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Uniqueness polynomials and bi-unique range sets for rational functions and non-Archimedean meromorphic functions

by

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1. Introduction. Let k be an algebraically closed field of characteristic $p \geq 0$, complete with respect to a non-Archimedean absolute value. Denote by $\mathcal{M}(k)$ the field of meromorphic functions in k. For $f \in \mathcal{M}(k)$ and $\mathcal{S}$ a subset of k, we let

$$E(f,\mathcal{S}) = \{(z,m) \in k \times \mathbb{Z}^+ : f(z) = a \in \mathcal{S} \text{ and } f(z) = a \text{ with multiplicity } m\},$$

$$E(f,\mathcal{S} \cup \{\infty\}) = E(f,\mathcal{S}) \cup \{(z,m) \in k \times \mathbb{Z}^+ : z \text{ is a pole of order } m \text{ of } f\}.$$

Let $\mathcal{F}$ be a non-empty subset of $\mathcal{M}(k)$. A subset S of $k \cup \{\infty\}$ is called a *unique range set* (*URS* for short) for $\mathcal{F}$ if $f = g$ for any $f, g \in \mathcal{F}$ such that $E(f,S) = E(g,S)$. Similarly, let S,T be two subsets of $k \cup \{\infty\}$ with $S \cap T = \emptyset$. (S,T) is called a *bi-URS* for $\mathcal{F}$ if $f = g$ for any $f, g \in \mathcal{F}$ such that $E(f,S) = E(g,S)$ and $E(f,T) = E(g,T)$.

Several interesting results about URS and bi-URS for non-Archimedean entire and meromorphic functions of zero characteristic have been obtained in [BE1-3], [BEH], [CY], [EHV], and [KA]. Cherry and Yang [CY] gave a characterization of a finite subset $\mathcal{S}$ of k to be an URS for p-adic entire functions by observing that it is equivalent to $\mathcal{S}$ being an URS for the polynomial ring over k. Recently, Khoai and An [KA] gave sufficient conditions for URS and bi-URS in terms of uniqueness polynomials and strong uniqueness polynomials for non-Archimedean meromorphic functions by using tools from p-adic Nevanlinna's theorem. In this paper we will give necessary and sufficient conditions for uniqueness polynomials and strong uniqueness polynomials for non-Archimedean meromorphic functions including the case of positive characteristic. Our approach is based on Cherry–Yang's observation

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and uses some basic results on plane curves and the truncated second main theorem for rational functions (cf. [W1, 2]).

We recall some definitions.

Definition. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{M}(k)$.

(1) A non-constant polynomial $P(X)$ over k is said to be a *uniqueness polynomial* for $\mathcal{F}$ if the identity $P(f) = P(g)$ implies $f = g$ for any pair of non-constant functions $f, g \in \mathcal{F}$.

(2) A non-constant polynomial $P(X)$ over k is said to be a *strong uniqueness polynomial* for $\mathcal{F}$ if the identity $P(f) = cP(g)$ implies $f = g$ for any pair of non-constant functions $f, g \in \mathcal{F}$ and for any non-zero constant c.

Let $P(X)$ be a polynomial of degree n in $k[X]$. We say it satisfies condition (I) if

(I) $P(X)$ is injective on the roots of $P'(X)$.

The basic ideas of the paper are as follows. Consider the plane curves $F(X,Y) = (P(X) - P(Y))/(X - Y) = 0$, and $F_c(X,Y) = P(X) - cP(Y)$ with $c \neq 0, 1$. If $P(f) = P(g)$ (resp. $P(f) = cP(g)$) for a pair of distinct non-constant non-Archimedean meromorphic functions f, g, then by Berkovich's non-Archimedean Picard theorem, the plane curve $F(X,Y) = 0$ (resp. $F_c(X,Y) = 0$) has a rational component. Therefore, $P(X)$ is a strong uniqueness polynomial if and only if the curves $F(X,Y) = 0$ and $F_c(X,Y) = 0$ for all $c \neq 0, 1$ have no rational components. In general, it is difficult to show a curve has no rational component if it has many multiple points; and even more difficult if it has non-ordinary multiple points. Therefore, we need to assume condition (I) to reduce the number of multiple points. Even so, there may still exist some non-ordinary multiple points for $F_c(X,Y) = 0$. We will use the truncated second main theorem for rational function fields to show that the local expansion of $F_c(X,Y) = 0$ at non-ordinary multiple points does not behave too badly for this. Indeed, one can perform a sequence of linear and quadratic transformations to transform it into a curve with only ordinary multiple points and having the same deficiency as $F_c(X,Y) = 0$.

The main results are as follows.

Theorem 1. *Let k be an algebraically closed field of characteristic $p \geq 0$, complete with respect to a non-Archimedean absolute value. Let $P(X)$ be a polynomial of degree n in $k[X]$, and $P'(X) = \lambda(X - \alpha_1)^{m_1} \ldots (X - \alpha_l)^{m_l}$ where λ is a non-zero constant. Suppose that $P(X)$ satisfies condition* (I). *Furthermore, for $p > 0$ assume that the multiplicity of $X - \alpha_i$ in $P(X) - P(\alpha_i)$ is $m_i + 1$, for $1 \leq i \leq l$; in addition if $p \,|\, n$ assume that the coefficient of X^{n-1} in $P(X)$ is not zero. Then the following are equivalent*:

(i) *$P(X)$ is a uniqueness polynomial for $\mathcal{M}(k)$,*

(ii) *$P(X)$ is a uniqueness polynomial for the field of rational functions in k,*

(iii) $(n-2)(n-3)/2 - \sum_{i=1}^{l} m_i(m_i-1)/2 > 0$,

(iv) *$l \geq 2$ if $p \mid n$; and $l \geq 3$ or $l = 2$ and $\min\{m_1, m_2\} \geq 2$ if $p = 0$ or $p \nmid n$.*

Definition. A subset S of k is called *affinely rigid* if no non-trivial affine transformation of k preserves S.

