

On the structure of the Selberg class, IV: basic invariants

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1. Introduction. This paper is a continuation of part II (see [6]) of the series. It is well known that the γ -factors in the functional equation of a function $F(s)$ in the extended Selberg class \mathcal{S}^\sharp (see below for definitions) are uniquely determined by $F(s)$ up to multiplicative constants, although their shape can be considerably changed by means of suitable identities satisfied by the Γ function; see Theorem 2.1 of Conrey–Ghosh [1]. Such a property of γ -factors leads to the notion of invariant (see [6]); we recall here that an *invariant* (resp. a *numerical invariant*) of $F \in \mathcal{S}^\sharp$ is an expression (resp. a number), defined in terms of the data in the functional equation of $F(s)$, depending only on the function $F(s)$ and not on the form of the functional equation. In [6] we obtained a transformation algorithm for γ -factors and, as a consequence, we gave a simple criterion characterizing invariants of functions in \mathcal{S}^\sharp . Amongst other things, such a criterion allowed us to exhibit several interesting new invariants.

In this paper we deal with the following two natural problems about invariants. The first problem is to construct a universal set of numerical invariants characterizing the functional equation; by universal we mean that the choice of such invariants is the same for every $F \in \mathcal{S}^\sharp$. Any such set of invariants will be called a set of *basic invariants*. Note that, given one such set, it is easy to build infinitely many sets of basic invariants. The second problem is to construct a form of the functional equation where all data are invariants, which we call an *invariant form*. Again, one can easily build infinitely many such forms of the functional equation from a given one. Note that the two problems are closely connected; for instance, a solution of the second problem immediately provides a set of numerical invariants, depending on each $F \in \mathcal{S}^\sharp$, characterizing the functional equation of $F(s)$. In this paper we deal first with the problem of constructing a set of basic

invariants, and then we turn to the second problem by constructing a special type of invariant form, called canonical form, of the functional equation.

We keep as much as possible the notation introduced in [5], [6] and in the survey paper [4]. In particular, we recall that a function $F \in \mathcal{S}^\sharp$ satisfies a functional equation of type

$$\gamma(s)F(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s),$$

where $|\omega| = 1$, $\bar{f}(s) = \overline{f(\bar{s})}$ and the γ -factor $\gamma(s)$ has the form

$$\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j), \quad Q > 0, \lambda_j > 0, \operatorname{Re} \mu_j \geq 0.$$

We also recall that degree, ξ -invariant, conductor and root number, defined by

$$d_F = 2 \sum_{j=1}^r \lambda_j, \quad \xi_F = 2 \sum_{j=1}^r (\mu_j - 1/2) = \eta_F + i\theta_F,$$

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}, \quad \omega_F^* = \omega e^{-i(\pi/2)(\eta_F+1)} \left(\frac{q_F}{(2\pi)^{d_F}} \right)^{i\theta_F/d_F} \prod_{j=1}^r \lambda_j^{-2i \operatorname{Im} \mu_j},$$

respectively, are invariants of $F(s)$. We also allow $d_F = 0$, in which case γ -factors simply reduce to $\gamma(s) = Q^s$. However, in what follows we consider only functions $F \in \mathcal{S}^\sharp$ with $d_F > 0$, since the whole theory of invariants is trivial in the case $d_F = 0$; see Theorem 1 of [5].

We introduce the following notation. As usual, we denote by $B_n(x)$ the n th Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi,$$

hence $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ and so on. Moreover, for a given $F \in \mathcal{S}^\sharp$ we write

$$K_F(z) = z \sum_{j=1}^r \frac{e^{z\mu_j/\lambda_j}}{e^{z/\lambda_j} - 1}, \quad z \in \mathbb{C},$$

and

$$H_F(n) = 2 \sum_{j=1}^r \frac{B_n(\mu_j)}{\lambda_j^{n-1}}, \quad n = 0, 1, \dots$$

Note that both $K_F(z)$ and $H_F(n)$ are additive with respect to the function $F(s)$, i.e., for $F, G \in \mathcal{S}^\sharp$ we have

$$K_{FG}(z) = K_F(z) + K_G(z), \quad H_{FG}(n) = H_F(n) + H_G(n).$$

We start with the construction of a set of basic invariants.

THEOREM 1. *The function $K_F(z)$ and the numbers $H_F(n)$, $n \geq 0$, are invariants of $F \in \mathcal{S}^\sharp$. Moreover, two functions $F, G \in \mathcal{S}^\sharp$ satisfy the same functional equation if and only if*

$$(1.1) \quad q_F = q_G, \quad \omega_F^* = \omega_G^* \quad \text{and} \quad H_F(n) = H_G(n) \quad \text{for all } n \geq 0.$$

The invariants $H_F(n)$, $n \geq 0$, are called the *H-invariants* of $F \in \mathcal{S}^\sharp$. Observe that degree and ξ -invariant are recovered as

$$(1.2) \quad d_F = H_F(0), \quad \xi_F = H_F(1).$$

Later on, see (2.8) below, we give a meaning to all *H-invariants* in terms of a certain asymptotic expansion; at present we leave open the problem of giving a meaning to the $H_F(n)$ in terms of $F(s)$, without explicit reference to the functional equation. In view of Theorem 1, q_F , ω_F^* and the *H-invariants* form a set of basic invariants. In Theorem 3 below, given any fixed $F \in \mathcal{S}^\sharp$, we construct a finite set of numerical invariants characterizing the functional equation of $F(s)$.

We remark that the proof of Theorem 1 shows that if $\gamma_1(s)$ and $\gamma_2(s)$ are γ -factors of $F(s)$ and $G(s)$, respectively, and (1.1) is replaced by the weaker assumptions

$$H_F(n) = H_G(n) \quad \text{for all } n \geq 0$$

and

$$q_F = q_G \quad \text{and} \quad H_F(n) = H_G(n) \quad \text{for all } n \geq 0,$$

then we obtain the weaker statements

$$\gamma_1(s) = e^{as+b}\gamma_2(s) \quad \text{with } a = \frac{1}{2} \log(q_F/q_G) \text{ and some constant } b \in \mathbb{C}$$

and

$$\gamma_1(s) = c\gamma_2(s) \quad \text{for some constant } c \in \mathbb{C},$$

respectively; see (2.5), (2.7) and (2.9) below. Moreover, the function $K_F(z)$ determines all *H-invariants* of $F(s)$ and *vice versa*, since

$$(1.3) \quad K_F(z) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n;$$

see (2.3) below. Hence $K_F(z)$ also determines the γ -factors of $F(s)$ up to factors e^{as+b} .

In Section 2 we give two proofs of the second part of Theorem 1. The first proof is simpler, while the second also provides the above mentioned interpretation of *H-invariants* as coefficients of a certain asymptotic expansion.

