Some product formulas for theta functions in one and two variables

by

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Dedicated to Harold Stark on the occasion of his 60th birthday

Introduction. For \( \tau \in \mathfrak{H} = \{ x + iy \mid y > 0 \} \), \( a, b \in \mathbb{R} \), we define the theta function in one variable \( z \in \mathbb{C} \) with characteristic vector \( \begin{bmatrix} a \\ b \end{bmatrix} \) by

\[
\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+a)^2 \tau + 2\pi i(n+a)(z+b)}.
\]

Let \( m \) be a positive integer. The following equation between modular forms is crucial to the construction of elliptic units:

\[
\prod_{\substack{0 \leq u, v < m \\ (u,v) \neq (0,0)}} \theta \begin{bmatrix} 1/2 + u/m \\ 1/2 + v/m \end{bmatrix}(0, \tau) = (-1)^{m-1} m \eta(\tau)^{m^2-1},
\]

where \( \eta(\tau) = e^{\pi i\tau/12} \prod_{n>0}(1 - e^{2\pi in\tau}) \). For example, Stark (see [S, p. 353]) proved (1) (in a disguised form) using Kronecker’s second limit formula, a tool which is not available for the study of theta functions in more than 1 variable.

Note that \( \eta(\tau)^{24} = \Delta(\tau) \), a cusp form of weight 12 related to the discriminant of the elliptic curve which has \( \tau \) as a period. No formula analogous to (1) holds for all \( m \) relating theta functions in two variables to the “discriminant modular form” attached to \( \tau \) in the Siegel upper half-space of degree 2 (see §3). One main purpose of this paper is to prove Theorem 2 of Section 3, which shows that an analogous formula does hold when \( m = 3 \) and \( m = 4 \).

That such an equation holds with an undetermined constant was shown for \( m = 3 \) in [Gr4], and independently for \( m = 3 \) and \( m = 4 \) by Goren [Go]. Our approach is different from that in [Go], using facts about Siegel modular forms, rather than considering the moduli of genus two curves.

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the end of Section 3 we briefly discuss how the two approaches compare, relating the product formulas of Theorem 2 to the fact that primitive 3- and 4-torsion points on the Jacobian of a curve of genus 2 do not lie on the embedded image of the curve under the Albanese map using a Weierstrass point as base point. This fact is also central to the arguments in [BoBa], [BaBo], and [Gr1], which for certain genus 2 curves defined over number fields, build units in number fields attached to torsion points. See [A] for similar results for genus 3 curves, and [dSG], [Gr2], [FK] and [Lec] for more on units attached to genus 2 curves. In particular, for one curve, [FK] builds units from 6-torsion points on the Jacobian from the point of view of theta functions. I hope that the type of product formulas given here will lead to a better understanding of the sort of norm computations done in [FK]. In some sense, Theorem 2 says that the arithmetic properties enjoyed by 3- and 4-torsion points on Jacobians of curves of genus 2 defined over number fields are reflected in the geometry of generic curves of genus 2.

Another product formula for theta functions in two variables is given in [Gr3] (see also [C]), and a seemingly unrelated product formula is given in [Bor].

The other main purpose of this paper is to derive generalizations of Jacobi’s derivative formula for theta functions in one variable, relating in Theorem 1 the products of derivatives at zero of theta functions with different rational characteristics to powers of $\eta(\tau)$. This is necessary for determining the constants in Theorem 2. For more on this theme, see [BG].

For the convenience of the reader, in Section 1 we recall certain properties of theta functions in several variables. Since we need it in what follows, in Section 2 we give a quick proof of (1) and a bevy of allied formulas. We also state and prove Theorem 1. Theorem 2 is stated and proved in Section 3.

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1. Properties of theta functions. Let $\mathcal{H}_g$ denote the Siegel upper half-space of degree $g$; that is, $g \times g$ symmetric complex matrices with positive-definite imaginary part. We let $\text{Sp}_{2g}(\mathbb{Z})$ denote the integral symplectic group of degree $g$; i.e., block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that
where $A, B, C$, and $D$ are integral $g \times g$ matrices, $I$ is the $g \times g$ identity, and $^t$ denotes the transpose. For $N > 0$, we let $\Gamma(N)$ denote the subgroup of matrices congruent to the identity mod $N$. Elements $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma = \Gamma(1)$ act on $\mathfrak{h}_g$ via $\gamma \circ \tau = (A\tau + B)(C\tau + D)^{-1}$. Let $k$ be a non-negative integer. Recall that $M_k(\Gamma(N))$, the space of Siegel modular forms of degree $g$, level $N$, and weight $k$, consists of holomorphic functions $f$ on $\mathfrak{h}_g$ satisfying

$$f(\gamma \circ \tau) = j_\gamma(\tau)^kf(\tau),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(N)$, where $j_\gamma(\tau) = \det(C\tau + D)$. When $g = 1$, we also require that $f$ be analytic on the compactification of $\Gamma(N) \backslash \mathfrak{h}_1$ gotten by adjoining points at the cusps.