Theorem 2. *Let k be an algebraically closed field of characteristic p, complete with respect to a non-Archimedean absolute value. Let $P(X)$ be a polynomial of degree n in $k[X]$, and $P'(X) = \lambda(X-\alpha_1)^{m_1}\dots(X-\alpha_l)^{m_l}$ where λ is a non-zero constant. Suppose that $P(X)$ satisfies condition* (I) *and has no multiple zeros. Furthermore, for $p > 0$ assume that the multiplicity of $X - \alpha_i$ in $P(X) - P(\alpha_i)$ is $m_i + 1$, for $1 \leq i \leq l$; in addition if $p \mid n$ assume that the coefficient of X^{n-1} in $P(X)$ is not zero. Let S be the set of roots of $P(X) = 0$. Then the following are equivalent*:

(i) *$(S, \{\infty\})$ is a bi-URS for $\mathcal{M}(k)$,*

(ii) *$P(X)$ is a strong uniqueness polynomial for $\mathcal{M}(k)$,*

(iii) *$P(X)$ is a strong uniqueness polynomial for the field of rational functions in k,*

(iv) *S is affinely rigid, and one of the following holds*:

(a) *if $p \mid n$, then either $l \geq 2$;*

(b) *if $p = 0$ or $p \nmid n$, then either $l \geq 3$ but $P(X)$ does not satisfy* (A), *or $l = 2$ and $\min\{m_1, m_2\} \geq 2$ but $P(X)$ does not satisfy* (B), *where* (A) *and* (B) *are as follows*:

$$\text{(A)}\qquad n = 4, \quad m_1 = m_2 = m_3 = 1 \quad \textit{and} \quad \frac{P(\alpha_1)}{P(\alpha_2)} = \frac{P(\alpha_2)}{P(\alpha_3)} = \frac{P(\alpha_3)}{P(\alpha_1)} = \omega,$$

where $\omega^2 + \omega + 1 = 0$;

$$\text{(B)}\qquad n = 5, \quad m_1 = m_2 = 2 \quad \textit{and} \quad P(\alpha_1) = -P(\alpha_2).$$

Remark. (1) The characterization of uniqueness polynomials of rational functions is independent of the constant field. To be more precise, under the same assumption, (ii), (iii) and (iv) in Theorem 1 are equivalent and (i), (iii) and (iv) in Theorem 2 are equivalent if k is replaced by an algebraically closed field of the same characteristic.

(2) When $p > 0$, if the multiplicity of $X - \alpha_i$ in $P(X) - P(\alpha_i)$ is $m_i + 1$, for $1 \leq i \leq l$, then $p \nmid m_i + 1$.

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In the message, Voloch gave a simpler proof of a result of [CY] which is included in the appendix. The author thanks J. F. Voloch for sharing his ideas and allowing her to include his proof. The author also thanks A. Escassut and the referee for helpful comments.

2. Singularities of two plane curves. Without loss of generality, we may assume that $P(X)$ is monic. Throughout the paper we let

$$P(X) = \sum_{i=0}^{n} a_i X^i, \quad a_i \in k,\ a_n = 1,$$

$$P'(X) = \begin{cases} n(X-\alpha_1)^{m_1}\dots(X-\alpha_l)^{m_l} & \text{if } p = 0 \text{ or} \\ & \quad p > 0 \text{ and } p \nmid n, \\ -a_{n-1}(X-\alpha_1)^{m_1}\dots(X-\alpha_l)^{m_l} & \text{if } p > 0 \text{ and } p \mid n, \end{cases}$$

$$F(X,Y) = (P(X)-P(Y))/(X-Y),$$

$$F_c(X,Y) = P(X) - cP(Y), \quad c \neq 0, 1.$$

Denote by $F(X,Y) = 0$ and $F_c(X,Y) = 0$ the algebraic curves in $\mathbb{P}^2(k)$ obtained by homogenizing these polynomials into homogeneous polynomials in three variables with the same degree. In this section, we will study the singularities of these plane curves.

We first discuss the singularities of $F(X,Y) = 0$. Let p be the characteristic of k. If $p = 0$ or $p > 0$ and $p \nmid n$, then $F(X,Y) = 0$ has $n-1$ distinct points at infinity, hence they are all non-singular. If $p \mid n$ and $a_{n-1} \neq 0$, then $(1,1,0)$ is the only point at infinity and its multiplicity is one.

For the affine points, we have

$$\frac{\partial F}{\partial X} = \frac{(X-Y)P'(X) - (P(X)-P(Y))}{(X-Y)^2},$$

$$\frac{\partial F}{\partial Y} = \frac{-(X-Y)P'(Y) + (P(X)-P(Y))}{(X-Y)^2}.$$

A point (x,y) with $x \neq y$ is in $F(X,Y) = 0$ if and only if $P(x) = P(y)$, and is a singular point if and only if $P'(x) = P'(y) = 0$. Hence, if $P(X)$ satisfies condition (I), then it is non-singular.

A point (x,x) is in $F(X,Y) = 0$ if and only if $P'(x) = 0$. If (x,x) is in $F(X,Y) = 0$, we may assume that $x = 0$ and $P'(0) = 0$ after changing variables. If $p = 0$ or $p > 0$, and the multiplicity of X in $P'(X)$ is m, then by assumption the multiplicity of X in $P(X) - P(0)$ is $m+1$. Clearly, $p \nmid m+1$, and the multiplicity of $(0,0)$ in $F(X,Y) = 0$ is m. Hence,

$$F(X,Y) = a_{m+1}(X^m + X^{m-1}Y + \dots + XY^{m-1} + Y^m) + a_{m+2}(X^{m+1} + \dots) + \dots$$

where $a_{m+1} \neq 0$, and $X^m + X^{m-1}Y + \dots + XY^{m-1} + Y^m$ factors into m

distinct linear forms. This shows that $(0,0)$ is an ordinary singularity. In conclusion, we have

Proposition 1. *Suppose that $P(X)$ satisfies all the conditions in Theorem 1. Then the set of singular points of $F(X,Y)=0$ is $\{(x,x) \mid x$ is a multiple root of $P'(X)=0\}$. Furthermore, every singular point in $F(X,Y)=0$ is an ordinary singularity with multiplicity equal to its root multiplicity in $P'(X)$.*

We now discuss the singularities of $F_c(X,Y)=0$. Similarly, if $P(X)$ satisfies the conditions in Theorem 1, then the curve has no singular points at infinity. For affine points, since

$$\frac{\partial F}{\partial X}=P'(X), \qquad \frac{\partial F}{\partial Y}=-cP'(Y), \qquad c\neq 0,1,$$

a point (x,x) is in $F_c(X,Y)=0$ if and only if $P(x)=0$, and is a singular point if and only if $P'(x)=0$. That is equivalent to x being a double root of $P(X)$. If $P(X)$ has no multiple roots, then (x,x) cannot be a singular point. On the other hand, an affine point (x,y) with $x\neq y$ in $F_c(X,Y)=0$ is a singular point if and only if $P(x)=cP(y)$ and $P'(x)=P'(y)=0$. If (α_i,α_j) and (α_i,α_k) are two points in $F_c(X,Y)=0$, then $P(\alpha_j)=P(\alpha_k)$. If $P(X)$ satisfies condition (I), then $\alpha_j=\alpha_k$. Therefore there are at most l possible singular points of this type $(\alpha_i,\alpha_{t(i)})$, $1\le i\le l$, where t is a permutation of $\{1,\dots,l\}$ and $t(i)\neq i$. In conclusion, we have

Proposition 2. *Suppose that $P(X)$ satisfies all the conditions on Theorem 2. Then $F_c(X,Y)=0$ has at most l singular points $(\alpha_i,\alpha_{t(i)})$, $1\le i\le l$, where t is a permutation of $\{1,\dots,l\}$ and $t(i)\neq i$. Furthermore, the multiplicity of $(\alpha_i,\alpha_{t(i)})$ in $F_c(X,Y)=0$ is less than or equal to $\min\{m_i,m_{t(i)}\}+1$.*

Lemma 1. *Assume that $P(X)$ satisfies all the conditions in Theorem 2. Suppose there exists a pair of non-constant rational functions (f,g) such that $P(f)=cP(g)$. Then $|m_i-m_{t(i)}|\le 1$ if $P(\alpha_i)=cP(\alpha_{t(i)})$.*

To prove this lemma, we will use the truncated second main theorem for rational functions which is stated as follows.