The function $K_F(z)$ gives rise to other interesting invariants. We first recall a few definitions and results from [6]. The γ -class number h_F is the number of \mathbb{Q} -equivalence classes arising from the coefficients $\lambda_1, \dots, \lambda_r$ of a γ -factor of $F \in \mathcal{S}^\sharp$, and h_F is an invariant. $F(s)$ is *reduced* if it has a γ -factor

with $0 \leq \operatorname{Re} \mu_j < 1$ for $j = 1, \dots, r$. The *canonical polynomial* $P_F(s)$ is the monic polynomial obtained by reducing a γ -factor $\gamma(s)$ of $F(s)$ by means of the factorial formula $\Gamma(s+1) = s\Gamma(s)$ as

$$\gamma(s) = c_\gamma P_F(s) \tilde{\gamma}(s),$$

where $c_\gamma \in \mathbb{C}$ and $\tilde{\gamma}(s)$ has $0 \leq \operatorname{Re} \tilde{\mu}_j < 1$ for $j = 1, \dots, r$. Moreover, we recall that $P_F(s)$ is an invariant, and $F(s)$ is reduced if and only if $P_F(s) = 1$ identically.

We split a γ -factor $\gamma(s)$ of $F \in \mathcal{S}^\sharp$ according to \mathbb{Q} -equivalence classes of the λ_j 's as

$$(1.4) \quad \gamma(s) = Q^s \prod_{j=1}^{h_F} \gamma_j(s)$$

and, analogously, we split $K_F(z)$ as

$$(1.5) \quad K_F(z) = \sum_{j=1}^{h_F} K_j(z).$$

The functions $K_j(z)$, $1 \leq j \leq h_F$, are almost invariants, in the sense that if functions $\tilde{K}_j(z)$ are constructed in the same way as the $K_j(z)$ starting from another γ -factor $\tilde{\gamma}(s)$ of $F(s)$, then

$$(1.6) \quad K_j(z) = \tilde{K}_j(z) + zE_j(z)$$

where $E_j(z)$ are exponential polynomials (see Section 3). We define the *canonical exponents* $\Lambda_1, \dots, \Lambda_{h_F}$ of $F(s)$ by

$$(1.7) \quad \Lambda_j = \max\{\Lambda \in \mathbb{R} : (e^{z/\Lambda} - 1)K_j(z) \text{ is entire}\}.$$

The canonical exponents exist, are positive and distinct, and of course are invariants by (1.6); see Lemma 3.2 below. We remark that the difficulty is to show that the functions $K_j(z)$ are not entire.

We say that an expression of the form

$$\gamma(s) = e^{as+b} \prod_{j=1}^{h_F} \prod_{k=1}^{M_j} \Gamma(\lambda_j s + \mu_{j,k}),$$

with $a \in \mathbb{R}$, $b \in \mathbb{C}$, $\lambda_j > 0$ and $\operatorname{Re} \mu_{j,k} \geq 0$, is a *balanced form* of the γ -factor $\gamma(s)$ if all ratios Λ_j/λ_j , $1 \leq j \leq h_F$, are equal. In such a case, the common ratio Λ_j/λ_j is called the *reduction factor*. Every γ -factor has a balanced form, and the reduction factor of any balanced form is a positive integer; see Lemma 3.3 below. We define the *reduction index* l_F of $F \in \mathcal{S}^\sharp$ as the minimal reduction factor of the balanced forms of γ -factors of $F(s)$. Clearly, the positive integer l_F is an invariant of $F(s)$.

Our motivation for introducing the notion of balanced form comes from the fact that the functional equations of all L -functions known to us have

γ -factors either in balanced form or easily transformable to balanced form. More precisely, such L -functions $F(s)$ have $h_F = 1$ and a γ -factor with all $\lambda_j = 1/2$; in [6] we conjectured that this is in fact the case for all functions in \mathcal{S}^\sharp .

Canonical exponents and reduction index can be used to obtain an invariant form of the functional equation where, in addition, the γ -factor is in balanced form. In fact, writing

$$(1.8) \quad Q_F = \left(q_F (2\pi)^{-d_F} l_F^{d_F} \prod_{j=1}^{h_F} \Lambda_j^{-2K_j \Lambda_j} \right)^{1/2},$$

$$\omega_F = \omega_F^* e^{i(\pi/2)(\eta_F + 1)} \left(\frac{q_F}{(2\pi)^{d_F}} \right)^{-i\theta_F/d_F} l_F^{-i\theta_F} \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Lambda_j^{2i \operatorname{Im} \mu_{j,k}},$$

we have

THEOREM 2. *Every $F \in \mathcal{S}^\sharp$ uniquely determines positive integers K_j ($1 \leq j \leq h_F$) such that*

$$(1.9) \quad \gamma_0(s)F(s) = \omega_F \bar{\gamma}_0(1-s) \bar{F}(1-s)$$

where

$$\gamma_0(s) = Q_F^s \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left(\frac{\Lambda_j}{l_F} s + \mu_{j,k}\right),$$

the $\mu_{j,k}$'s ($1 \leq j \leq h_F$, $1 \leq k \leq l_F K_j$) are uniquely determined (mod \mathbb{Z}) complex numbers with $\operatorname{Re} \mu_{j,k} \geq 0$, and Q_F , ω_F are given by (1.8). Moreover, the $\mu_{j,k}$'s are uniquely determined if $h_F = 1$ or if $F(s)$ is reduced, and $l_F = 1$ in the latter case.

Note that functional equation (1.9) has γ -factor in balanced form, but it is not necessarily in invariant form. In the cases where all data of (1.9) are invariants, i.e., when the $\mu_{j,k}$'s are uniquely determined, we say that (1.9) is the *canonical form* of the functional equation of $F(s)$. In this case $\gamma_0(s)$ is denoted by $\gamma_F(s)$. We recall from [6] that, conjecturally, $h_F = 1$ for every $F \in \mathcal{S}^\sharp$, and hence we expect that *the canonical form exists for every $F \in \mathcal{S}^\sharp$.*

We remark that the non-uniqueness of the $\mu_{j,k}$'s can be eliminated at the cost of the explicit presence of the canonical polynomial in the functional equation. In fact, the proof of Theorem 2 (see (3.13) below) shows that every $F \in \mathcal{S}^\sharp$ satisfies the functional equation

$$(1.10) \quad \gamma_1(s)F(s) = \omega_1 \bar{\gamma}_1(1-s) \bar{F}(1-s),$$

$$\gamma_1(s) = Q_1^s P_F(s) \prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \Gamma(\Lambda_j s + \tilde{\mu}_{j,h})$$

with $0 \leq \operatorname{Re} \tilde{\mu}_{j,h} < 1$. In this case the $\tilde{\mu}_{j,h}$'s are *uniquely determined*, and Q_1 and ω_1 have the expressions of Q_F and ω_F in (1.8), respectively, with $l_F = 1$ and $\mu_{j,k}$ replaced by $\tilde{\mu}_{j,h}$. Although (1.10) is not *a priori* of the type allowed by the axioms of the Selberg class, it can be transformed into such a functional equation, by absorbing $P_F(s)$ into suitable Γ -factors. In particular, (1.10) can be transformed into the canonical form, when it exists.

