\sum_{x<n\le x+y} f(n) \tag{1}$$

where $10 \le y \le x$ are large real numbers with $y = o(x)$ as $x \to \infty$. In particular, a result analogous to the well-known Halász theorem for the short-sum case would be of great importance in some problems of number theory. However, this question still remains open and even the weaker case of functions satisfying the Wirsing conditions is still unsolved.

The usual tools from analytic number theory used for sums over long intervals may not work for sums of the type (1) or may provide results weaker than expected. For instance, using Srinivasan's multidimensional exponent pairs, Varbanec [11] showed that

$$\sum_{x<n\le x+y} \mu_2(n) = \frac{y}{\zeta(2)} + O(x^{0.2196} + y^{1/2}),$$

whilst using elementary means the authors in [2] proved that the main term $x^{0.2196}$ may be improved to $x^{1/5}\log x$. Here and below, $\mu_2(n) = 1$ whenever n is squarefree and 0 otherwise.

In [1], we proved that if f is a $[0,1]$-valued multiplicative function satisfying $f(p) = 1$ for all primes p, then, for all $\varepsilon > 0$, we have

$$\sum_{x<n\le x+y} f(n) = y\mathcal{M}_f + O(x^{1/15+\varepsilon}y^{2/3})$$

2010 *Mathematics Subject Classification*: 11A25, 11L07.
Key words and phrases: short interval estimates, integer points close to smooth curves.

DOI: 10.4064/aa157-1-1

uniformly for $2 \le y \ll x^{1/2}$, where

$$\mathcal{M}_f = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^\alpha}\right). \tag{2}$$

Recently, Kapoor [9] extended this estimate to a larger class of complex-valued multiplicative functions f such that

$$|f(p) - 1| \le A/p^\theta \quad \text{and} \quad |f(p^\alpha)| \le A$$

for some $A > 0$ and $\theta \ge 4/5$.

The aim of this work is both to generalize Varbanec's estimate and to extend the above results. To this end, let $k \ge 2$ be an integer, $A \ge 0$ and $\varepsilon > 0$ be real numbers, and define the class $\mathcal{S}(k; A, \varepsilon)$ to be the set of complex-valued multiplicative functions f satisfying

$$|f(p^\alpha) - f(p^{\alpha-1})| \le A/p^\alpha \quad \text{if } 1 \le \alpha < k, \tag{3}$$

$$|(f \star \mu)(n)| \le c_{\varepsilon,k} n^\varepsilon \quad \text{if } n \text{ is } k\text{-full}, \tag{4}$$

for some constant $c_{\varepsilon,k} > 0$ depending at most on ε and k and where $f \star g$ is the usual Dirichlet convolution product of the two arithmetic functions f and g. Recall that an integer $n = p_1^{e_1} \cdots p_r^{e_r}$ is said to be *k-full* if $e_i \ge k$ for all i. If $e_i < k$ for all i, then n is *k-free* and μ_k is the characteristic function of the set of k-free numbers.

Before stating our main result, note that if $g = f \star \mu$ is the Eratosthenes transform of f, then by (3) and (4) we readily get

$$|g(n)| \le \begin{cases} A^{\omega(n)} n^{-1} & \text{if } n \text{ is } k\text{-free}, \\ c_{\varepsilon,k} n^\varepsilon & \text{if } n \text{ is } k\text{-full}. \end{cases} \tag{5}$$

We are now in a position to state our main theorem.

THEOREM 1.1. *Let $k \ge 2$ be an integer, $A \ge 0$ and $\varepsilon > 0$ be real numbers and let $f \in \mathcal{S}(k; A, \varepsilon)$. Then uniformly for $4^k \le y < x$ we have*

$$\sum_{x < n \le x+y} f(n) = y\mathcal{M}_f + O_{\varepsilon,k,A}\left(x^{\frac{1}{2k+1}+\varepsilon} + yx^{-\frac{1}{6(4k-1)(2k-1)}+\varepsilon} + y^{1-\frac{2(k-1)}{k(3k-1)}} x^\varepsilon\right)$$

where $\mathcal{M}_f$ is defined in (2)*.*

This article is organized as follows. Section 2 supplies some recent results from the theory of counting integer points lying very near certain smooth curves. This theory has essentially been developed by Huxley–Sargos [6, 7, 8] and Filaseta–Trifonov [2, 3] in order to investigate distribution problems in number theory. Section 3 is devoted to the proof of some crucial lemmas needed in the proof of the main result, which is postponed to Section 4. Finally, several applications of Theorem 1.1 will be given in Section 5.