Truncated Second Main Theorem. *Let $\mathfrak{f}$ be a non-constant rational function over k, and assume $\mathfrak{f}$ is not a pth power if the characteristic p of k is positive. Let $\mathfrak{f}$ be a ratio of two relative prime polynomials $\mathfrak{f}_1$ and $\mathfrak{f}_2$, and let $c_1,\dots,c_q$ be q distinct elements in k. Then*

$$(q-2)\max\{\deg\mathfrak{f}_1,\deg\mathfrak{f}_2\}\le\sum_{i=1}^{q}N_1(\mathfrak{f}-c_i)-1$$

where $N_1(\mathfrak{f}-c_i)$ is the number of distinct zeros of $\mathfrak{f}-c_i$.

We refer to [W1] or [W2] for the proof and a more general statement of this theorem.

Proof of Lemma 1. Let $f = f_1/f_2$ and $g = g_1/g_2$ where f_i and g_i are polynomials over k, f_1 is prime to f_2 and g_1 is prime to g_2. Since $P(f) = cP(g)$, the pole order of f at any point of k equals the pole order of g. Therefore g_2 is a non-zero constant multiple of f_2. By adjusting the coefficients of g_1, we may assume that $g_2 = f_2$. Then $P(f) = cP(g)$ gives

$$f_1^n - cg_1^n + a_{n-1}(f_1^{n-1} - cg_1^{n-1})f_2 + \ldots + a_0(1-c)f_2^n = 0.$$

If $\deg f_1 > \deg f_2$, then $\deg g_1 = \deg f_1$. If $\deg f_1 \le \deg f_2$, then $\deg g_1 \le \deg f_2$. Therefore,

$$\max\{\deg f_1, \deg f_2\} = \max\{\deg g_1, \deg f_2\}. \tag{2.1}$$

If $P(\beta) = P(\alpha_i)$ and $\beta \ne \alpha_i$, then $P'(\beta) \ne 0$ because $P(X)$ satisfies condition (I). Hence, for each $1 \le i \le l$, we have

$$P(X) = P(\alpha_i) + (X - \alpha_i)^{m_i+1}(X - \beta_{i1}) \ldots (X - \beta_{in-m_i-1}).$$

If $P(\alpha_i) = cP(\alpha_{t(i)})$, then

$$\begin{aligned}(f - \alpha_i)^{m_i+1}&(f - \beta_{i1}) \ldots (f - \beta_{in-m_i-1}) \\ &= (g - \alpha_{t(i)})^{m_{t(i)}+1}(g - \beta_{t(i)1}) \ldots (g - \beta_{t(i)n-m_{t(i)}-1}).\end{aligned} \tag{2.2}$$

Since α_i, β_{ij}, $1 \le j \le n - m_i - 1$, are distinct, $f - \alpha_i$ and $f - \beta_{ij}$, $1 \le j \le n - m_i - 1$, have no common zero pairwise. Similarly, $g - \alpha_{t(i)}$ and $g - \beta_{t(i)j}$, $1 \le j \le n - m_{t(i)} - 1$, have no common zero pairwise.

By assumption, f is not constant. When $p > 0$, we may write $f = \mathfrak{f}^{p^r}$ where $\mathfrak{f}$ is not a pth power and r is a non-negative integer. Since k is algebraically closed, the left hand side of (2.2) is a p^rth power. Since $g - \alpha_{t(i)}$ and $g - \beta_{t(i)j}$, $1 \le j \le n - m_{t(i)} - 1$, have no common zero pairwise, g is also a p^rth power function. Let $g = \mathfrak{g}^{p^r}$. For $\alpha \in k$, denote by α^{1/p^r} the unique solution of $X^{p^r} = \alpha$. Then from (2.2), we have

$$\begin{aligned}N_1(\mathfrak{f} - \alpha_i^{1/p^r}) &+ \sum_{j=1}^{n-m_i-1} N_1(\mathfrak{f} - \beta_{ij}^{1/p^r}) \\ &= N_1(\mathfrak{g} - \alpha_{t(i)}^{1/p^r}) + \sum_{j=1}^{n-m_{t(i)}-1} N_1(\mathfrak{g} - \beta_{t(i)j}^{1/p^r}).\end{aligned}$$

Now we will apply the truncated second main theorem for the non-pth power rational function $\mathfrak{f}$ and $n - m_i$ distinct points α_i^{1/p^r} and β_{ij}^{1/p^r}, $j = 1, \ldots, n - m_i - 1$. We have

$$\begin{aligned}(n-m_i-2)p^{-r}&\max\{\deg f_1,\deg f_2\}\\ &\le N_1(\mathfrak{f}-\alpha_i^{1/p^r})+\sum_{j=1}^{n-m_i-1}N_1(\mathfrak{f}-\beta_{ij}^{1/p^r})-1\\ &=N_1(\mathfrak{g}-\alpha_{t(i)}^{1/p^r})+\sum_{j=1}^{n-m_{t(i)}-1}N_1(\mathfrak{g}-\beta_{t(i)j}^{1/p^r})-1\\ &\le (n-m_{t(i)})p^{-r}\max\{\deg g_1,\deg f_2\}-1.\end{aligned}$$

This implies $m_{t(i)}-m_i\le 1$, by (2.1). Similarly, we have $m_i-m_{t(i)}\le 1$.

When $p=0$, since f is not constant we may apply the truncated second main theorem directly to derive the same result. ∎

We will need the following lemma for later computation.