Writing $\mathbb{R}^g$ and $\mathbb{Z}^g$ as column vectors, for any $a, b \in \mathbb{R}^g$, $\tau \in \mathfrak{h}_g$, we let

$$\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t(n+a)\tau(n+a)+2\pi i^t(n+a)(z+b)}$$

denote the theta function in $g$ variables $z \in \mathbb{C}^g$ with characteristic vector $\begin{bmatrix} a \\ b \end{bmatrix}$. In particular, if $a, b \in \frac{1}{2}\mathbb{Z}^g$, we call $\begin{bmatrix} a \\ b \end{bmatrix}$ a theta characteristic. We call a theta characteristic even or odd depending respectively on whether $\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ is an even or odd function of $z$, i.e., whether $e^{4\pi i^tab}$ is $1$ or $-1$. We identify theta characteristics mod $1$. It follows immediately from (2) that

$$\theta \begin{bmatrix} a+p \\ b+q \end{bmatrix}(z, \tau) = e^{2\pi i^t(aq)}\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau),$$

for $p, q \in \mathbb{Z}^g$. Hence if $a, b \in \frac{1}{m}\mathbb{Z}^g$, then $\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)^m$ depends only on $\begin{bmatrix} a \\ b \end{bmatrix}$ mod $1$. Therefore we lose at most a sign when we identify theta characteristics mod $1$.

For any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, theta functions transform as ([I2, pp. 85, 176, 182])

$$\theta \begin{bmatrix} a \\ b \end{bmatrix}^\gamma (C\tau + D)^{-1}z, \gamma \circ \tau)$$

$$= \zeta(\gamma)j_\gamma(\tau)^{1/2}e^{\pi i^t\tau(C\tau+D)^{-1}Cz}\theta \begin{bmatrix} a \\ b \end{bmatrix}(z, \tau),$$

where

$$\zeta(\gamma) = \varrho(\gamma)e^{-\pi i^t([a^tBDa-2a^tBCb+b^tACb-(a^tD-b^tC)(A^tB)o]o)};$$

$$\varrho(\gamma) = \text{an eighth root of } 1 \ (= \text{a fourth root of } 1 \text{ for } \gamma \in \Gamma(2)), $$

$$\begin{bmatrix} a \\ b \end{bmatrix}^\gamma = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (C^tD)0 \\ (A^tB)0 \end{bmatrix},$$
and where for a matrix $M$, $(M)_0$ denotes the column vector consisting of the diagonal entries of $M$, and $j_{\gamma}(\tau)^{1/2}$ is a choice of branch of square root of $j_{\gamma}(\tau)$.

For $\gamma \in \Gamma$, the map $\left[ \begin{array}{c} a \\ b \end{array} \right] \mapsto \left[ \begin{array}{c} a \\ b \end{array} \right]^{\gamma} \mod 1$ gives an action on characteristic vectors mod 1 we call the symplectic action. It is clear that the theta characteristics are stable under the symplectic action, but it can be shown ([I2, p. 213]) that the subsets of even and odd theta characteristics are also left stable by the symplectic action.

For any positive integer $m$, and theta characteristic $\delta$, we let $\text{prim}(m)$ be the set of characteristic vectors mod 1 defined by

$$\text{prim}(m) = \left\{ \left[ \begin{array}{c} a \\ b \end{array} \right] \mod 1 \middle| ma, mb \in \mathbb{Z}^g, (ma, mb, m) = 1 \right\},$$

and set

$$\text{char}_{\delta}(m) = \delta + \text{prim}(m) \mod 1.$$

Note that if $\sigma_g(m)$ is the cardinality of $\text{prim}(m)$, then

$$\sigma_g(m) = m^{2g} \prod_{p | m} \left( 1 - \frac{1}{p^{2g}} \right),$$

the product being over all primes dividing $m$.

Let $\alpha$ be the involution on characteristic vectors mod 1 that sends $\left[ \begin{array}{c} a \\ b \end{array} \right]$ to $\left[ \begin{array}{c} -a \\ -b \end{array} \right]$, and for any set $S$ of characteristic vectors mod 1 upon which $\alpha$ acts, let $S/\alpha$ denote the quotient set of $S$ modulo the action of $\alpha$.

**Lemma 1.** (i) For $m$ odd, $\delta$ a theta characteristic, and $\gamma \in \Gamma(2)$, the symplectic action of $\gamma$ on characteristic vectors mod 1 leaves $\text{char}_{\delta}(m)$ stable. This induces an action of $\Gamma$ on the sets

$$\text{even}(m) = \bigcup_{\delta \text{ even}} \text{char}_{\delta}(m), \quad \text{odd}(m) = \bigcup_{\delta \text{ odd}} \text{char}_{\delta}(m).$$

(ii) For $m$ a multiple of 4, $\text{char}_{\delta}(m) = \text{prim}(m)$ for all theta characteristics $\delta$. Furthermore, the symplectic action on characteristic vectors mod 1 gives an action of $\Gamma$ on $\text{prim}(m)$.

(iii) The symplectic action on characteristic vectors mod 1 induces an action of $\Gamma(2)$ on $\text{char}_{\delta}(m)/\alpha$. This induces actions of $\Gamma$ on $\text{even}(m)/\alpha$ and $\text{odd}(m)/\alpha$ when $m$ is odd, and on $\text{prim}(m)$ when $m$ is a multiple of 4.