From the definition of l_F we see that (1.9) has the minimum number of Γ -factors amongst the γ -factors of $F(s)$ in balanced form. Observe that if $h_F = 1$ then a γ -factor is in balanced form provided its λ -coefficients are all equal. Therefore, *if $h_F = 1$ (which we believe to be the general case in \mathcal{S}^\sharp) the canonical form exists and has the minimum number of Γ -factors amongst the γ -factors in balanced form.*

The proof of Theorem 2 shows that the data h_F , l_F , A_j 's, K_j 's and $\mu_{j,k}$'s in the canonical form are determined by $K_F(s)$. Hence, in view of (1.3), Theorem 2 provides a constructive proof of Theorem 1, of course in those cases where the canonical form exists.

In Section 3, after the proof of Theorem 2, we outline an algorithm for the computation of (1.9) starting from any given form of the functional equation of $F \in \mathcal{S}^\sharp$. For example, the standard functional equations of $\zeta(s)$ and $L(s, \chi)$, χ primitive Dirichlet character, are already in canonical form, while for the Dedekind zeta function associated with a number field K with $[K : \mathbb{Q}] = n = r_1 + 2r_2$ we have

$$h_F = 1, \quad l_F = 1, \quad A_1 = 1/2, \quad K_1 = n,$$

$$\mu_{1,k} = \begin{cases} 0 & \text{for } 1 \leq k \leq r_1 + r_2, \\ 1/2 & \text{for } r_1 + r_2 + 1 \leq k \leq n. \end{cases}$$

Therefore, the canonical form is different from the standard one in this case. Similar computations can be easily performed in the holomorphic modular case as well, thus showing that, of course after normalization to meet the axioms of the Selberg class, the standard functional equation coincides with the canonical form in this case.

In Theorem 1 we proved that conductor, root number and H -invariants form a set of basic invariants. Now we show that for any given $F \in \mathcal{S}^\sharp$ there exists a finite set of numerical invariants, depending on $F(s)$, characterizing the functional equation of $F(s)$. With the notation in Theorem 2 we write

$$r_F = l_F \sum_{j=1}^{h_F} K_j, \quad n_F = 2^{2h_F-1} r_F - 2h_F + 1.$$

Moreover, we recall from [6] the strong λ conjecture asserting that every $F \in \mathcal{S}^\sharp$ has a γ -factor with λ -coefficients all equal to $1/2$; see also Section 9 of [4]. We have

THEOREM 3. (i) *The functional equation of $F \in \mathcal{S}^\sharp$ is characterized by the invariants h_F, r_F ,*

$$(1.11) \quad q_F, \quad \omega_F^* \quad \text{and} \quad H_F(n) \quad \text{with} \quad n \leq n_F.$$

(ii) *Assuming the strong λ conjecture, the functional equation of $F \in \mathcal{S}^\sharp$ is characterized by the invariants in (1.11) with n_F replaced by d_F . Moreover, this is best possible in the sense that for every integer $d \geq 1$ there exist $F, G \in \mathcal{S}_d^\sharp$ for which the invariants in (1.11) with n_F replaced by $d - 1$ are equal, but $F(s)$ and $G(s)$ satisfy different functional equations.*

As a consequence of Theorem 3(ii), given any $F \in \mathcal{S}_d^\sharp$ we expect that q_F, ω_F^* and the H -invariants with $n \leq d$ characterize the functional equation of $F(s)$. We remark that in [5] we proved that the functional equation of $F \in \mathcal{S}_1^\sharp$ is characterized by the triplet (q_F, ω_F^*, ξ_F) . In view of (1.2), this confirms the above expectation in the case $d = 1$.

We finally remark that it will be clear from the next sections that none of the proofs in this paper relies in any way on the assumption that the functional equations under consideration have solutions in \mathcal{S}^\sharp . Therefore, a completely similar set of results can be developed by considering only functional equations, irrespective of whether they have solutions in \mathcal{S}^\sharp or not. This is basically a matter of taste, and our choice of referring to functions in \mathcal{S}^\sharp comes from the view that L -functions are the main concern in our investigations.

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2. Proof of Theorem 1. We start with a uniqueness principle for generalized Dirichlet series with complex exponents, of type

$$(2.1) \quad f(z) = \sum_{n=1}^{\infty} a_n e^{\varrho_n z}$$

with $\varrho_n = \alpha_n + i\beta_n, a_n \in \mathbb{C}, \varrho_n \neq \varrho_m$ if $n \neq m$ and $0 \leq \alpha_1 \leq \dots \leq \alpha_n \rightarrow \infty$. We remark that, in general, the analogue of the standard uniqueness principle for ordinary Dirichlet series fails in the case of generalized Dirichlet series. A counterexample is given in Section 1.3 of Chapter II of Leontiev's book [7]. However, such a uniqueness principle holds if the series is absolutely convergent on a sufficiently large domain.

LEMMA 2.1. *Let $f(z)$ be as in (2.1) and absolutely convergent for $z = x \leq x_0$. If $f(z) = 0$ for $z = x \leq x_0$, then $a_n = 0$ for every $n \in \mathbb{N}$.*

Proof. By contradiction, let n_0 be the smallest n such that $a_n \neq 0$. Moreover, let m_0 be the largest n such that $\alpha_n = \alpha_{n_0}$. We may assume without loss of generality that $\alpha_{n_0} = 0$. Hence for $x \rightarrow -\infty$ we have

$$(2.2) \quad 0 = f(x) = \sum_{n=n_0}^{m_0} a_n e^{\alpha_n x} + \sum_{n=m_0+1}^{\infty} a_n e^{\alpha_n x} = \sum_{n=n_0}^{m_0} a_n e^{i\beta_n x} + o(1).$$

From (2.2) we have

$$\sum_{n=n_0}^{m_0} |a_n|^2 = \lim_{X \rightarrow \infty} \frac{1}{X} \int_{-2X}^{-X} \left| \sum_{n=n_0}^{m_0} a_n e^{i\beta_n x} \right|^2 dx = 0,$$

a contradiction. ■

Let $\gamma(s)$ be a γ -factor of $F \in \mathcal{S}^\sharp$, $|z| < 2\pi \min_{j=1, \dots, r} |\lambda_j|$ and $z_j = z/\lambda_j$. From the definitions of $K_F(z)$, $H_F(n)$ and Bernoulli polynomials we have

$$(2.3) \quad \begin{aligned} 2K_F(z) &= 2 \sum_{j=1}^r \lambda_j \frac{z_j e^{\mu_j z_j}}{e^{z_j} - 1} \\ &= 2 \sum_{j=1}^r \lambda_j \sum_{n=0}^{\infty} B_n(\mu_j) \frac{z_j^n}{n!} = \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n. \end{aligned}$$

Since a power series uniquely determines its Taylor coefficients, from (2.3) we see that the invariance of the $H_F(n), n \geq 0$, follows from the invariance of $K_F(z)$.