2. Integer points near smooth curves. We summarize in the next lemma the main tools picked up from the work of Huxley–Sargos [8, Théorème 2] and Filaseta–Trifonov [3, Theorem 7]. In this section, N is a large positive integer, δ is a small positive real number and $\mathcal{R}(f,N,\delta)$ is the number of integers $n \in [N,2N]$ such that $\|f(n)\| < \delta$, where as usual $\|x\|$ is the distance between the real number x and its nearest integer.

Lemma 2.1.

(i) *Let $s \geq 3$ be an integer, $\delta \in (0,1/4)$ and $\varphi \in C^s[N,2N]$ such that there exist $\lambda_{s-1}, \lambda_s > 0$ such that, for all $x \in [N,2N]$, we have*

$$|\varphi^{(s-1)}(x)| \asymp \lambda_{s-1}, \qquad |\varphi^{(s)}(x)| \asymp \lambda_s \quad \textit{and} \quad \lambda_{s-1} = N\lambda_s.$$

Then

$$\mathcal{R}(\varphi,N,\delta) \ll N\lambda_s^{\frac{2}{s(s+1)}} + N(\delta\lambda_{s-1})^{\frac{2}{s(s-1)+2}} + N\delta^{\frac{2}{(s-1)(s-2)}} + (\delta/\lambda_{s-1})^{\frac{1}{s-1}} + 1.$$

(ii) *Let $r \geq 2$ be an integer, $\delta > 0$ and $x \geq 1$ such that $4 \leq N \leq x^{1/r}$. Then there exists a constant $c_r > 0$ depending only on r such that if $N^{r-1}\delta \leq c_r$, then*

$$\mathcal{R}(x/n^r,N,\delta) \ll x^{\frac{1}{2r+1}} + x^{\frac{1}{6r+3}}\delta N^{r-1/3}.$$

Lemma 2.2. *Let $k \geq 2$ be an integer, $A \geq 0$, $\varepsilon > 0$, $f \in \mathcal{S}(k;A,\varepsilon)$ and let $g = f \star \mu$. For any real number $z \geq 1$, we have*

$$\sum_{n \leq z} |g(n)| \ll_{\varepsilon,k,A} z^{1/k+\varepsilon} \quad \textit{and} \quad \sum_{n > z} \frac{|g(n)|}{n} \ll_{\varepsilon,k,A} z^{-1+1/k+\varepsilon}.$$

Proof. Writing uniquely $n = ab$ with a k-free, b k-full and $(a,b)=1$ and using (5), we get

$$\sum_{n \leq z} |g(n)| \leq \sum_{\substack{a \leq z \\ a\ k\text{-free}}} \frac{A^{\omega(a)}}{a} \sum_{\substack{b \leq z/a \\ b\ k\text{-full}}} |g(b)| \ll z^{2\varepsilon} \sum_{a \leq z} \frac{1}{a} \sum_{\substack{b \leq z/a \\ b\ k\text{-full}}} 1$$
$$\ll z^{1/k+2\varepsilon} \sum_{a \leq z} \frac{1}{a^{1+1/k}} \ll_{\varepsilon,k,A} z^{1/k+2\varepsilon}.$$

The second inequality follows at once by partial summation. We leave the details to the reader. ∎

3. Fundamental lemmas

Lemma 3.1. *Let $r \geq 2$ be a fixed integer. Uniformly for $x \geq 1$ and $0 < y < x$ we have*

$$\sum_{(2y)^{1/r} < n \leq (2x)^{1/r}} \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right) \ll_r \left(x^{\frac{1}{2r+1}} + y x^{-\frac{1}{6r(2r+1)}} + y^{1/r} \right) \log(ex).$$