LEMMA 2. *Let $d>0$ and $e_i\ge 2$ be integers and $(d-1)(d-2)=\sum_{i=1}^h e_i(e_i-1)+2g$, where $g=0$ or $g=1$. If $h\ge 2$, then $\sum_{i=1}^h e_i\ge d+h-1-g$; if $h=1$ and $g=0$, then $e_1=d-1$.*

Proof. We have

$$\sum_{i=1}^h e_i(e_i-1)=\Big(\sum_{i=1}^h e_i\Big)^2-\sum_{i=1}^h e_i-2\sum_{1\le i<j\le h}e_ie_j.$$

It is easy to see that

$$\sum_{1\le i<j\le h}e_ie_j\ge\frac{h-1}{2}\min_i\{e_i\}\sum_{1\le i\le h}e_i\ge(h-1)\sum_{1\le i\le h}e_i.$$

If $g=1$ and $h\ge 2$, or $g=0$ and $h\ge 1$, then

$$\begin{aligned}(d-1)(d-2)&\le\Big(\sum_{i=1}^h e_i\Big)^2-(2h-1)\sum_{i=1}^h e_i+2g\\ &\le\Big(\sum_{i=1}^h e_i\Big)^2-(2h-1)\sum_{i=1}^h e_i+h(h-1)\\ &=\Big(\sum_{i=1}^h e_i-h+1\Big)\Big(\sum_{i=1}^h e_i-h\Big).\end{aligned}$$

Therefore, $\sum_{i=1}^h e_i\ge d+h-2$. It is also clear that the inequality is strict when $g=0$ and $h\ge 2$. In this case we have $\sum_{i=1}^h e_i\ge d+h-1$. Therefore, $\sum_{i=1}^h e_i\ge d+h-1-g$ when $h\ge 2$. The other assertion is clear. ∎

3. The proof of Theorem 1. Let $Q(X,Y)$ be a polynomial in two variables over k, and $\deg Q$ be the highest total degree. Then it can be homogenized into a homogeneous polynomial $\overline{Q}(X,Y,Z)$ of the same degree. For simplicity, we sometimes use $Q(X,Y)=0$ to denote the plane curve given by $\overline{Q}(X,Y,Z)=0$. Denote by δ_Q the deficiency of the plane curve $\overline{Q}(X,Y,Z)=0$, which is

$$\frac{1}{2}(\deg Q-1)(\deg Q-2)-\frac{1}{2}\sum_P m_P(m_P-1)$$

where the sum is taken over all points in $\overline{Q}(X,Y,Z)=0$ and m_P is the multiplicity of $\overline{Q}(X,Y,Z)=0$ at P.

Lemma 3. *Suppose that $P(X)$ satisfies all the conditions in Theorem 1. Then $F(X,Y)$ has an irreducible polynomial factor which defines a plane curve of genus zero if and only if*

$$\delta_F=\frac{(n-2)(n-3)}{2}-\sum_{i=1}^{l}\frac{m_i(m_i-1)}{2}\le 0.$$

Proof. From Proposition 1, $(\alpha_1,\alpha_1),\dots,(\alpha_l,\alpha_l)$ are the only possible singular points of $F(X,Y)=0$, and they are all ordinary under our assumption. If $F(X,Y)$ is irreducible, then δ_F is the genus of the defining curve. Therefore the assertion is clear.

Assume that $F(X,Y)$ is reducible. Let $H(X,Y)\in k[X,Y]$ be a proper irreducible factor of $F(X,Y)$, and write $F(X,Y)=G(X,Y)H(X,Y)$. Then $H(X,Y)\nmid G(X,Y)$ because $F(X,Y)=0$ has only finitely many multiple points. In this case (α_i,α_i), $1\le i\le l$, are the only possible multiple points of $G(X,Y)=0$ and $H(X,Y)=0$, and they are also the only possible points in the intersection of these two curves. Let m_i^G and m_i^H be the multiplicity of (α_i,α_i) in $G(X,Y)=0$ and $H(X,Y)=0$ respectively. Then the genus of the curve defined by $H(X,Y)=0$ equals its deficiency δ_H. Since $P'(X)=F(X,X)=G(X,X)H(X,X)$, we have

$$\sum_{i=1}^{l} m_i^H=d \quad\text{and}\quad m_i\le d-1,$$

where d is the total degree of $H(X,Y)$. Without loss of generality, we may assume that the m_i^H are arranged in decreasing order. From Lemma 2 one can show easily that $\delta_H=0$ if and only if (i) $d=1$ and $m_1^H=1$, $m_i^H=0$ for $2\le i\le l$ or (ii) $d\ge 2$ and $m_1^H=d-1$, $m_2^H=1$, and $m_i^H=0$ for $3\le i\le l$. By Bézout's theorem,

$$d(n-d-1)=\sum_{i=1}^{l} m_i^G m_i^H.$$

If (i) holds, we get $m_1^G=n-d-1=\deg G$, which implies that $\delta_G<0$. If

(ii) holds, then

$$d(n-d-1) = (d-1)m_1^G + m_2^G \leq (d-1)(m_1^G + m_2^G) \leq (d-1)(n-d-1),$$

which is impossible. This shows that if $\delta_H = 0$, then $\delta_G < 0$. From Bézout's theorem and simple counting one can easily verify that

$$\delta_F = \delta_G + \delta_H - 1.$$

Therefore we may conclude that if $\delta_H = 0$, then $\delta_F < 0$.

On the other hand, since $F(X,Y) = 0$ has $n-1$ distinct points at infinity, $F(X,Y)$ cannot have multiple irreducible factors. Let $F(X,Y) = \prod_{i=1}^{j} H_i(X,Y)$ where $H_i(X,Y)$ are distinct irreducible polynomials. Then $\delta_F = \sum_{i=1}^{j} \delta_{H_i} - j + 1$. If $\delta_F \leq 0$, then at least one of the δ_{H_i} has to be zero. Hence $F(X,Y)$ has an irreducible factor of genus zero. ∎

Proof of Theorem 1. (i) implies (ii) trivially. The proof of (ii) implying (i) was already given in [CY]. We include the proof for reader's convenience. Assume that $P(X)$ is a uniqueness polynomial for the ring of rational functions. If f and g are two distinct non-constant non-Archimedean meromorphic functions such that $P(f) = P(g)$, then $F(f,g) = 0$. Let $H(X,Y)$ be an irreducible factor of $F(X,Y)$ such that $H(f,g) = 0$. Since f and g are not constant, by Berkovich's non-Archimedean Picard theorem (cf. [Ber] and also [CW] for a more elementary proof), $H(X,Y) = 0$ is a curve of genus zero. Since k is algebraically closed, this curve is rationally parametrized. In other words, there exist non-constant rational functions $r(t)$ and $s(t)$ and $R(X,Y)$ such that $t = R(X,Y)$ and $H(r(t), s(t)) = 0$. Let $h = R(X,Y)$, so that $f = r(h)$ and $g = s(h)$. Since $P(X)$ is a uniqueness polynomial for the rational function fields, $f = r(h) = s(h) = g$.