**Remark.** For $m = 2m'$, $m'$ odd, we have $\text{prim}(m) = \bigcup_{\delta \neq [\delta]} \text{char}_{\delta}(m')$, the union being over all non-zero theta characteristics.

**Proof.** (i) Suppose $m$ is odd, $\left[ \begin{array}{c} a \\ b \end{array} \right] \in \text{char}_{\delta}(m)$, so

$$\left[ \begin{array}{c} a \\ b \end{array} \right] = \delta + \left[ \begin{array}{c} c \\ d \end{array} \right] \mod 1,$$
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for some $[\frac{a}{c}] \in \text{prim}(m)$. The symplectic action of $\gamma \in \Gamma = (A B \atop C D)$ on $[\frac{a}{b}]$ from (5) is such that

$$\left[\begin{array}{cc} a & \gamma \\ b & \end{array}\right] = \delta^\gamma + \left(\begin{array}{cc} D & -C \\ -B & A \end{array}\right) \left[\begin{array}{c} c \\ d \end{array}\right] \mod 1.$$ 

It is easy to check that the symplectic action of $\Gamma(2)$ fixes $\delta \mod 1$, and that the *multiplicative action* on characteristic vectors mod 1 defined by

$$[\begin{array}{c} c \\ d \end{array}] \rightarrow \left(\begin{array}{cc} D & -C \\ -B & A \end{array}\right) [\begin{array}{c} c \\ d \end{array}] \mod 1$$

is an action of $\Gamma$ on $\text{prim}(m)$. Therefore the symplectic action of $\Gamma(2)$ on characteristic vectors mod 1 defines an action on $\text{char}_\delta(m)$. Hence the symplectic action of $\Gamma$ on characteristic vectors mod 1 defines actions on the sets $\text{even}(m)$ and $\text{odd}(m)$.

(ii) For $m$ a multiple of 4, and $\delta$ a theta characteristic, it is clear that $\text{char}_\delta(m) = \text{prim}(m)$. For $\gamma = (A B \atop C D) \in \Gamma$, the multiplicative action on characteristic vectors mod 1 permutes $\text{prim}(m)$, and differs from the symplectic action by the addition of an element in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. But $\text{prim}(m)$ is invariant under the addition of such elements.

(iii) We have $\alpha(\delta) = \delta$ for any theta characteristic $\delta$, so $\alpha$ acts on $\text{char}_\delta(m)$. It follows as in the proof of (i) that the symplectic action of $\Gamma(2)$ commutes with $\alpha$, so gives an action on $\text{char}_\delta(m)/\alpha$. The rest follows as in the proofs of (i) and (ii).

Let $\text{Prim}(m)$ and $\text{Char}_\delta(m)$ denote respectively sets of representatives for the classes of characteristic vectors mod 1 in $\text{prim}(m)$ and $\text{char}_\delta(m)$. Let $\text{Prim}(m)/\alpha$ and $\text{Char}_\delta(m)/\alpha$ denote respectively sets of representatives for the classes of characteristic vectors mod 1 and modulo $\alpha$ in $\text{prim}(m)/\alpha$ and $\text{char}_\delta(m)/\alpha$.

**Proposition 1.** Let $m$ be any positive integer.

(i) For any theta characteristic $\delta$, and any $[\frac{a}{b}] \in \text{Char}_\delta(m)$,

$$\theta \left[\begin{array}{c} a \\ b \end{array}\right] (0, \tau)_{2m} \in M_{2m}(\Gamma(2m)),$$

and is independent of the choice of $[\frac{a}{b}] \mod 1$.

(ii) For any theta characteristic $\delta$, and $m$ odd, if

$$\phi_{\delta,m}(\tau) = \prod_{[\frac{a}{b}] \in \text{Char}_\delta(m)} \theta \left[\begin{array}{c} a \\ b \end{array}\right] (0, \tau),$$
then \((\phi_{\delta,m}(\tau))^{4m}\) is a modular form of level 2. For \(m\) a multiple of 4, if
\[
\phi_m(\tau) = \prod_{[\frac{a}{b}] \in \text{Prim}(m)} \theta\left[\frac{a}{b}\right](0, \tau),
\]
then \((\phi_m(\tau))^{4m}\) is a modular form of level 2.

(iii) For \(m\) odd, if
\[
\phi_{\text{even},m}(\tau) = \prod_{\delta \text{ even}} \phi_{\delta,m}(\tau) \quad \text{and} \quad \phi_{\text{odd},m}(\tau) = \prod_{\delta \text{ odd}} \phi_{\delta,m}(\tau),
\]
then \((\phi_{\text{odd},m}(\tau))^{4m}\) and \((\phi_{\text{even},m}(\tau))^{4m}\) are modular forms of level 1.