Proof. ► *Case* $0 < y \leq 4^r$. For any integer $m \leq x + 4^r$, we have

$$\tau(m) \leq \exp\left(\frac{2^{2r+1}(2r+1)}{\log 2} \right) m^{\frac{1}{2r+1}} \ll_r x^{\frac{1}{2r+1}}$$

so that

$$\sum_{(2y)^{1/r} < n \leq (2x)^{1/r}} \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right) = \sum_{x < m \leq x+y} \sum_{\substack{n^r \mid m \\ (2y)^{1/r} < n \leq (2x)^{1/r}}} 1$$

$$\leq \sum_{x < m \leq x + 4^r} \tau(m) \ll_r x^{\frac{1}{2r+1}}.$$

► *Case* $4^r \leq y \leq x^{1/(2r)}$. Let $c_r > 0$ be the constant appearing in Lemma 2.1(ii) and write

$$\sum_{(2y)^{1/r} < n \leq (2x)^{1/r}} \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right)$$

$$= \Bigg(\sum_{(2y)^{1/r} < n \leq c_r^{-1} x^{1/(2r)}} + \sum_{c_r^{-1} x^{1/(2r)} < n \leq x^{1/r}} + \sum_{x^{1/r} < n \leq (2x)^{1/r}} \Bigg) \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right)$$

$$= \Bigg(\sum_{(4y)^{1/r} < n \leq c_r^{-1} x^{1/(2r)}} + \sum_{c_r^{-1} x^{1/(2r)} < n \leq x^{1/r}} + \sum_{x^{1/r} < n \leq (2x)^{1/r}} \Bigg) \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right)$$

$$+ O(y^{1/r})$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + O(y^{1/r}).$$

▷ For Σ_3, we easily see that $[x/n^r] = 0$ and $[(x+y)/n^r] = 1$ if $x^{1/r} < n \leq (x+y)^{1/r}$ and 0 otherwise, so that

$$\Sigma_3 = \sum_{x^{1/r} < n \leq (x+y)^{1/r}} 1 \leq (x+y)^{1/r} - x^{1/r} + 1 \ll y x^{-1+1/r} + 1 \ll y^{1/r}.$$

▷ Since $y \leq x^{1/(2r)}$, we may apply Lemma 2.1(ii) to Σ_2, which yields

$$\Sigma_2 \ll \max_{c_r^{-1} x^{1/(2r)} < N \leq x^{1/r}} \mathcal{R}(x/n^r, N, y/N^r) \log(ex)$$

$$\ll \max_{c_r^{-1} x^{1/(2r)} < N \leq x^{1/r}} \left(x^{\frac{1}{2r+1}} + x^{\frac{1}{6r+3}} y N^{-1/3} \right) \log(ex)$$

$$\ll \left(x^{\frac{1}{2r+1}} + y x^{-\frac{1}{6r(2r+1)}} \right) \log(ex).$$

▷ The sum Σ_1 is estimated using Lemma 2.1(i) with $s = 2r$. This gives

$$\Sigma_1 \ll \max_{(4y)^{1/r} < N \le c_r^{-1} x^{1/(2r)}} \Big(x^{\frac{1}{r(2r+1)}} N^{\frac{2r-2}{2r+1}} + (xy)^{\frac{1}{2r^2-r+1}} N^{\frac{(r-2)(2r-1)}{2r^2-r+1}} + N^{\frac{2r^2-4r+1}{(r-1)(2r-1)}} y^{\frac{1}{(r-1)(2r-1)}} + N(yx^{-1})^{\frac{1}{2r-1}} \Big) \log(ex)$$
$$\ll \Big(x^{\frac{1}{2r+1}} + x^{\frac{1}{r} - \frac{2r+1}{2(2r^2-r+1)}} y^{\frac{1}{2r^2-r+1}} + x^{\frac{1}{2r} - \frac{1}{2(r-1)(2r-1)}} y^{\frac{1}{(r-1)(2r-1)}} + \big(yx^{-\frac{1}{2r}}\big)^{\frac{1}{2r-1}} \Big) \log(ex)$$

and we check that every secondary term is absorbed by the first one since $y \le x^{1/(2r)}$.