$P(X)$ is a uniqueness polynomial for rational functions if and only if $F(X,Y)$ has no irreducible polynomial factors which define a plane curve of genus zero. Therefore (ii) and (iii) are equivalent by Lemma 3.

Without loss of generality, we may assume that m_i's are in decreasing order. If $p = 0$ or $p > 0$ and $p \nmid n$, then $\sum_{i=1}^{l} m_i = n-1$. By Lemma 2, $\delta_F \leq 0$ if and only if (a) $m_1 = n-1$, and $m_i = 0$ for $i \geq 2$ or (b) $m_1 = n-2$, $m_2 = 1$ and $m_i = 0$ for $i \geq 3$. Similarly, if $p \mid n$ and $a_{n-1} \neq 0$, then $\sum_{i=1}^{l} m_i = n-1$. Similarly, $\delta_F \leq 0$ if and only if $m_1 = n-2$, and $m_i = 0$ for $i \geq 2$. Therefore (iii) and (iv) are equivalent. ∎

4. The proof of Theorem 2. Since the plane curve $F_c(X,Y) = 0$ may have non-ordinary multiple points, its deficiency does not necessarily equal its genus when $F_c(X,Y)$ is irreducible. However, if $P(X)$ satisfies all the conditions in Theorem 2 and there exists a pair of non-constant rational functions (f,g) such that $P(f) = cP(g)$, then the deficiency of the irreducible plane curve $F_c(X,Y) = 0$ does equal its genus. We will deduce this fact by showing that there exists a sequence of linear and quadratic transformations which takes $F_c(X,Y)=0$ to a curve which has only ordinary singularities and has the same deficiency as $F_c(X,Y)=0$. Furthermore, we will show that Bézout's theorem still holds in the sense of Lemma 4(2).

LEMMA 4. *Assume that $P(X)$ satisfies all the conditions in Theorem 2. If $P(f)=cP(g)$ for a pair of non-constant rational functions (f,g), then*:

(1) *There exists a sequence of linear and quadratic transformations which takes $F_c(X,Y)=0$ to a curve which has only ordinary singularities and has the same deficiency as $F_c(X,Y)=0$.*

(2) *If $F_c(X,Y)$ is reducible, say $F_c(X,Y)=H(X,Y)G(X,Y)$, then*

$$\deg H \deg G = \sum_P m_P^H m_P^G,$$

where the sum is taken over the intersection of $H=0$ and $G=0$, and m_P^H (resp. m_P^G) is the multiplicity of H (resp. G) at P.

Proof. From Lemma 1, we see that $(\alpha_1,\alpha_{t(1)}),\dots,(\alpha_l,\alpha_{t(l)})$ are the only possible singular points of $F_c(X,Y)=0$. Moreover, if $P(\alpha_i)=cP(\alpha_{t(i)})$, then $|m_i-m_{t(i)}|\le 1$. By assumption,

$$\begin{aligned}F_c(X,Y)&=\nu_i(X-\alpha_i)^{m_i+1}+\{\text{higher order terms in } X-\alpha_i\}\\&\quad+\mu_i(Y-\alpha_{t(i)})^{m_{t(i)}+1}+\{\text{higher order terms in } Y-\alpha_{t(i)}\},\end{aligned}$$

where $\nu_i,\mu_i\neq 0$, and $p\nmid m_i+1$ if $p>0$. If $m_i=m_{t(i)}$, then $(\alpha_i,\alpha_{t(i)})$ is an ordinary singularity. If $|m_i-m_{t(i)}|=1$, then it is not an ordinary singularity and the multiplicity is $\min\{m_i,m_{t(i)}\}+1$. In what follows we will perform a sequence of linear and quadratic transformations to obtain a curve with only ordinary singularities and with the same deficiency as $F_c(X,Y)=0$. We refer to [Ful, Chapter 5] for notation and terminology. For simplicity of notation we denote by $F_c(X,Y,Z)$ the homogenization of $F_c(X,Y)$. Suppose that $m_{t(i)}=m_i+1$. We first make a linear transformation which takes the curve in an excellent position, and the point $(\alpha_i,\alpha_{t(i)})$ to origin. Let

$$\begin{aligned}Q_0(X,Y,Z)&=F_c(X+\alpha_iZ+Y,Y+\alpha_{t(i)}Z,Z)\\&=\nu_i(X+Y)^{m_i+1}Z^{n-m_i-1}+\nu_{i1}(X+Y)^{m_i+2}Z^{n-m_i-2}\\&\qquad+\dots+(X+Y)^n+\mu_iY^{m_i+2}Z^{n-m_i-2}\\&\qquad+\mu_{i1}Y^{m_i+3}Z^{n-m_i-3}+\dots-c(X+Y)^n\end{aligned}$$

where ν_{ij}'s and μ_{ij}'s are in k. We then perform a quadratic transformation to get $Q_0(YZ,XZ,XY)=Z^{m_i+1}Q_1(X,Y,Z)$, with

$$\begin{aligned}Q_1(X,Y,Z) = \nu_i(X+Y)^{m_i+1}(XY)^{n-m_i-1}\\ + \nu_{i1}(X+Y)^{m_i+2}(XY)^{n-m_i-2}Z + \dots\\ + (X+Y)^n Z^{n-m_i-1} + \mu_i X^n Y^{n-m_i-2} Z\\ + \mu_{i1} X^n Y^{n-m_i-3} Z^2 + \dots - cX^n Z^{n-m_i-1}.\end{aligned}$$

Similar to [Ful, Chapter 5], the three fundamental points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ become ordinary multiple points of $Q_1(X,Y,Z)=0$ with multiplicities $n-m_i$, $n-m_i$, and n respectively. It is easy to check that the only non-fundamental point lies in the intersection of $Q_1(X,Y,Z)$ and the union of three exceptional lines $\{X=0\}$, $\{Y=0\}$, and $\{Z=0\}$ is $(1,-1,0)$. Since

$$\begin{aligned}Q_1(1,Y,Z) = \mu_i Y^{n-m_i-2} Z + \mu_{i1} Y^{n-m_i-3} Z^2 + \dots - cZ^{n-m_i-1}\\ + \nu_i(1+Y)^{m_i+1}(Y)^{n-m_i-1} + \dots + (1+Y)^n Z^{n-m_i-1}\end{aligned}$$

where ν_{ij}'s are in k, the multiplicity of this point $(1,-1,0)$ is one.

For points of $Q_1(X,Y,Z)=0$ outside of the union of three exceptional lines, this transformation preserves the multiplicities and ordinary multiple points. One can easily show that the deficiency of $Q_1(X,Y,Z)=0$ equals δ_{F_c}.