(iv) If \(m\) is odd, take
\[
f(\tau) = \psi_{\text{even},m}(\tau) = \prod_{\delta \text{ even}} \prod_{[\frac{a}{b}] \in \text{Char}_\delta(m)/\alpha} \theta\left[\frac{a}{b}\right](0, \tau), \quad \text{or}
\]
\[
f(\tau) = \psi_{\text{odd},m}(\tau) = \prod_{\delta \text{ odd}} \prod_{[\frac{a}{b}] \in \text{Char}_\delta(m)/\alpha} \theta\left[\frac{a}{b}\right](0, \tau).
\]
If \(m\) is a multiple of 4 take
\[
f(\tau) = \psi_m(\tau) = \prod_{[\frac{a}{b}] \in \text{Prim}(m)/\alpha} \theta\left[\frac{a}{b}\right](0, \tau).
\]

- If \(g = 1\) and \(m \geq 3\), then \(f(\tau)^{\gcd(8m,12)}\) is a modular form of level 1.
- If \(g = m = 1\), then \(f(\tau)^8\) is a modular form of level 1.
- If \(g = 2\), then \(f(\tau)^2\) is a modular form of level 1.
- If \(g \geq 3\), then \(f(\tau)\) is a modular form of level 1.

Proof. (i) Since \(\Gamma(2m) \subset \Gamma(2)\), for \(\gamma \in \Gamma(2m)\), \(\rho(\gamma)^4 = 1\). It is easy to verify then that \(\zeta(\gamma)^{4m} = 1\). Further, \(\left[\frac{a}{b}\right]^{\gamma} \equiv \left[\frac{a}{b}\right] \mod 1\), so by (3) and (4),
\[
\left[\frac{a}{b}\right]^{\gamma 2m} \theta\left[\frac{a}{b}\right](0, \tau)^{4m} = \left(\theta\left[\frac{a}{b}\right](0, \gamma \circ \tau)^{4m} = \left(\theta\left[\frac{a}{b}\right](0, \gamma \circ \tau)^{4m},
\]
and by (3), \(\theta\left[\frac{a}{b}\right](0, \tau)^{4m}\) depends only on \(\left[\frac{a}{b}\right] \mod 1\).

(ii) By Lemma 1, and part (i), this is just the product of modular forms which are permuted under the action of \(\Gamma(2)/\Gamma(2m)\) (or \(\Gamma/\Gamma(2m)\) for \(m\) a multiple of 4), where the action is \(f(\tau) \mapsto f(\gamma \circ \tau)/j_\gamma(\tau)^{2m}\). This drops the level to 2 for \(m\) odd, and to 1 for \(m\) a multiple of 4.

(iii) For \(m\) odd, as in (ii), this is just the product under the action of \(\Gamma/\Gamma(2)\) of modular forms on \(\Gamma(2)\), which drops the level to 1.

(iv) If \(g(\tau)^n\) is a modular form of weight \(nk\), \(k\) an integer, then the map \(\gamma \mapsto g(\gamma \circ \tau)/(g(\tau)j_\gamma(\tau)^k)\) is a character on \(\Gamma\). Let \(m\) be odd or a multiple of 4.
Since $\theta\left[\begin{array}{c}a \\ b \end{array}\right](0, \tau) = \theta\left[\begin{array}{c}a \\ 0 \end{array}\right](0, \tau)$ for $m \geq 3$, $f(\tau)^2$ differs by at most a multiplicative constant from $\phi_{\text{odd},m}(\tau)$, $\phi_{\text{even},m}(\tau)$, or $\phi_m(\tau)$. So for $m \geq 3$, since it is easy to check that $\sigma_{2g}(m)$ is a multiple of 4, we see from (4) that $f(\tau)^{8m}$ is a modular form whose weight is divisible by $8m$. It is known ([M, p. 169]) that the number of even and odd theta characteristics is $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g - 1)$, respectively. Hence when $m = 1$, since $\alpha$ pointwise fixes theta characteristics, $f(\tau)$ differs by at most a multiplicative constant from $\phi_{\text{odd},1}(\tau)$ or $\phi_{\text{even},1}(\tau)$, so if $g > 1$, $f(\tau)^4$ is a modular form whose weight is divisible by 4.

So in any case, unless $g = m = 1$, we find that $f(\tau)$ is a modular form with character on $\Gamma$. For $g = 2$ every character of $\Gamma$ is of order dividing 2; and for $g \geq 3$ there are no non-trivial characters of $\Gamma$ ([K, pp. 43–44]). For $g = 1$, [Leh, p. 349] shows that for $m \geq 3$, $f(\tau)$ times some power of $\eta(\tau)$ is a modular form, so $f(\tau)^{12}$ is a modular form.

Finally, if $g = m = 1$, $\psi_{\text{odd},1}(\tau) = 0$ and Lemma 2(i) below shows that $\psi_{\text{even},1}(\tau)$ is a constant multiple of $\eta(\tau)^3$, so $\psi_{\text{even},1}(\tau)^8$ is a modular form.

REMARK. We will see in Proposition 2 that if $g = 1$, and $c(f)$ is the number of theta functions in the product defining $f(\tau)$ in Proposition 1(iv), then $f(\tau)$ is a constant times $\eta(\tau)^{c(f)}$.