▶ *Case* $x^{1/(2r)} < y < x$. We split the sum into the following subsums

$$\sum_{(2y)^{1/r} < n \le (2x)^{1/r}} \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right)$$
$$= \Big(\sum_{(4y)^{1/r} < n \le c_r^{-1} y} + \sum_{c_r^{-1} y < n \le x^{1/r}} + \sum_{x^{1/r} < n \le (2x)^{1/r}} \Big) \left(\left[\frac{x+y}{n^r} \right] - \left[\frac{x}{n^r} \right] \right) + O(y^{1/r})$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + O(y^{1/r})$$

where we assume by convention that $\Sigma_2 = 0$ whenever $y \ge c_r x^{1/r}$.

▷ As above, we have $\Sigma_3 \ll y^{1/r}$.

▷ Lemma 2.1(ii) applied to Σ_2 yields

$$\Sigma_2 \ll \big(x^{\frac{1}{2r+1}} + x^{\frac{1}{6r+3}} y^{2/3} \big) \log(ex)$$

and since $x < y^{2r}$ we get

$$\Sigma_2 \ll \big(x^{\frac{1}{2r+1}} + yx^{-\frac{1}{6r(2r+1)}} \big) \log(ex).$$

▷ As above, we apply Lemma 2.1(i) with $s = 2r$ to the sum Σ_1, which gives

$$\Sigma_1 \ll \big(x^{\frac{1}{r(2r+1)}} y^{\frac{2r-2}{2r+1}} + x^{\frac{1}{2r^2-r+1}} y^{\frac{(r-1)(2r-3)}{2r^2-r+1}} + y^{\frac{2r-2}{2r-1}} + x^{-\frac{1}{2r-1}} y^{\frac{2r}{2r-1}} \big) \log(ex),$$

and the inequality $x < y^{2r}$ implies that

$$\Sigma_1 \ll y^{\frac{2r}{2r+1}} \log x \ll yx^{-\frac{1}{6r(2r+1)}} \log(ex).$$

The proof is complete. ■

Lemma 3.2. *Let $k \ge 2$ be a fixed integer and $\varepsilon > 0$ be a small real number. Uniformly for $X \ge 1$ and $0 < Y < X$ we have*

$$\sum_{\substack{2Y < b \le 2X \\ b\ k\text{-full}}} \left(\left[\frac{X+Y}{b} \right] - \left[\frac{X}{b} \right] \right) \ll_{k,\varepsilon} X^{\varepsilon} \Big\{ X^{\frac{1}{2k+1}} + YX^{-\frac{1}{6(4k-1)(2k-1)}} + Y^{1-\frac{2(k-1)}{k(3k-1)}} \Big\}.$$

Proof. Let S be the sum on the left-hand side. One may uniquely write every k-full integer b as

$$b = \prod_{i=1}^{k} a_i^{k+i-1} = a_j^{k+j-1} d_j \quad \text{where} \quad d_j = \prod_{\substack{i=1 \\ i \neq j}}^{k} a_i^{k+i-1} \quad (j \in \{1, \dots, k\})$$

with $a_2 \cdots a_k$ squarefree and $(a_\ell, a_m) = 1$ for all $2 \leq \ell < m \leq k$. Since $b > Y$, there exists an integer $j \in \{1, \dots, k\}$ such that $a_j > Y^{2/(k(3k-1))}$. The contribution S_j of these integers to S satisfies

$$S_j \ll \sum_{Y^{\frac{2}{k(3k-1)}} < a_j \leq (2X)^{\frac{1}{k+j-1}}} \sum_{m \in \mathcal{I}_j} \sum_{d_j \mid m} 1$$

where d_j is defined above and

$$\mathcal{I}_j = \left(\frac{X}{a_j^{k+j-1}}, \frac{X+Y}{a_j^{k+j-1}} \right] \cap \mathbb{Z}.$$

We infer that

$$S_j \ll_\varepsilon X^\varepsilon \sum_{Y^{\frac{2}{k(3k-1)}} < a_j \leq (2X)^{\frac{1}{k+j-1}}} \left(\left[\frac{X+Y}{a_j^{k+j-1}} \right] - \left[\frac{X}{a_j^{k+j-1}} \right] \right)$$

and hence

$$S \leq \sum_{j=1}^{k} S_j \ll_{k,\varepsilon} X^\varepsilon \sum_{j=1}^{k} \sum_{Y^{\frac{2}{k(3k-1)}} < a_j \leq (2X)^{\frac{1}{k+j-1}}} \left(\left[\frac{X+Y}{a_j^{k+j-1}} \right] - \left[\frac{X}{a_j^{k+j-1}} \right] \right).$$