If $(\alpha_i,\alpha_{t(i)})$ is the only non-ordinary multiple points in $F_c(X,Y)=0$, then we are done. Otherwise, there is another non-ordinary multiple point $(\alpha_j,\alpha_{t(j)})$ in $F_c(X,Y)=0$. Clearly, it does not lie in those three exceptional lines. For points of $Q_1(X,Y,Z)=0$ outside of the union of three exceptional lines, we may write $Q_1(X,Y,1) = (XY)^{2n}Q(1/X,1/Y,1) = (XY)^{2n}F_c(1/X+\alpha_i+1/Y, 1/Y+\alpha_{t(i)})$. This implies that the quadratic transformation does not change the local expansion of points outside of the union of three exceptional lines. To be more precise, in this step we first make a linear change of coordinates of the form $X'=r_1X-s_1$ and $Y'=r_2Y-s_2$ such that $X'=0$ and $Y'=0$ is a singularity given by previous transformations of $(\alpha_j,\alpha_{t(j)})$, and the local expansion is of the form

$$\begin{aligned}Q_{20}(X',Y',1) &= Q_1\left(\frac{1}{r_1}(X+s_1), \frac{1}{r_2}(Y+s_2), 1\right)\\ &= \nu_j(X'+\varrho Y')^{m_j+1}(X'Y')^{n-m_j-1}\\ &\quad + \nu_{j1}(X'+\varrho Y')^{m_j+2}(X'Y')^{n-m_j-2} + \dots\\ &\quad + \mu_j X'^n Y'^{n-m_j-2} + \mu_{j1} X'^n Y'^{n-m_j-3} + \dots\end{aligned}$$

where ϱ is a non-zero constant. Therefore when we perform another quadratic transformation to resolve the singularity corresponding to $(\alpha_i,\alpha_{t(i)})$, there is still only one non-fundamental point in the intersection of $Q_{20}(X',Y',Z)$ and the union of three exceptional lines $X'=0$, $Y'=0$, and $Z=0$, and its multiplicity is one. Similarly, $Q_{20}(X',Y',Z)=0$ has the same deficiency as $Q_1(X,Y,Z) = 0$. Since the number of non-ordinary multiple points is finite, after finitely many linear and quadratic transformations we may obtain a curve $Q(X,Y,Z) = 0$ with only ordinary singularities and with the same deficiency as $F_c(X,Y) = 0$. Therefore, the genus of the plane curve $F_c(X,Y) = 0$ equals its deficiency.

We now prove the second assertion. Let $F_c(X,Y) = G(X,Y)H(X,Y)$ where $H(X,Y) \in k[X,Y]$ is a proper factor of $F(X,Y)$ of degree d. Since $F_c(X,Y) = 0$ has only finitely many multiple points, $H(X,Y)$ and $G(X,Y)$ have no common factor. Clearly, $(\alpha_1, \alpha_{t(1)}), \ldots, (\alpha_l, \alpha_{t(l)})$ are also the only possible points in the intersection of $G(X,Y) = 0$ and $H(X,Y) = 0$. Let m_i^G and m_i^H be the multiplicity of $(\alpha_i, \alpha_{t(i)})$ in $G(X,Y) = 0$ and $H(X,Y) = 0$ respectively.

Recall the previous construction of resolving singularities where we first transform $F_c(X,Y) = 0$ into $Q_1(X,Y,Z) = 0$ by linear and quadratic transformations. Assume that in this step $H(X,Y)$ and $G(X,Y)$ are transformed to $H_1(X,Y)$ and $G_1(X,Y)$ respectively. Then the degree of $H_1(X,Y)$ is $2d - m_1^H$, and the multiplicities of the three fundamental points in $H_1(X,Y) = 0$ are d, $d - m_1^H$, and $d - m_1^H$ respectively. Similarly, the degree of $G_1(X,Y)$ is $2n - 2d - m_1^H$, multiplicities of the three fundamental points in $G_1(X,Y) = 0$ are $n - d$, $n - d - m_1^G$, and $n - d - m_1^G$ respectively. Since the multiplicity of the only non-fundamental point in the intersection of $Q_1(X,Y,Z) = 0$ and the union of three exceptional lines is one, it is not in the intersection of $H_1(X,Y) = 0$ and $G_1(X,Y) = 0$. Let $m_P^{H_1}$ and $m_P^{G_1}$ be the multiplicity of the point P in $H_1(X,Y) = 0$ and $G_1(X,Y) = 0$ respectively. One can check easily that

$$
\begin{aligned}
\text{(4.1)} \qquad \deg H_1 \deg G_1 - \sum_P m_P^{H_1} m_P^{G_1}
&= (2d - m_1^H)(2n - 2d - m_1^G) - d(n-d) \\
&\quad - 2(d - m_1^H)(n - d - m_1^G) - \sum_{i \ge 2} m_i^H m_i^G \\
&= d(n-d) - \sum_{i \ge 1} m_i^H m_i^G .
\end{aligned}
$$

If the plane curve $F_c(X,Y) = 0$ has only one non-ordinary multiple points, then the multiple points of $Q_1(X,Y,Z) = 0$ are ordinary. Hence the left hand side of (4.1) equals zero by Bézout's theorem. In general, after a sequence of linear and quadratic transformations $F_c(X,Y) = 0$ can be transformed into a curve $Q(X,Y) = 0$ with only ordinary multiple points. Since an equation of the form (4.1) holds for each transformation, by Bézout's theorem $d(n-d) = \sum_{i \ge 1} m_i^H m_i^G$. ■

Lemma 5. *Suppose that $P(X)$ satisfies all the conditions in Theorem 2 and $F_c(X,Y)$ has no linear factor. Then $F_c(X,Y)$ has an irreducible factor of genus zero if and only if*

$$\delta_{F_c} = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{l} \frac{e_i(e_i-1)}{2} \le 0,$$

where e_i is the multiplicity of $F_c(X,Y)=0$ at $(\alpha_i, \alpha_{t(i)})$.

Proof. If $F_c(X,Y)$ is irreducible, then δ_{F_c} is the genus by Lemma 4.1. Hence the assertion is clear.