2. Theta functions in one variable. Here $\Gamma = \text{SL}_2(\mathbb{Z})$. The only odd theta characteristic is represented by $[1/2]$. We take $[0]$, $[0, 1/2]$, and $[1/2]$ as representatives for the three even theta characteristics. We recall some classic facts about modular forms of degree 1 (see, e.g. [M]). For $\tau \in \mathfrak{h} = \mathfrak{h}_1$, set $q = e^{2\pi i \tau}$. For any modular form, its “$q$-expansion” is its Fourier series at $i\infty$ in $q$. If $f(\tau)$ is holomorphic on $\mathfrak{h}$ and $f(\tau)^n$ is a modular form, then $f(\tau)$ has a $q$-expansion in $q^{1/n} = e^{2\pi i \tau/n}$. The exponent of $q$ in the lead term of the $q$-expansion is the order of zero of a form at $i\infty$. Recall we define

$$
\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n) \quad \text{and} \quad \Delta(\tau) = \eta(\tau)^{24}.
$$

We let $\theta'[\begin{array}{c}a \\ b \end{array}](z, \tau)$ denote $\frac{d}{dz} \theta\left[\begin{array}{c}a \\ b \end{array}\right](z, \tau)$.

**Lemma 2.** (i) We have

$$
\frac{1}{2\pi i} \theta'[\begin{array}{c}1/2 \\ 1/2 \end{array}](0, \tau) = \frac{i}{2} \theta\left[\begin{array}{c}0 \\ 0 \end{array}\right](0, \tau) \theta\left[\begin{array}{c}1/2 \\ 0 \end{array}\right](0, \tau) \theta\left[\begin{array}{c}0 \\ 1/2 \end{array}\right](0, \tau) = i\eta^3(\tau).
$$

(ii) $\Delta(\tau)$ is a modular form of level 1 and weight 12. It has no zeros on $\mathfrak{h}$ and a simple zero at $i\infty$.

(iii) The only modular forms of any level which have weight 0 are constants.
Proof. See [M, p. 42 and pp. 64–72]. (i) is Jacobi’s derivative formula. For any positive integer \( m \), we define

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{1}{2} + u/m \right] \left( 1/2 + v/m \right) (0, \tau),
\]

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{u/m}{v/m} \right] (0, \tau),
\]

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{1}{2} + u/m \right] \left( 1/2 + v/m \right) (0, \tau),
\]

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{u/m}{1/2 + v/m} \right] (0, \tau).
\]

Lemma 3. The following are lead terms of \( q \)-expansions:

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{1}{2} + u/m \right] \left( 1/2 + v/m \right) (0, \tau),
\]

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{u/m}{1/2 + v/m} \right] (0, \tau).
\]

Proof. For any theta function \( \theta\left[ \frac{a}{b} \right] (0, \tau) \) with \( a, b \in \mathbb{Q} \), we can compute its \( q \)-expansion directly from its definition (2). Alternatively, one can use the product expansion for theta functions (see [M, p. 69]). We leave the verification of the lemma to the reader.

Proposition 2.

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{1}{2} + u/m \right] \left( 1/2 + v/m \right) (0, \tau),
\]

\[
\prod_{0 \leq u, v < m} \theta\left[ \frac{u/m}{1/2 + v/m} \right] (0, \tau).
\]
$\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau) = (-1)^{(m-1)/2}\eta(\tau)^{m^2-1} (-1)^{(m-2)/2}m\eta(\tau)^{m^2-1}$

$\prod \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(m)(\tau) = \eta(\tau)^{m^2-1} - mn(\tau)^{m^2-1}$

**Proof.** First let us consider $\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau)$. Since $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ represents the only odd theta characteristic when $g = 1$, applying Proposition 1 to all the factors on the right hand side of

$$\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau) = \prod \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(m)(\tau)$$

we see that $(\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau))^{4m}$ is a modular form of level 1, and weight $2m(m^2 - 1)$. By Lemma 3, the lead term of its $q$-expansion is a constant times $q^{m(m^2-1)/6}$. Therefore, by Lemma 2,

$$\frac{(\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau))^{4m}}{\Delta(\tau)^{m(m^2-1)/6}}$$

is a modular form of level 1 and weight 0, and hence a constant. Since $\mathfrak{h}$ is connected, $\prod \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(m)(\tau)$ and $\eta(\tau)^{m^2-1}$ differ only by a constant. The constant is determined by the $q$-expansion in Lemma 3.

For any $\delta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$, all of which represent even theta characteristics, the same argument only shows, a priori, that $(\prod[\delta](m)(\tau))^{4m}$ is of level 2. But since each image under the action of $\Gamma$ has a $q$-expansion whose lead term is a constant times $q^{m(m^2-1)/6}$, we deduce again that $(\prod[\delta](m)(\tau))^{4m}/\Delta(\tau)^{m(m^2-1)/6}$ is a modular form of level 2 and weight 0, and hence a constant. Therefore, again $\prod[\delta](m)(\tau)$ and $\eta(\tau)^{m^2-1}$ differ only by a constant, determined by the $q$-expansions in Lemma 3.

**PROPOSITION 3.** Let $m$ be any positive integer.

(i) For any theta characteristic $\delta$, and any $\begin{bmatrix} a \\ b \end{bmatrix} \in \text{Char}_\delta(m)$,

$$\theta' \begin{bmatrix} a \\ b \end{bmatrix}(0, \tau)^{4m} \in M_{6m}(\Gamma(2m))$$

and is independent of the choice of $\begin{bmatrix} a \\ b \end{bmatrix} \mod 1$. 
(ii) For any theta characteristic \( \delta \), and \( m \) odd, if
\[
\Phi_{\delta,m}(\tau) = \prod_{\charm \in \Char(m)} \theta^\prime \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau),
\]
then \( (\Phi_{\delta,m}(\tau))^{4m} \) is a modular form of level 2. For \( m \) a multiple of 4, if
\[
\Phi_{m}(\tau) = \prod_{\charm \in \Prim(m)} \theta^\prime \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau),
\]
then \( (\Phi_{m}(\tau))^{4m} \) is a modular form of level 1.