We now split the inner sum into

$$\sum_{Y^{\frac{2}{k(3k-1)}} < a_j \leq (2Y)^{\frac{1}{k+j-1}}} + \sum_{(2Y)^{\frac{1}{k+j-1}} < a_j \leq (2X)^{\frac{1}{k+j-1}}} ;$$

estimating the first one trivially leads to

$$S \ll_{k,\varepsilon} X^\varepsilon \left(\Sigma_k + Y^{1 - \frac{2(k-1)}{k(3k-1)}} \right)$$

where

$$\Sigma_k := \sum_{j=1}^{k} \sum_{(2Y)^{\frac{1}{k+j-1}} < a_j \leq (2X)^{\frac{1}{k+j-1}}} \left(\left[\frac{X+Y}{a_j^{k+j-1}} \right] - \left[\frac{X}{a_j^{k+j-1}} \right] \right).$$

Lemma 3.1 applied to Σ_k with $r = k + j - 1 \geq 2$ yields

$$\begin{aligned} \Sigma_k &\ll_{k,\varepsilon} \sum_{j=1}^{k} \left(X^{\frac{1}{2k+2j-1}} + Y X^{-\frac{1}{6(k+j-1)(2k+2j-1)}} + Y^{\frac{1}{k+j-1}} \right) \log(eX) \\ &\ll_{k,\varepsilon} \left(X^{\frac{1}{2k+1}} + Y X^{-\frac{1}{6(4k-1)(2k-1)}} + Y^{1/k} + Y^{\frac{1}{2k-1}} \right) \log(eX), \end{aligned}$$

concluding the proof since $Y^{1/k} + Y^{\frac{1}{2k-1}} \le 2X^{\frac{1}{2k+1}}$ if $0 < Y \le 1$ and $Y^{1/k} + Y^{\frac{1}{2k-1}} \le 2Y^{1-\frac{2(k-1)}{k(3k-1)}}$ whenever $Y > 1$. ∎

4. Proof of Theorem 1.1. Using the Möbius inversion formula and interchanging the order of summations we get

$$\begin{aligned}\sum_{x<n\le x+y} f(n) &= \sum_{d\le x+y} g(d)\left(\left[\frac{x+y}{d}\right]-\left[\frac{x}{d}\right]\right)\\ &= \Big(\sum_{d\le 2y}+\sum_{2y<d\le x+y}\Big)g(d)\left(\left[\frac{x+y}{d}\right]-\left[\frac{x}{d}\right]\right)\\ &= \varSigma_1+\varSigma_2.\end{aligned}$$

▷ By Lemma 2.2, the series $\sum_{d\ge 1} g(d)d^{-1}$ is absolutely convergent and we get

$$\begin{aligned}\varSigma_1 &= y\sum_{d=1}^{\infty}\frac{g(d)}{d}+O\Big(\sum_{n\le 2y}|g(n)|\Big)+O\Big(y\sum_{n>2y}\frac{|g(n)|}{n}\Big)\\ &= y\mathcal{M}_f+O_{k,\varepsilon}(y^{1/k+\varepsilon}).\end{aligned}$$

▷ Using (5) we get, as in the proof of Lemma 2.2,

$$\begin{aligned}|\varSigma_2| &\le \sum_{2y<d\le 2x}|g(n)|\left(\left[\frac{x+y}{d}\right]-\left[\frac{x}{d}\right]\right)\\ &\le \sum_{\substack{a\le 2x\\ a\ k\text{-free}}}\frac{A^{\omega(a)}}{a}\sum_{\substack{2y/a<b\le 2x/a\\ b\ k\text{-full}}}|g(b)|\left(\left[\frac{(x+y)/a}{b}\right]-\left[\frac{x/a}{b}\right]\right)\\ &\ll x^{2\varepsilon}\sum_{a\le 2x}\frac{\mu_k(a)}{a}\sum_{\substack{2y/a<b\le 2x/a\\ b\ k\text{-full}}}\left(\left[\frac{(x+y)/a}{b}\right]-\left[\frac{x/a}{b}\right]\right)\\ &\ll x^{2\varepsilon}\Big(\sum_{a\le x}+\sum_{x<a\le 2x}\Big)\frac{\mu_k(a)}{a}\sum_{\substack{2y/a<b\le 2x/a\\ b\ k\text{-full}}}\left(\left[\frac{(x+y)/a}{b}\right]-\left[\frac{x/a}{b}\right]\right)\\ &= \varSigma_{21}+\varSigma_{22}.\end{aligned}$$