Assume that $F_c(X,Y) = G(X,Y)H(X,Y)$ where $H(X,Y) \in k[X,Y]$ is a proper irreducible factor of $F(X,Y)$ and the genus of $H(X,Y)=0$ is zero. Since a plane curve of genus zero is parametrized by rational functions, this implies that $P(f) = cP(g)$ for some non-constant pair of rational functions (f,g). By Lemma 4.1, we may transform $F_c(X,Y)=0$ to another curve $Q(X,Y)=0$ with only ordinary singularities and having deficiency equal to δ_{F_c}. Suppose that $H(X,Y)=0$ was taken to $H_Q(X,Y)=0$ under this transformation. Since they are birational to each other and the deficiency δ_H of $H(X,Y)=0$ is preserved under this transformation, this implies $\delta_H = 0$. Clearly, $(\alpha_1, \alpha_{t(1)}), \ldots, (\alpha_l, \alpha_{t(l)})$ are the only possible singular points for $G(X,Y)=0$ and $H(X,Y)=0$, and are the only possible points in the intersection of these two curves. Let m_i^G and m_i^H be the multiplicities of $(\alpha_i, \alpha_{t(i)})$ in $G(X,Y)=0$ and $H(X,Y)=0$ respectively. Then

$$0 = \delta_H = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{l} \frac{m_i^H(m_i^H-1)}{2}, \tag{4.2}$$

where $d \ge 2$ is the total degree of $H(X,Y)$. Without loss of generality, we may assume that m_i's are in descending order and exactly h of them are non-zero. Since $\delta_H = 0$, Lemma 2 implies either (i) $m_1^H = d-1$, $m_2^H = \ldots = m_h^H = 1$ or (ii) $\sum_{i=1}^{l} m_i^H \ge d+h-1$. Since $\sum_{i=1}^{h} m_i^H + \sum_{i=1}^{h} m_i^G \le n+h-1$, we have $\sum_{i=1}^{h} m_i^G \le n-d+1$ for (i) and $\sum_{i=1}^{h} m_i^G \le n-d$ for (ii). Then for (i), the following can be deduced from Lemma 4(2):

$$d(n-d) = (d-1)m_1^G + \sum_{i=2}^{h} m_i^G = (d-2)m_1^G + \sum_{i=1}^{h} m_i^G.$$

Therefore

$$(d-2)m_1^G \ge d(n-d) - (n-d+1) = (d-1)(n-d) - 1.$$

If $d=2$, this implies that $n \le 3$. Hence $n=3$, $P'(X)$ has two single roots and $\delta_G = 0$. Therefore $F_c(X,Y)$ has only ordinary singularities, and $\delta_{F_c} = \delta_H + \delta_G - 1 < 0$.

For (ii), if $m_1^H = m_2^H = \ldots = m_h^H$, then by Lemma 4(2)

$$d(n-d) = m_1^H \sum_{i=1}^{h} m_i^G \le m_1^H (n-d).$$

This implies that $m_1^H \ge d$, which is impossible. Otherwise, we may assume $m_1^H = m_2^H = \ldots = m_{r-1}^H > m_r^H > 0$. By Lemma 4(2)

$$\begin{aligned} d(n-d) &= m_1^H \sum_{i=1}^{r-1} m_i^G + \sum_{i=r}^{h} m_i^H m_i^G \\ &\le (m_1^H - m_r^H) \sum_{i=1}^{r-1} m_i^G + m_r^H (n-d). \end{aligned}$$

This implies that

$$\sum_{i=1}^{r-1} m_i^G \ge (n-d) \frac{d - m_r^H}{m_1^H - m_r^H} > n-d,$$

which contradicts the fact that $\sum_{i=1}^{h} m_i^G \le n-d$.

Conversely, let $F_c(X,Y) = \prod_{i=1}^{j} H_i(X,Y)$ where each H_i is irreducible with total degree no less than 2. Similarly, H_i are all distinct. One can check easily that $\delta_{F_c} \ge \delta_{H_1} + \ldots + \delta_{H_j} - j + 1$. If $\delta_{F_c} \le 0$, then at least one of the δ_{H_i} is zero. Hence the genus of the plane curve $H_i(X,Y) = 0$ is zero. ■

Lemma 6. *Suppose that $P(X)$ has no multiple zeros and S is the set of zeros of $P(X)$. Then S is affinely rigid if and only if neither $F(X,Y)$ nor $F_c(X,Y)$, $c \ne 0,1$, has a linear factor.*

Proof. Since $F_c(X,X) = P(X) - cP(X) \ne 0$, $X - Y$ is not a linear factor of $F_c(X,Y)$. It is not a linear factor of $F(X,Y)$ either, because $F(X,X) = P'(X)$ is not zero, otherwise $P(X)$ would be a pth power, contrary to the assumption that $P(X)$ has no multiple zeros. If $rX + s - Y$ divides $F_c(X,Y)$ for some $c \ne 0,1$, or $F(X,Y)$, then $(r,s) \ne (1,0)$ and $P(X) = bP(rX+s)$, $b = 1$ or c.

Let $S = \{\beta_1, \ldots, \beta_n\}$. Then

$$(X - \beta_1) \ldots (X - \beta_n) = b(rX + s - \beta_1) \ldots (rX + s - \beta_n).$$

Hence, $r^{-n} = b$ and $rS + s = S$. Therefore S is not affinely rigid. The converse is clear. ■

When the characteristic of k is zero, Khoai and An show that $P(X)$ is a strong uniqueness polynomial if it has no multiple zeros, satisfies condition (I), $l \geq 3$, and

$$\sum_{i=1}^{l} P(\alpha_i) \neq 0. \tag{G}$$

The following result shows that under the assumption of Theorem 2, if (G) holds, then S is affinely rigid.

PROPOSITION 3. *Assume that $P(X)$ satisfies all the conditions in Theorem 2.*

(1) *If $F(X,Y)$ has a linear factor, then $\delta_F < 0$.*
(2) *If $F_c(X,Y)$ has a linear factor, then $\sum_{i=1}^{l} P(\alpha_i) = 0$.*

Proof. The first assertion follows from the proof of Lemma 3. For the second assertion, we will use the notation from the proof of Lemma 6. Let H be a linear factor of $F_c(X,Y)$, and $m_1^H = m_2^H = \ldots = m_h^H = 1$. Then by Lemma 4(2),

$$\sum_{i=1}^{h} m_i^G = n - 1. \tag{4.3}$$

Since $m_i^G \leq \min\{m_i, m_{t(i)}\}$ and $\sum_{i=1}^{l} m_i = n-1$, (4.3) implies $h = l$, and for each $1 \leq i \leq l$, $P(\alpha_i) = cP(\alpha_{t(i)})$ and $m_i = m_{t(i)}$.

Without loss of generality, we may assume that $H = Y - rX - s$. Clearly, $r \neq 0$ and $(r,s) \neq (1,0)$. Then $P(X) = cP(rX+s)$. Therefore, $P'(X) = crP'(rX+s)$. Clearly, $P(\alpha_i) = cP(r\alpha_i + s)$ and $P'(r\alpha_i + s) = 0$. Since $P(X)$ satisfies condition (I), this implies $r\alpha_i + s = \alpha_{t(i)}$. Therefore, $\sum_{i=1}^{l} P(\alpha_i) = c\sum_{i=1}^{l} P(r\alpha_i + s) = c\sum_{i=1}^{l} P(\alpha_i)$. Since $c \neq 1$, this shows that $\sum_{i=1}^{l} P(\alpha_i) = 0$.