(iii) For \( m \) odd, if
\[
\Phi_{\text{even},m}(\tau) = \prod_{\charm \in \Char(m)/\alpha} \Phi_{\delta,m}(\tau) \quad \text{and} \quad \Phi_{\text{odd},m}(\tau) = \Phi_{\delta,m}(\tau)
\]
for \( \delta = \left[ \frac{1}{2} \right] \mod 1 \), then \( (\Phi_{\text{odd},m}(\tau))^{4m} \) and \( (\Phi_{\text{even},m}(\tau))^{4m} \) are modular forms of level 1.

(iv) If \( m \) is odd, take
\[
F(\tau) = \Psi_{\text{even},m}(\tau) = \prod_{\charm \in \Char(m)/\alpha} \theta^\prime \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau), \quad \text{or} \quad \Psi_{\text{odd},m}(\tau) = \prod_{\charm \in \Char(m)/\alpha} \theta^\prime \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau)
\]
for \( \delta = \left[ \frac{1}{2} \right] \mod 1 \). If \( m \) is a multiple of 4, take
\[
F(\tau) = \Psi_{m}(\tau) = \prod_{\charm \in \Prim(m)/\alpha} \theta^\prime \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau).
\]

Then if \( m \geq 3 \), \( F(\tau) \) times some power of \( \eta(\tau)^2 \) is a modular form of level 1, so \( F(\tau)^{\gcd(8m,12)} \) is a modular form of level 1. If \( m = 1 \), \( F(\tau)^8 \) is a modular form of level 1.

Proof. These follow just as Proposition 1 by differentiating (4) and using the resulting formula at \( z = 0 \).

Unlike the products in Proposition 2, the products \( F(\tau) \) in Proposition 3(iv) are not necessarily constants times a power of \( \eta(\tau) \). However, we will show in Theorem 1 that this is true for \( m = 3 \) and \( m = 4 \). For an analysis of these products for all \( m \), see [BG].

For any representative \( \left[ \begin{array}{c} a \\ b \end{array} \right] \) of a theta characteristic, we let
\[
\text{derivprod} \left[ \begin{array}{c} a \\ b \end{array} \right] (3)(\tau) = \prod_{\substack{0 \leq u, v < 3 \\ (u,v) \neq (0,0)}} \theta^\prime \left[ \begin{array}{c} a + u/3 \\ b + v/3 \end{array} \right] (0, \tau),
\]
and set

$$\text{derivprod}(4)(\tau) = \prod_{0 \leq u, v < 4, (u, v) \neq (0,0), (0,2), (2,0), (2,2)} \theta^\prime \left[ \frac{u}{4}, \frac{v}{4} \right](0, \tau).$$

Part (ii) of the following theorem can be considered a generalization of Jacobi’s derivative formula (Lemma 2(i)).

**Theorem 1.** For $\left[ \begin{array}{c} a \\ b \end{array} \right] = \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right]$:

(i) The lead term of the $q$-expansions of $\theta^{\left[ \begin{array}{c} a \\ b \end{array} \right]}(0, \tau)^8 \text{derivprod}^{\left[ \begin{array}{c} a \\ b \end{array} \right]}(3)(\tau)$ is

$$\frac{(-1)^{2a+1}2^8\pi^8}{3^5} \cdot q^{4/3}.$$

The lead term of the $q$-expansion of $\text{derivprod}(4)(\tau)$ is $(-\pi^{12}/2^3)q^{3/2}$.

(ii)

$$\theta^{\left[ \begin{array}{c} a \\ b \end{array} \right]}(0, \tau)^8 \text{derivprod}^{\left[ \begin{array}{c} a \\ b \end{array} \right]}(3)(\tau) = \frac{(-1)^{2a+1}2^8\pi^8}{3^5} \eta(\tau)^{32}$$

and

$$\text{derivprod}(4)(\tau) = \frac{-\pi^{12}}{2^3} \eta(\tau)^{36}.$$  

**Proof.** (i) This is a computation whose verification we leave to the reader.

(ii) This is entirely similar to the proof of Proposition 2.

### 3. Theta functions in two variables.

Here $\Gamma = \text{Sp}_4(\mathbb{Z})$. The structure of the ring $\bigcup_{k \geq 0} M_k(\Gamma)$ was determined by Igusa [I1] and subsequently by Hammond [H], and Freitag [F].

There are six odd theta characteristics, represented by

$$\left[ \begin{array}{c} 1/2 \\ 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right].$$

Representatives for the 10 even theta characteristics are

$$\left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right].$$
We write
\[ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \quad \text{for } \tau \in \mathfrak{h}_2. \]

Let \( Z \) denote the image of the subvariety \( \tau_{12} = 0 \) of \( \mathfrak{h}_2 \) under the action by \( \Gamma \).