Now we observe that the poles of $\gamma(s)$ are at $s = -(\mu_j + k)/\lambda_j, j = 1, \dots, r$ and $k = 0, 1, \dots$, therefore for $\text{Re } z < 0$ we have

$$(2.4) \quad K_F(z) = -z \sum_{j=1}^r \sum_{k=0}^{\infty} e^{\frac{\mu_j+k}{\lambda_j} z} = -z \sum_{\varrho} e^{-\varrho z},$$

where ϱ runs over the poles of $\gamma(s)$. Hence $K_F(z)$ is an invariant of $F \in \mathcal{S}^\sharp$, since the poles of a γ -factor of $F(s)$ are uniquely determined by $F(s)$ itself.

Let $F, G \in \mathcal{S}^\sharp$ have γ -factors $\gamma_1(s)$ and $\gamma_2(s)$, respectively. In order to prove the second part of Theorem 1 we first assume that $F(s)$ and $G(s)$ have the same H -invariants and observe that (2.3), (2.4) and Lemma 2.1 imply that $\gamma_1(s)$ and $\gamma_2(s)$ have the same poles. Therefore $\gamma_1(s)/\gamma_2(s)$ is an entire function of order 1 without zeros, and hence

$$\gamma_1(s) = e^{as+b} \gamma_2(s), \quad a, b \in \mathbb{C}.$$

Moreover, using Stirling's formula with fixed σ and $t \rightarrow \pm\infty$ we see that in

fact

$$(2.5) \quad \gamma_1(s) = e^{as+b} \gamma_2(s), \quad a \in \mathbb{R}, b \in \mathbb{C}.$$

Assume now in addition that $F(s)$ and $G(s)$ have the same conductor. With obvious notation, inserting (2.5) in the functional equation of $F(s)$ we see that $F(s)$ also satisfies the functional equation

$$(2.6) \quad e^{as} \gamma_2(s) F(s) = \omega_1 e^{\bar{b}} e^{-b} e^{a(1-s)} \bar{\gamma}_2(1-s) \bar{F}(1-s).$$

Hence, recalling (1.2) and denoting by d the common degree of $F(s)$ and $G(s)$, computing the conductor of $F(s)$ by means of (2.6) and equating with the conductor of $G(s)$ we obtain

$$(2\pi)^d e^{2a} Q_2^2 \prod_{j=1}^{r_2} \lambda_{j,2}^{2\lambda_{j,2}} = (2\pi)^d Q_2^2 \prod_{j=1}^{r_2} \lambda_{j,2}^{2\lambda_{j,2}}.$$

Therefore $a = 0$ and hence

$$(2.7) \quad \gamma_1(s) = c \gamma_2(s), \quad c \in \mathbb{C}.$$

Finally, assuming in addition that $F(s)$ and $G(s)$ have the same root number, arguing as above we can show that the constant c in (2.7) is real, and hence $F(s)$ and $G(s)$ satisfy the same functional equation. Theorem 1 is therefore proved, since it is obvious that (1.1) holds if $F(s)$ and $G(s)$ satisfy the same functional equation. ■

Now we give an interpretation of the H -invariants as coefficients of a certain asymptotic expansion. Such an analysis provides another proof of the second part of Theorem 1.

Let $F \in \mathcal{S}^\sharp$ and $\gamma(s)$ be a γ -factor of $F(s)$. From Stirling's asymptotic expansion

$$\begin{aligned} \log \Gamma(s+a) &= (s+a-\frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \sum_{n=1}^N (-1)^{n+1} \frac{B_{n+1}(a)}{n(n+1)} s^{-n} \\ &\quad + O(|s|^{-N-1}) \end{aligned}$$

valid for all $N \in \mathbb{N}$, $a \in \mathbb{C}$ and $|\arg s| < \pi$ (see Section 1.18 of [2]), we obtain

$$(2.8) \quad \begin{aligned} \log \gamma(s) &= \frac{1}{2} H_F(0) s \log s \\ &\quad + \frac{1}{2} (\log q_F - H_F(0) \log 2\pi e) s + \frac{1}{2} H_F(1) \log s \\ &\quad + c(\gamma) + \frac{1}{2} \sum_{n=1}^N \frac{(-1)^{n+1}}{n(n+1)} H_F(n+1) s^{-n} + O(|s|^{-N-1}) \end{aligned}$$

for all $N \in \mathbb{N}$ and $|\arg s| < \pi$, where

$$c(\gamma) = \sum_{j=1}^r \left(\mu_j - \frac{1}{2}\right) \log \lambda_j + \frac{r}{2} \log 2\pi.$$

From (2.8) we see that if $F, G \in \mathcal{S}^\sharp$ with γ -factors $\gamma_1(s)$ and $\gamma_2(s)$, respectively, have the same H -invariants then

$$(2.9) \quad \log \frac{\gamma_1(s)}{\gamma_2(s)} = \frac{1}{2}s \log \frac{q_F}{q_G} + c(\gamma_1) - c(\gamma_2) + O(|s|^{-N}) = as + b + O(|s|^{-N}),$$

say, for all $N \in \mathbb{N}$ and $|\arg s| < \pi$. However, (2.9) is not enough for an immediate deduction of the second part of Theorem 1. To perform such a deduction we proceed as follows.

Starting from the formula

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \int_0^{\infty e^{i\beta}} \psi(z) e^{-zs} \frac{dz}{z},$$

$$\psi(z) = \frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2},$$

uniformly for $-\pi/2 < \beta < \pi/2$ and $-(\pi/2 + \beta) < \arg s < \pi/2 - \beta$ (see (5) of Section 1.9 of [2]), we can prove

LEMMA 2.2. *Let $a \in \mathbb{C}$, $\delta > 0$ and $c \in \mathbb{R}$ sufficiently large be fixed constants. Then, uniformly for $|\arg s| \leq \pi - \delta$ and $|s| > (c/\delta)|a|$, we have*

$$\log \Gamma(s+a) = \left(s+a - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \int_0^{\infty e^{i\beta}} \psi(z, a) e^{-zs} \frac{dz}{z} + O(e^{-\kappa|s|}),$$

where $\kappa = \kappa(\delta) > 0$ is a certain constant, $-\pi/2 < \beta < \pi/2$ is such that $|\arg s + \beta| < \pi/2 - \delta$ and

$$\psi(z, a) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} B_n(a) z^{n-1}, \quad |z| < 2\pi.$$

We omit the proof of Lemma 2.2, which can be obtained by rather long but standard computations along the lines of the proof of Stirling's asymptotic expansion, and only outline the second proof of the second part of Theorem 1.