When $x < a \le 2x$, we have $2x/a < 2$ so that 1 is the unique value taken by b in the inner sum of $\varSigma_{22}$ and hence

$$\varSigma_{22} = \sum_{x<a\le 2x}\frac{\mu_k(a)}{a}\left(\left[\frac{x+y}{a}\right]-\left[\frac{x}{a}\right]\right) \le \sum_{x<a\le 2x}\frac{\mu_k(a)}{a} \ll 1.$$

For $\varSigma_{21}$, we apply Lemma 3.2 to the inner sum with $X = x/a$ and $Y = y/a$, which gives

$$\begin{aligned}\Sigma_{21} &\ll x^{3\varepsilon} \sum_{a \le x} \frac{\mu_k(a)}{a} \\ &\quad \times \left\{ \left(\frac{x}{a} \right)^{\frac{1}{2k+1}} + y x^{-\frac{1}{6(4k-1)(2k-1)}} a^{-1+\frac{1}{6(4k-1)(2k-1)}} + \left(\frac{y}{a} \right)^{1-\frac{2(k-1)}{k(3k-1)}} \right\} \\ &\ll x^{3\varepsilon} \big(x^{\frac{1}{2k+1}} + y x^{-\frac{1}{6(4k-1)(2k-1)}} + y^{1-\frac{2(k-1)}{k(3k-1)}} \big)\end{aligned}$$

as required. ∎

5. Applications. Let $k \ge 2$ be a fixed integer.

▷ Define the multiplicative function

$$\tau^{(k)}(n) = \sum_{d^k \mid n} 1.$$

Hence $\tau^{(k)}(p^\alpha) = 1 + [\alpha/k]$ and thus

$$(\tau^{(k)} \star \mu)(p^\alpha) = \left[\frac{\alpha}{k} \right] - \left[\frac{\alpha - 1}{k} \right] \in \{0, 1\}.$$

▷ Let β_k be the number of k-full divisors. We have

$$\beta_k(p^\alpha) = \begin{cases} 1 & \text{if } 1 \le \alpha < k, \\ \alpha - k + 2 & \text{if } \alpha \ge k, \end{cases}$$

so that $\beta_k \star \mu$ is the characteristic function of the set of k-full numbers.

▷ Let γ_k be the greatest k-free divisor. Then

$$\gamma_k(p^\alpha) = \begin{cases} p^\alpha & \text{if } 1 \le \alpha < k, \\ p^{k-1} & \text{if } \alpha \ge k, \end{cases}$$

so that

$$(\varphi \gamma_k \operatorname{Id}^{-2} \star \mu)(p^\alpha) = \begin{cases} -p^{-1} & \text{if } \alpha = 1, \\ 0 & \text{if } 2 \le \alpha < k, \\ -p^{k-\alpha-2}(p-1)^2 & \text{if } \alpha \ge k. \end{cases}$$

▷ Let M_k be the maximal k-full divisor, so that

$$M_k(p^\alpha) = \begin{cases} 1 & \text{if } 1 \le \alpha < k, \\ p^\alpha & \text{if } \alpha \ge k. \end{cases}$$

This implies that

$$(M_k^{-1} \star \mu)(p^\alpha) = \begin{cases} 0 & \text{if } 1 \le \alpha < k, \\ p^{-k} - 1 & \text{if } \alpha = k, \\ p^{-\alpha} - p^{1-\alpha} & \text{if } \alpha > k. \end{cases}$$

▷ Let a be the number of non-isomorphic abelian groups. It is well-known (see [10] for instance) that $a(p^\alpha) = P(\alpha)$ where P is the unrestricted partition function. Hence $(a \star \mu)(p) = P(1) - 1 = 0$ and, for any $\alpha \geq 2$, we have

$$0 \leq (a \star \mu)(p^\alpha) = P(\alpha) - P(\alpha - 1) \leq P(\alpha) = a(p^\alpha)$$

where the first inequality is given by [5]. We infer that

$$0 \leq \frac{(a \star \mu)(n)}{n^\varepsilon} \leq \frac{a(n)}{n^\varepsilon} \xrightarrow{n \to \infty} 0$$

where the limit comes from [10, Proposition 2].