We define
\[ \Delta_2(\tau) = 2^{-12} \prod_{\delta \text{ even}} \theta[\delta]^2(0, \tau). \]

We need to accumulate some facts.

**Lemma 4.** (i) \( \Delta_2(\tau) \) is a modular form of level 1 and weight 10, which has zeros of order 2 along \( Z \) and no other zeros.

(ii) A modular form of any level which is of weight 0 is a constant.

**Proof.** These can be found in [K, pp. 115, 119].

We call \( \Delta_2(\tau) \) the discriminant modular form (of degree 2). The reason for the name is that via Thomae’s formula, it can be shown for \( \tau \not\in Z \) that \( \Delta_2(\tau) \) differs only by a multiplicative constant from the discriminant of the curve of genus 2 whose period matrix is \( (\sigma, \rho) \) (see [Gr3]).

It follows from the definition (2) that if \( \tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathfrak{h}_2 \), then for \( a = [a_1 \ b_1], b = [b_1 \ b_2], a_i, b_i \in \mathbb{R}, i = 1, 2, \) we have
\[ \theta\left[ \begin{array}{c} a \\ b \end{array} \right](0, \tau)\bigg|_{\tau_{12}=0} = \theta\left[ \begin{array}{c} a_1 \\ b_1 \end{array} \right](0, \tau_{11})\theta\left[ \begin{array}{c} a_2 \\ b_2 \end{array} \right](0, \tau_{22}). \]

Recall that \( \theta\left[ \begin{array}{c} a_i \\ b_i \end{array} \right](0, \tau_{ii}) = 0 \) if and only if \( \left[ \begin{array}{c} a_i \\ b_i \end{array} \right] \equiv [1/2] \mod 1 ([M, p. 11]).\)

**Theorem 2.** For any odd theta characteristic \( \delta = \left[ \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} \right] \),
\[ f_{\delta}(\tau) := \prod_{0 \leq u_i, v_i < 3 \atop (u_1, u_2, v_1, v_2) \neq (0,0,0)} \theta\left[ \begin{array}{c} u_1/3 \\ u_2/3 \\ v_1/3 \\ v_2/3 \end{array} \right] (0, \tau) = c_3(\delta) \Delta_2(\tau)^4, \]
where \( c_3(\delta) = (-1)^{2a_1+2a_2}3^4 \), and
\[ g(\tau) := \prod_{0 \leq u_i, v_i < 4 \atop (u_1, u_2, v_1, v_2) \text{ not all even}} \theta\left[ \begin{array}{c} u_1/4 \\ u_2/4 \\ v_1/4 \\ v_2/4 \end{array} \right] (0, \tau) = c_4 \cdot \Delta_2(\tau)^{12}, \]
where \( c_4 = 2^{24} \).
Proof. By Proposition 1, \( f_\delta(\tau)^{12} \) is a modular form of level 2 and weight 480. Note that by (6), for every odd characteristic \( \delta \), 8 terms in the product \( f_\delta(\tau) \) vanish when \( \tau_{12} = 0 \). Since the 6 choices of \( f_\delta(\tau)^{12} \) are permuted by \( \Gamma \), for each \( \delta \), \( f_\delta(\tau)^{12} \) vanishes at least to order 96 on \( Z \). So by Lemma 4, \( f_\delta(\tau)^{12}/(\Delta_2(\tau)^{48}) \) is a modular form of level 2 and weight 0, which is necessarily a constant. Therefore \( f_\delta(\tau)^{12} \) differs by a constant from \( \Delta_2(\tau)^{48} \), and since \( h_2 \) is connected, \( f_\delta(\tau) \) differs by a constant from \( \Delta_2(\tau)^4 \). A calculation with (3) shows that the constant is independent of the choice of representative for \( \delta \) mod 1.

By Proposition 1, \( g(\tau)^2 \) is a modular form of weight 240 and level 1. Of the 240 terms in the product \( g \), 24 vanish along \( Z \). Therefore \( g(\tau)^2/\Delta_2(\tau)^{24} \) is a modular form of weight 0 and level 1, and hence a constant. Therefore \( g(\tau) \) and \( \Delta_2(\tau)^{12} \) differ by a constant.

It remains to compute \( c_3(\delta) \) and \( c_4 \). For this we need to take the Taylor expansion of \( \theta^a(b)(0, \tau) \) in \( \tau_{12} \) at 0. If the function does not vanish on \( \tau_{12} = 0 \), the lead term in the expansion is given by (6). If it does vanish, the lead term is given by \( \tau_{12} \) times

\[
\frac{d}{d\tau_{12}} \left[ \theta^a(b)(0, \tau) \right]_{\tau_{12}=0} = \frac{1}{2\pi i} \theta' \left[ \begin{array}{c} a_1 \\ b_1 \end{array} \right] (0, \tau_{11}) \theta' \left[ \begin{array}{c} a_2 \\ b_2 \end{array} \right] (0, \tau_{22}).
\]

For starters, we compute the lead term of the Taylor expansion of \( \Delta_2(\tau) \) as

\[
2^{-12} (2\pi i)^2 \left( \frac{1}{2\pi i} \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (0, \tau_{11}) \right)^2 \left( \frac{1}{2\pi i} \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (0, \tau_{22}) \right)^2 \times \left( \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (0, \tau_{11}) \theta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (0, \tau_{11}) \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, \tau_{11}) \right)^6 \times \left( \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (0, \tau_{22}) \theta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (0, \tau_{22}) \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, \tau_{22}) \right)^6 (\tau_{12})^2
\]

by Lemma 2.