Assuming that $F, G \in \mathcal{S}^\sharp$ have the same H -invariants and using Lemma 2.2 instead of Stirling's asymptotic expansion we obtain the following improved version of (2.9):

$$(2.10) \quad \log \frac{\gamma_1(s)}{\gamma_2(s)} = as + b + O(e^{-\kappa|s|})$$

uniformly for $|s|$ sufficiently large and $|\arg s| \leq \pi - \delta$, where κ and δ are as in Lemma 2.2. Moreover, writing

$$f(t) = \frac{\gamma_1(1+it)}{\gamma_2(1+it)} e^{-a(1+it)-b} - 1, \quad t \in \mathbb{R},$$

and

$$(2.11) \quad \begin{aligned} \mu_k(f) &= \int_{-\infty}^{\infty} f(t)t^k dt \\ &= \frac{1}{i^{k+1}} \int_{1-i\infty}^{1+i\infty} \left(\frac{\gamma_1(s)}{\gamma_2(s)} e^{-as-b} - 1 \right) (s-1)^k ds, \quad k = 0, 1, \dots, \end{aligned}$$

from (2.10) we have

$$(2.12) \quad f(t) = O(e^{-\kappa|t|}) \quad \text{and} \quad \mu_k(f) = 0 \quad \text{for } k = 0, 1, \dots,$$

the last statement being easily verified by shifting the line of integration in (2.11) to the right and using (2.10).

Consider now the Fourier transform $\widehat{f}(z)$ of $f(t)$. From (2.12) we see that $\widehat{f}(z)$ is holomorphic for $|\text{Im } z| < \kappa/(2\pi)$ and, moreover,

$$\widehat{f}^{(k)}(0) = (2\pi i)^k \mu_k(f) = 0, \quad k = 0, 1, \dots$$

Therefore $\widehat{f}(z) = 0$ identically and hence $f(t) = 0$ identically. By the definition of $f(t)$, the principle of analytic continuation and (2.9), this gives (2.5) with $a = \frac{1}{2} \log(q_F/q_G)$, and the second part of Theorem 1 follows easily.

3. Proof of Theorem 2. We first state the following sharper version of Ritt's [8] theorem on exponential polynomials, due to Shields [9]. In fact, Shields [9] proves a stronger form of Lemma 3.1 below. We recall that an *exponential polynomial* is an expression of type

$$f(z) = \sum_{n \leq N} a_n e^{\alpha_n z}$$

with $a_n \in \mathbb{C}$ and distinct $\alpha_n \in \mathbb{C}$.

LEMMA 3.1 (Shields [9]). *Let $f(z)$ and $g(z)$ be exponential polynomials and suppose that $f(z)/g(z)$ has only finitely many poles. Then $f(z)/g(z)$ is an exponential polynomial.*

Let $\gamma(s)$ be a γ -factor of $F \in \mathcal{S}^\sharp$. We split $\gamma(s)$ and $K_F(z)$ into \mathbb{Q} -equivalence classes as in (1.4) and (1.5) and, with abuse of notation, write

$$(3.1) \quad \gamma_j(s) = \prod_{i=1}^{r_j} \Gamma(\lambda_i s + \mu_i), \quad j = 1, \dots, h_F.$$

It is easy to see that two Γ -factors belonging to different \mathbb{Q} -equivalence classes may have at most one common pole. Therefore, apart from a finite number of elements, the sets of poles of the $\gamma_j(s)$, $1 \leq j \leq h_F$, are invariants of $F(s)$. Hence the functions $K_j(z)$ are uniquely determined up to addition of an exponential polynomial times z , as in (1.6). Thus, the A_j 's defined

by (1.7) are also invariants, if they exist. The first step in the proof of Theorem 2 is to show that the Λ_j 's do exist.

LEMMA 3.2. *Let $F \in \mathcal{S}^\sharp$. Then the Λ_j 's defined by (1.7) exist, are positive and belong to distinct \mathbb{Q} -equivalence classes.*

Proof. Let $\gamma(s)$ be a γ -factor of $F(s)$. For $1 \leq j \leq h_F$, we recall that Λ_j denotes the maximal Λ such that

$$(3.2) \quad (e^{z/\Lambda} - 1)K_j(z)$$

is entire. We start observing that the λ_i 's in (3.1), $1 \leq i \leq r_j$, are all \mathbb{Q} -equivalent, and hence we can write

$$\lambda_i = n_i \eta_j, \quad 1 \leq i \leq r_j,$$

with certain positive integers n_i and a positive $\eta_j \in \mathbb{R}$. Therefore, by the Legendre–Gauss multiplication formula

$$(3.3) \quad \Gamma(s) = m^{s-1/2} (2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma\left(\frac{s+k}{m}\right), \quad m = 1, 2, \dots,$$

we rewrite $\gamma_j(s)$ in the form

$$(3.4) \quad \gamma_j(s) = e^{a_j s + b_j} \prod_{i=1}^{r'_j} \Gamma(\eta_j s + c_i), \quad \text{Re } c_i \geq 0,$$

and computing $K_j(z)$ using (3.4) we obtain

$$K_j(z) = \frac{z}{e^{z/\eta_j} - 1} \sum_{i=1}^{r'_j} e^{c_i z / \eta_j}.$$

Hence the set of Λ 's such that (3.2) is entire is not empty. Moreover, the poles of $K_j(z)$ are contained in the set $\{2\pi i k \eta_j : k \in \mathbb{Z}\}$.