Proof of Theorem 2. We first show that (i) is equivalent to (ii). Suppose that f and g are two non-constant meromorphic functions on k satisfying $E(f,S) = E(g,S)$ and $E(f,\infty) = E(g,\infty)$. Then $P(f)/P(g) = c$ for some non-zero constant. If $P(X)$ is a strong uniqueness polynomial for $\mathcal{M}(k)$, then $f = g$. Hence, $(S, \{\infty\})$ is a bi-URS for $\mathcal{M}(k)$. Conversely, suppose that f and g are two non-constant meromorphic functions on k such that $P(f) = cP(g)$ for some non-zero constant. Then $E(f,\infty) = E(g,\infty)$. Let $S = \{\beta_1, \ldots, \beta_n\}$. Then

$$(f - \beta_1) \ldots (f - \beta_n) = c(g - \beta_1) \ldots (g - \beta_n).$$

Suppose that $\mathrm{ord}_a(f - \beta_1) > 0$ for some $a \in k$. Since $\beta_1, \ldots, \beta_n$ are distinct, we have $\mathrm{ord}_a(f - \beta_1) = \mathrm{ord}_a(g - \beta_m) > 0$ for some m, and $\mathrm{ord}_a(f - \beta_i) = \mathrm{ord}_a(g - \beta_j) = 0$ for $i \neq 1$, $j \neq m$. This shows that $E(f,S) = E(g,S)$.

Therefore, if $(S, \{\infty\})$ is a bi-URS for $\mathcal{M}(k)$, then $f = g$. This shows that $P(X)$ is a strong uniqueness polynomial.

Similarly to Theorem 1, (ii) and (iii) are equivalent. By Theorem 1, Lemma 5 and Lemma 6, (iii) is equivalent to S being affinely rigid, $\delta_F > 0$ and $\delta_{F_c} > 0$ for each $c \neq 0, 1$. Let e_i be the multiplicity of $F_c(X, Y) = 0$ at $(\alpha_i, \alpha_{t(i)})$. Since $e_i \le m_i + 1$, and $\sum_{i=1}^{l} m_i$ equals $n-2$ if $p \mid n$, and $n-1$ otherwise we have

$$\begin{aligned}
\delta_{F_c} &= \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{l} \frac{e_i(e_i-1)}{2} \\
&\ge \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{l} \frac{m_i(m_i+1)}{2} \\
&= \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{l} \frac{m_i(m_i-1)}{2} - \sum_{i=1}^{l} m_i \\
&= \begin{cases} \delta_F & \text{if } p \mid n, \\ \delta_F - 1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.4}$$

Therefore, if $p \mid n$, then $\delta_{F_c} \le 0$ if and only if $\delta_F \le 0$.

For $p = 0$ or $p > 0$ and $p \nmid n$, if $\delta_F > 0$, then by (4.4) $\delta_{F_c} \le 0$ only if $\delta_F = 1$ and $e_i = m_i + 1 = m_{t(i)} + 1$ for each $1 \le i \le l$. Suppose that $m_1 \ge m_2 \ge \ldots \ge m_h \ge 2 > m_{h+1} \ge \ldots$ If $h \ge 2$, then by Lemma 2, $\delta_F = 1$ implies that $\sum_{i=1}^{l} m_i \ge n-1+h-2$. Since $\sum_{i=1}^{l} m_i \le n-1$, $h \le 2$. If $\delta_{F_c} \le 0$, then $m_1 = m_{t(1)}$ by (4.4). Since $t(1) \neq 1$, h cannot be 1. If $h = 0$ and $\delta_F = 1$, then $n = 4$ and $P'(X) = (X-\alpha_1)(X-\alpha_2)(X-\alpha_3)$, where $\alpha_i,\ i = 1, 2, 3$ are distinct; and $P(\alpha_1) = cP(\alpha_2)$, $P(\alpha_2) = cP(\alpha_3)$, $P(\alpha_3) = cP(\alpha_1)$, or $P(\alpha_1) = cP(\alpha_3)$, $P(\alpha_2) = cP(\alpha_1)$, $P(\alpha_3) = cP(\alpha_2)$. In either case, we have $c = \omega$ where $\omega^2 + \omega + 1 = 0$. If $h = 2$, and $\delta_F = 1$, then $n = 5$ and $P'(X) = (X-\alpha_1)^2(X-\alpha_2)^2$, where $\alpha_1 \neq \alpha_2$; and $P(\alpha_1) = cP(\alpha_2)$, $P(\alpha_2) = cP(\alpha_1)$. Therefore, $c = -1$. One could also check that $\delta_{F_c} \le 0$ for these exceptional cases. Together with Theorem 1, we see that (iv) is equivalent to (iii). ■

Appendix. URS for non-Archimedean entire functions. For the sake of completeness, we include the following result on the URS of non-Archimedean entire functions.

THEOREM. *Let k be an algebraically closed field of characteristic $p \ge 0$, complete with respect to a non-Archimedean absolute value. Let S be a finite set in k, with n elements. Assume $p \nmid n$ if $p > 0$. Then S is a URS for non-Archimedean entire functions over k if and only if S is affinely rigid.*

When the characteristic of k is zero, this result was obtained in [BEH] for the case of polynomials and in [CY] for the general case. An important step in [CY] is making use of Berkovich's Picard theorem. Here we will include a short proof which is due to Voloch based on Cherry and Yang's observation.

Proof. Cherry and Yang [CY] have shown that S is a URS for non-Archimedean entire functions if and only if S is a URS for polynomials over k. Let f and g be two polynomials over k. If $E(f,S) = E(g,S)$, then $P(f) = P(g)$ or $P(f) = cP(g)$ for some $c \neq 0, 1 \in k$. Clearly, $\deg f(x) = \deg g(x) = d$. Consider the curves $F(X,Y) = 0$ and $F_c(X,Y) = 0$ which have $n-1$ and n distinct points at infinity respectively, if $p = 0$ or $p > 0$ and $p \nmid n$. On the other hand, $(f(x), g(x), 1)$ defines a morphism from $\mathbb{P}^1(k)$ to a plane curve in $\mathbb{P}^2(k)$ which has exactly one d-fold point at infinity. Therefore, $(f(x), g(x))$ can only be a solution of a linear irreducible factor of $F(X,Y)$ or $F_c(X,Y)$. Therefore $F(X,Y)$ or $F_c(X,Y)$ must have a linear factor, which is equivalent to S not being affinely rigid by Lemma 6. ∎

When $p \mid n$ the characterization of a unique range set is more complicated. In a recent joint work of Boutabaa, Cherry, and Escassut [BCE], they give some examples and counterexamples concerning URS for non-Archimedean entire functions.

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