▷ Let $\tau^{(e)}$ be the number of exponential divisors. We have $\tau^{(e)}(p^\alpha) = \tau(\alpha)$ and hence $(\tau^{(e)} \star \mu)(p) = \tau(1) - 1 = 0$ and, for any $\alpha \geq 2$, we have

$$|(\tau^{(e)} \star \mu)(p^\alpha)| = |\tau(\alpha) - \tau(\alpha - 1)| \leq \alpha + 1 = \tau(p^\alpha).$$

▷ Let $\mu^{(e)}$ be the analog of the Möbius function for the exponential divisors. Then $\mu^{(e)}(p^\alpha) = \mu(\alpha)$, and therefore as above we obtain

$$|(\mu^{(e)} \star \mu)(p^\alpha)| = \begin{cases} 0 & \text{if } \alpha = 1, \\ \mu(\alpha) - \mu(\alpha - 1) & \text{if } \alpha \geq 2. \end{cases}$$

Putting all this together and using Theorem 1.1, we get the following asymptotic formulae.

Corollary 5.1. *Let $k \geq 2$ be a fixed integer and $\varepsilon > 0$ be a small real number. Define*

$$\mathcal{R}_k = \mathcal{R}_k(x, y) = x^{\frac{1}{2k+1}} + yx^{-\frac{1}{6(4k-1)(2k-1)}} + y^{1 - \frac{2(k-1)}{k(3k-1)}}.$$

Then uniformly for all $4^k \leq y < x$ we have

$$\sum_{x < n \leq x+y} \mu^{(e)}(n) = y \prod_p \left(1 + \sum_{\alpha=2}^{\infty} \frac{\mu(\alpha) - \mu(\alpha - 1)}{p^\alpha}\right) + O_\varepsilon(\mathcal{R}_2 x^\varepsilon),$$

$$\sum_{x < n \leq x+y} \tau^{(e)}(n) = y \prod_p \left(1 + \sum_{\alpha=2}^{\infty} \frac{\tau(\alpha) - \tau(\alpha - 1)}{p^\alpha}\right) + O_\varepsilon(\mathcal{R}_2 x^\varepsilon),$$

$$\sum_{x < n \leq x+y} a(n) = y \prod_{j=2}^{\infty} \zeta(j) + O_\varepsilon(\mathcal{R}_2 x^\varepsilon),$$

$$\sum_{x < n \leq x+y} \mu_k(n) = \frac{y}{\zeta(k)} + O_{\varepsilon,k}(\mathcal{R}_k x^\varepsilon),$$

$$\sum_{x < n \leq x+y} \tau^{(k)}(n) = y\zeta(k) + O_{\varepsilon,k}(\mathcal{R}_k x^\varepsilon),$$

$$\sum_{x < n \leq x+y} \beta_k(n) = y \prod_p \left(1 + \frac{1}{p^{k-1}(p-1)}\right) + O_{\varepsilon,k}(\mathcal{R}_k x^\varepsilon),$$

$$\sum_{x<n\le x+y}\frac{1}{M_k(n)}=y\prod_p\left(1-\frac{1}{p^k}+\frac{1}{p^{2k-1}(p+1)}\right)+O_{\varepsilon,k}(\mathcal{R}_k x^\varepsilon),$$
$$\sum_{x<n\le x+y}\frac{\varphi(n)\gamma_k(n)}{n^2}=y\prod_p\left(1-\frac{1}{p^2}-\frac{p-1}{p^k(p+1)}\right)+O_{\varepsilon,k}(\mathcal{R}_k x^\varepsilon).$$

Acknowledgments. I express my gratitude to the referee for his careful reading of the manuscript and the many valuable suggestions and corrections he made.

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Olivier Bordellès
2 allée de la Combe
43000 Aiguilhe, France
E-mail: borde43@wanadoo.fr

Received on 29.11.2011
and in revised form on 9.6.2012 (6899)