By (2.4) applied to $K_j(z)$, the function $K_j(z)/z$ is not an exponential polynomial. Hence, by Lemma 3.1, $K_j(z)$ has at least one pole, say at $z = 2\pi i k_0 \eta_j$ with some $k_0 > 0$. Such a pole has therefore to be cancelled by a zero of $e^{z/\Lambda} - 1$, hence admissible Λ 's have the shape

$$(3.5) \quad \Lambda = k_0 \eta_j / k, \quad k \in \mathbb{Z}.$$

Thus $\Lambda \leq k_0 \eta_j$ and the set of admissible Λ 's has a maximum, therefore Λ_j exists for $1 \leq j \leq h_F$. Finally, by (3.5) it is clear that the Λ_j 's are positive and belong to distinct \mathbb{Q} -equivalence classes. ■

For $1 \leq j \leq h_F$ let

$$(3.6) \quad f_j(z) = (e^{z/\Lambda_j} - 1)K_j(z).$$

By (2.4) applied to $K_j(z)$ we have

$$K_j(z) = -z \sum_{\varrho} e^{-\varrho \Lambda_j z / \Lambda_j},$$

where ϱ runs over the poles of $\gamma_j(s)$. Hence for $\operatorname{Re} z < 0$ we can write

$$(3.7) \quad f_j(z) = z \sum_{l=1}^{\infty} n_j(l) e^{m_{l,j} z / \Lambda_j}$$

with some $n_j(l) \in \mathbb{Z}$ and $m_{l,j} \in \mathbb{C}$ satisfying $\operatorname{Re} m_{l,j} \geq 0$, $\operatorname{Im} m_{l,j} \ll 1$ and

$$0 < A_j \leq \frac{\operatorname{Re} m_{l,j}}{l} \leq B_j$$

for l sufficiently large, where A_j, B_j , $1 \leq j \leq h_F$, are suitable constants. Observe now that $f_j(z)/z$ is a quotient of two exponential polynomials and, by definition of the Λ_j 's, it has at most one pole. Hence by Lemma 3.1 we have $n_j(l) = 0$ for l sufficiently large. Therefore the sum in (3.7) is finite and, rearranging and repeating terms if necessary, we write (3.7) as

$$(3.8) \quad f_j(z) = z \sum_{l=1}^{N_j} n_j(l) e^{m_{l,j} z / \Lambda_j}$$

with $n_j(l) = -1$ for $1 \leq l \leq M_j$ and $n_j(l) = 1$ for $M_j + 1 \leq l \leq N_j$. From (3.6) and (3.8) we therefore obtain

$$(3.9) \quad K_j(z) + z \sum_{l=1}^{M_j} \frac{e^{m_{l,j} z / \Lambda_j}}{e^{z / \Lambda_j} - 1} = z \sum_{l=M_j+1}^{N_j} \frac{e^{m_{l,j} z / \Lambda_j}}{e^{z / \Lambda_j} - 1}.$$

With the data in (3.9) we build the products

$$\prod_{l=1}^{M_j} \Gamma(\Lambda_j s + m_{l,j}), \quad \prod_{l=M_j+1}^{N_j} \Gamma(\Lambda_j s + m_{l,j})$$

and hence the argument in the proof of Theorem 1, showing that $K_F(z)$ determines $\gamma(s)$ up to a factor e^{as+b} (see also the remark after Theorem 1), gives in this case the identities

$$(3.10) \quad \gamma_j(s) \prod_{l=1}^{M_j} \Gamma(\Lambda_j s + m_{l,j}) = e^{a^{(j)}s+b^{(j)}} \prod_{l=M_j+1}^{N_j} \Gamma(\Lambda_j s + m_{l,j}),$$

$1 \leq j \leq h_F,$

with some $a^{(j)} \in \mathbb{R}$ and $b^{(j)} \in \mathbb{C}$. Observing that for every $1 \leq l \leq M_j$ the left hand side of (3.10) has a pole at $s = -m_{l,j}/\Lambda_j$, we deduce that there exist an $l' = l'(l)$, $M_j + 1 \leq l' \leq N_j$, and an integer $k_{l,j} \geq 0$ such that

$$m_{l,j} = m_{l',j} + k_{l,j}, \quad l = 1, \dots, M_j.$$

Therefore, applying $k_{l,j}$ times the factorial formula to $\Gamma(\Lambda_j s + m_{l,j})$ on the left hand side of (3.10), dividing both sides by the common Γ -factors obtained in this way, reducing (again by the factorial formula) the remaining Γ -factors on the right hand side and finally multiplying all identities for $1 \leq j \leq h_F$, we end up with an identity of the form

$$(3.11) \quad \gamma(s) = e^{a_1 s + b_1} R(s) \prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \Gamma(\Lambda_j s + \tilde{\mu}_{j,h})$$

where $a_1 \in \mathbb{R}$, $b_1 \in \mathbb{C}$, $R(s)$ is a rational function and $0 \leq \operatorname{Re} \tilde{\mu}_{j,h} < 1$.

Reducing $\gamma(s)$ we have

$$\gamma(s) = c_\gamma P_F(s) \tilde{\gamma}(s)$$

where $c_\gamma \in \mathbb{C}$, $P_F(s)$ is the canonical polynomial of $F(s)$ and $\tilde{\gamma}(s)$ is the reduced part of $\gamma(s)$ (see Section 3 of [6]), and hence comparing with (3.11) we obtain

$$(3.12) \quad \tilde{\gamma}(s) = e^{a_2 s + b_2} \frac{R(s)}{P_F(s)} \prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \Gamma(\Lambda_j s + \tilde{\mu}_{j,h})$$

with $a_2 \in \mathbb{R}$ and $b_2 \in \mathbb{C}$. Since the Γ -factors on both sides of (3.12) are reduced, from Lemma 2.3(ii) of [6] we see that

$$R(s) = c_1 P_F(s)$$

with $c_1 \in \mathbb{C}$, and hence (3.11) becomes

$$(3.13) \quad \gamma(s) = e^{a_3 s + b_3} P_F(s) \prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \Gamma(\Lambda_j s + \tilde{\mu}_{j,h})$$

with $a_3 \in \mathbb{R}$ and $b_3 \in \mathbb{C}$. We remark at this point that a_3 , the K_j 's and the $\tilde{\mu}_{j,h}$'s in (3.13) are *uniquely determined by $F(s)$* . In fact, if $\gamma'(s)$ is a γ -factor of $F(s)$ we have $\gamma(s) = c_2 \gamma'(s)$ for some $c_2 \in \mathbb{C}$, and hence (3.13) leads to an identity of type

$$\prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \Gamma(\Lambda_j s + \tilde{\mu}_{j,h}) = e^{a_4 s + b_4} \prod_{j=1}^{h_F} \prod_{h=1}^{K'_j} \Gamma(\Lambda_j s + \tilde{\mu}'_{j,h})$$

with $a_4 \in \mathbb{R}$, $b_4 \in \mathbb{C}$ and $0 \leq \operatorname{Re} \tilde{\mu}'_{j,h} < 1$. Our assertion follows then by the argument leading to Lemma 2.3(ii) of [6].

Since the left hand side of (3.13) has no zeros, the zeros of $P_F(s)$ must be cancelled by poles of the Γ -factors. Therefore we can write

$$(3.14) \quad P_F(s) = c_3 \prod_{(j,h,l)} (\Lambda_j s + \tilde{\mu}_{j,h} + l),$$

where $c_3 \in \mathbb{R}$ and the product is over a certain (possibly empty) finite set of triplets (j, h, l) with $1 \leq j \leq h_F$, $1 \leq h \leq K_j$ and $l = 0, 1, \dots$. Observe that the expression of $P_F(s)$ in (3.14) is *not necessarily unique*, since Γ -factors corresponding to different \mathbb{Q} -equivalence classes may have a common pole. Applying the Legendre–Gauss multiplication formula (3.3) with

$$m = \text{maximum } l \text{ in (3.14)} + 1$$

to each Γ -factor in (3.13), from (3.13) and (3.14) we get

$$\gamma(s) = e^{a_5 s + b_5} \prod_{(j,h,l)} \left(\frac{\Lambda_j s + \tilde{\mu}_{j,h} + l}{m} \right) \prod_{j=1}^{h_F} \prod_{h=1}^{K_j} \prod_{l=0}^{m-1} \Gamma \left(\frac{\Lambda_j s + \tilde{\mu}_{j,h} + l}{m} \right)$$

with $a_5 \in \mathbb{R}$ and $b_5 \in \mathbb{C}$. Hence by repeated applications of the factorial formula we can absorb the canonical polynomial into new Γ -factors as

$$(3.15) \quad \gamma(s) = e^{a_6 s + b_6} \prod_{j=1}^{h_F} \prod_{k=1}^{mK_j} \Gamma \left(\frac{\Lambda_j}{m} s + \mu'_{j,k} \right)$$

with $a_6 \in \mathbb{R}$, $b_6 \in \mathbb{C}$ and $\text{Re } \mu'_{j,k} \geq 0$.

We have

LEMMA 3.3. *Let $F \in \mathcal{S}^\sharp$. Then every γ -factor of $F(s)$ has a balanced form. Moreover, the reduction factor of any balanced form is a positive integer.*

Proof. The first assertion follows from (3.15). In order to prove the second, let

$$\gamma(s) = e^{as+b} \prod_{j=1}^{h_F} \prod_{k=1}^{M_j} \Gamma(\lambda_j s + \mu_{j,k})$$

be a balanced form of $\gamma(s)$. Clearly, the function $(e^{z/\lambda_j} - 1)K_j(z)$ is entire for $1 \leq j \leq h_F$, hence $\lambda_j \leq \Lambda_j$ and we write

$$\frac{1}{\lambda_j} = \frac{k}{\Lambda_j} + \frac{\theta}{\Lambda_j}$$

with a positive integer k and $0 \leq \theta < 1$. We therefore have

$$(e^{z/\lambda_j} - 1)K_j(z) = (e^{kz/\Lambda_j} - 1)e^{\theta z/\Lambda_j} K_j(z) + (e^{\theta z/\Lambda_j} - 1)K_j(z),$$

and the first term on the right hand side is easily seen to be entire. Hence the function $(e^{\theta z/\Lambda_j} - 1)K_j(z)$ is entire as well, thus $\theta = 0$ by the maximality of Λ_j , and Lemma 3.3 follows. ■

Observing that for $1 \leq j \leq h_F$ the contribution of the Γ -factors in the j th \mathbb{Q} -equivalence class to the degree d_F of $F(s)$ is an invariant, from

Lemma 3.3, the definition of l_F and (3.15) we get

$$(3.16) \quad \gamma(s) = e^{as+b} \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left(\frac{\Lambda_j}{l_F} s + \mu_{j,k}\right)$$

with $a \in \mathbb{R}$, $b \in \mathbb{C}$ and $\text{Re } \mu_{j,k} \geq 0$. Moreover, l_F is the minimal integer such that (3.16) holds. Note that the $\mu_{j,k}$'s are uniquely determined (mod \mathbb{Z}). In fact, if

$$\gamma(s) = e^{as+b} \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left(\frac{\Lambda_j}{l_F} s + \mu_{j,k}\right) = e^{a's+b'} \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left(\frac{\Lambda_j}{l_F} s + \mu'_{j,k}\right),$$

comparing poles we see that $-l_F(\mu_{j,k}+h)/\Lambda_j$ coincides with $-l_F(\mu'_{j,k}+l)/\Lambda_j$ for suitable integers h and l , therefore the $\mu_{j,k}$'s are unique (mod \mathbb{Z}).

Rewriting (3.16) as

$$(3.17) \quad \gamma(s) = c\gamma_0(s), \quad \gamma_0(s) = Q_F^s \prod_{j=1}^{h_F} \prod_{k=1}^{l_F K_j} \Gamma\left(\frac{\Lambda_j}{l_F} s + \mu_{j,k}\right),$$

we see that $\gamma_0(s)$ is a γ -factor of $F(s)$. Moreover, Q_F and ω_F are easily computed by means of the definition of conductor and of root number, respectively, thus yielding the first part of Theorem 2.

Observe that if $F(s)$ is reduced then the canonical polynomial is trivial. Hence Theorem 2 follows already from (3.13), by observing that $l_F = 1$ in this case. Finally, Theorem 2 follows from (3.17) and Lemma 2.3(i) of [6] if $h_F = 1$. ■

The proof of Theorem 2 allows the explicit computation of (1.9) from any given form of the functional equation of $F \in \mathcal{S}^\sharp$. A sketch of the steps of such a computation can be extracted from the proof of Theorem 2 as follows.

- (i) For $j = 1, \dots, h_F$ compute the Λ_j by means of their definition (1.7).
- (ii) For $j = 1, \dots, h_F$ compute $f_j(z)$ in the form (3.8). Then write (3.10) (without explicit computation of the factor $e^{a^{(j)}s+b^{(j)}}$).
- (iii) Follow the procedure described between (3.10) and (3.11), thus getting (3.13) (without explicit computation of the factor $e^{a_3s+b_3}$).
- (iv) By means of the multiplication and factorial formulae, determine l_F and transform (3.13) into (3.16) (without explicit computation of the factor e^{as+b}).

The form (1.9) of the functional equation is then obtained after computation of Q_F and ω_F by means of the expressions given by (1.8).

We end the section with three examples illustrating Theorem 2 and its proof. First of all, observe that the identity

$$\Gamma(s/2)\Gamma(s+1) = 2\Gamma(s/2+1)\Gamma(s)$$

shows why the λ 's in the canonical form of the functional equation have to be equal in each \mathbb{Q} -equivalence class, in order to have uniquely determined μ 's. Moreover, the identity

$$s\Gamma(s)\Gamma(\sqrt{2}s) = \Gamma(s+1)\Gamma(\sqrt{2}s) = \frac{1}{\sqrt{2}}\Gamma(s)\Gamma(\sqrt{2}s+1)$$

shows that the μ 's in (1.9) are not necessarily unique if $h_F > 1$ and $F(s)$ is not reduced.

Suppose now that

$$\gamma(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+3}{2}\right).$$

In this case $h_F = 1$, $A_1 = 1$, $K_1 = 1$, $P_F(s) = s + 1$ and (3.13) becomes

$$\gamma(s) = e^{as+b}(s+1)\Gamma(s),$$

thus showing the need of a further application of the multiplication formula to (3.13) in order to absorb the canonical polynomial into new Γ -factors.

4. Proof of Theorem 3. For any fixed $N \in \mathbb{N}$ consider the symmetric polynomials

$$p_k(X_1, \dots, X_N) = \sum_{j=1}^N X_j^k, \quad k \in \mathbb{N}.$$

We need the following lemma, which is a classical result on symmetric functions based on Newton's formulae (see Section 1.5 of [3]).

LEMMA 4.1. (i) If $\mathbf{z} = (z_1, \dots, z_N)$, $\mathbf{w} = (w_1, \dots, w_N) \in \mathbb{C}^N$ satisfy

$$p_k(\mathbf{z}) = p_k(\mathbf{w}), \quad k = 1, \dots, N,$$

then there exists a permutation σ such that $z_j = w_{\sigma(j)}$ for $j = 1, \dots, N$.

(ii) There exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ such that

$$p_k(\mathbf{x}) = p_k(\mathbf{y}) \quad \text{for } k = 1, \dots, N-1 \quad \text{and} \quad p_N(\mathbf{x}) \neq p_N(\mathbf{y}).$$

Let $F \in \mathcal{S}^\sharp$. In order to prove Theorem 3(i) we observe that by Theorem 1 it is sufficient to show that if $G \in \mathcal{S}^\sharp$ has $h_G = h_F$, $r_G = r_F$ and

$$(4.1) \quad H_G(n) = H_F(n) \quad \text{for } n \leq n_F (= n_G)$$

then

$$(4.2) \quad H_G(n) = H_F(n) \quad \text{for all } n \geq 0.$$

Denote by A_1, \dots, A_{h_F} and A'_1, \dots, A'_{h_F} the canonical exponents of $F(s)$ and $G(s)$, respectively. Computing $K_F(z)$ and $K_G(z)$ by means of the form of

the functional equation given in Theorem 2 we see that the entire function

$$(4.3) \quad f(z) = \frac{1}{z} \prod_{j=1}^{h_F} (e^{J_F z / \Lambda_j} - 1)(e^{J_G z / \Lambda'_j} - 1)(K_F(z) - K_G(z))$$

can be written in the form

$$(4.4) \quad f(z) = \sum_{l=1}^L e^{\alpha_l z} - \sum_{l=1}^L e^{\beta_l z}$$

with $\alpha_l, \beta_l \in \mathbb{C}$ and

$$(4.5) \quad L \leq r_F 2^{2h_F - 1}.$$

From (1.3) and (4.1) we see that the coefficients of the Taylor expansion of $K_F(z) - K_G(z)$ vanish for $n \leq n_F$, hence from (4.3) we deduce that $f(z)$ has a zero of order at least $n_F + 2h_F$ at $z = 0$. Therefore

$$f^{(k)}(0) = 0, \quad k \leq n_F + 2h_F - 1$$

and hence computing $f^{(k)}(0)$ by means of (4.4) we obtain

$$(4.6) \quad \sum_{l=1}^L \alpha_l^k = \sum_{l=1}^L \beta_l^k, \quad k \leq n_F + 2h_F - 1.$$

From Lemma 4.1(i), (4.5) and (4.6) we see that if

$$n_F = r_F 2^{2h_F - 1} - 2h_F + 1$$

then the α_l 's are a permutation of the β_l 's, therefore $f(z) = 0$ identically by (4.4) and hence $K_F(z) = K_G(z)$ by (4.3). As a consequence, (4.2) holds and hence Theorem 3(i) follows.

We turn to the proof of Theorem 3(ii). In this case $F(s)$ and $G(s)$ have a functional equation with exactly d_F Γ -factors with λ -coefficients all equal to $1/2$. If we replace $f(z)$ in (4.3) by

$$f(z) = \frac{e^{2z} - 1}{z} (K_F(z) - K_G(z)),$$

(4.4) becomes

$$(4.7) \quad f(z) = \sum_{l=1}^{d_F} e^{\alpha_l z} - \sum_{l=1}^{d_F} e^{\beta_l z}.$$

If we assume (4.1) with n_F replaced by d_F , the argument leading to (4.6) gives in this case

$$(4.8) \quad \sum_{l=1}^{d_F} \alpha_l^k = \sum_{l=1}^{d_F} \beta_l^k, \quad k \leq d_F,$$

and the first part of Theorem 3(ii) follows from Lemma 4.1(i), (4.7) and (4.8), as before.

Now we construct examples of L -functions of integer degree $d \geq 1$ satisfying (1.11) with n_F replaced by $d - 1$ and having different functional equations. For $d = 1$ we just consider

$$F(s) = L(s, \chi), \quad G(s) = L(s + i\theta, \chi)$$

with a primitive Dirichlet character $\chi \pmod{q}$, $q > 1$, and $\theta \neq 0$. Recalling that conductor and root number are also shift-invariants while the ξ -invariant $H_F(1)$ is not shift-invariant (see Section 4 of [6]), we see that $F(s)$ and $G(s)$ provide the required example for $d = 1$.

For $d \geq 2$ we fix an even primitive Dirichlet character $\chi \pmod{q}$, $q > 1$, and write

$$L(s, \mathbf{x}, \chi) = \prod_{j=1}^d L(s + ix_j, \chi), \quad \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

By Lemma 4.1(ii) there exist $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d), \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ such that

$$(4.9) \quad \sum_{j=1}^d \theta_j^k = \sum_{j=1}^d \xi_j^k \quad \text{for } k = 1, \dots, d-1 \quad \text{and} \quad \sum_{j=1}^d \theta_j^d \neq \sum_{j=1}^d \xi_j^d,$$

and we consider

$$F(s) = L(s, \boldsymbol{\theta}, \chi), \quad G(s) = L(s, \boldsymbol{\xi}, \chi).$$

Again, $\omega_F^* = \omega_G^*$ and $q_F = q_G = q^d$ by shift-invariance, while (4.9) implies that

$$(4.10) \quad H_F(n) = H_G(n) \quad \text{for } n = 1, \dots, d-1 \quad \text{and} \quad H_F(d) \neq H_G(d).$$

In fact, in this case the function $f(z)$ in (4.7) becomes

$$f(z) = \sum_{j=1}^d e^{2i\theta_j z} - \sum_{j=1}^d e^{2i\xi_j z}$$

and hence (4.9) implies that

$$f^{(k)}(0) = 0 \quad \text{for } k = 1, \dots, d-1 \quad \text{and} \quad f^{(d)}(0) \neq 0.$$

Therefore $K_F(z) - K_G(z)$ has a zero of order exactly d at $z = 0$, thus (4.10) follows. The proof of Theorem 3 is now complete, since by Theorem 1 the functional equations of $F(s)$ and $G(s)$ are different.

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