The formulas in Section 2 now give enough ammunition to calculate \( c_3(\delta) \) and \( c_4 \).

We will compute the Taylor expansion of \( f_{\delta_0}(\tau) \) when \( \delta_0 = \left[ \begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array} \right] \). The other choices for odd theta characteristics are treated similarly. The lead term of the expansion is

\[
\left( \frac{1}{2\pi i} \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (0, \tau_{22}) \right)^8 \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, \tau_{11})^8 \text{derivprod} \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (3)(\tau_{11})
\]
\[
\times \left( \prod \frac{1/2}{1/2} (3)(\tau_{22}) \right)^9 \left( \prod \frac{1/2}{0} (3)(\tau_{11}) \right)^8 (\tau_{12})^8 \\
= 3^4 2^8 \pi^8 \Delta(\tau_{11})^4 \Delta(\tau_{22})^4 (\tau_{12})^8,
\]
so \( c_3(\delta_0) = 3^4 \).

Finally, the lead term of the Taylor expansion of \( g(\tau) \) is
\[
\left( \frac{1}{2\pi i} \theta' \left[ \frac{1/2}{1/2} \right] (0, \tau_{11}) \right)^{12} \left( \frac{1}{2\pi i} \theta' \left[ \frac{1/2}{1/2} \right] (0, \tau_{22}) \right)^{12} \\
\times \text{derivprod}(4)(\tau_{11}) \text{derivprod}(4)(\tau_{22}) \\
\times \left( \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (0, \tau_{11}) \theta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (0, \tau_{11}) \right)^{15} \\
\times \left( \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (4)(\tau_{11}) \right)^{15} (\tau_{12})^{24} \\
= 2^{48} \pi^{24} \Delta(\tau_{11})^{12} \Delta(\tau_{22})^{12} (\tau_{12})^{24},
\]
so \( c_4 = 2^{24} \).

**Remark.** For any \( \tau \in h_2 \) not in \( Z \), \( \tau \) is the period matrix of some complex curve \( C \) of genus 2. The curve has six Weierstrass points, \( w_k, 1 \leq k \leq 6 \), and the canonical divisor class is \( 2w_k \) for any \( k \). Fix one choice of \( k \).

We can pick a symplectic basis \( A_1, A_2, B_1, B_2 \) for \( H_1(C, Z) \) (i.e., such that \( A_1 \cdot A_2 = B_1 \cdot B_2 = 0, A_i \cdot B_j = \delta_{ij} \)), and a normalized basis \( \mu_1, \mu_2 \) of holomorphic differentials of \( C \) such that
\[
\int_{A_i} \mu_j = I, \quad \int_{B_i} \mu_j = \tau.
\]
Then we have an embedding
\[
C \overset{\Phi_k}{\rightarrow} \mathbb{C}^2 / L
\]
given by
\[
P \mapsto \int_{w_k} (\mu_1, \mu_2) \text{ mod } L,
\]
where \( L \) is the lattice in \( \mathbb{C}^2 \) generated by the columns of \( I \) and \( \tau \). (For background and details, see [Gr5].) The map \( \Phi_k \) extends by linearity to divisors of \( C \), and the Abel–Jacobi Theorem says that if \( D \) is a divisor of degree 0, then \( D \) is the divisor of a function if and only if \( \Phi_k(D) \) is the origin.
in $C^2/L$. It follows that $\phi_k(w_j)$, $j = 1, \ldots, 6$, are precisely the 2-torsion points of $C^2/L$ which lie on $\phi_k(C)$.

A fundamental theorem of Riemann says that there is an odd theta characteristic $\delta = \delta(k)$ such that $\theta[\delta](z, \tau)$, $z \in C^2$, has a zero of order 1 along the pullback of $\phi_k(C)$ to $C^2$ and no other zeros. For $a, b \in \mathbb{R}^2$, since $\theta[\delta + \frac{a}{b}](0, \tau)$ differs by an exponential from $\theta[\delta](\tau a + b, \tau)$, we see that $\theta[\delta + \frac{a}{b}](0, \tau) = 0$ if and only if $\tau a + b \in \phi_k(C)$. Theorem 2 says $\theta[\delta + \frac{c}{d}](0, \tau) \neq 0$ for $\tau \notin Z$, when $3c \equiv 3d \equiv 0 \mod 1$, and $c$ or $d \neq 0 \mod 1$, and that $\theta[\frac{c}{d}](0, \tau) \neq 0$ for $\tau \notin Z$ when $4c \equiv 4d \equiv 0 \mod 1$, and $2c$ or $2d \neq 0 \mod 1$. With this we get

**Corollary.** There is no point $P$ on $C$, $P \neq w_k$, such that $3(P - w_k)$ is the divisor of a function, and there is no point $P$ on $C$, $P \neq w_j$, $1 \leq j \leq 6$, such that $4(P - w_k)$ is the divisor of a function.

This corollary can easily be derived from the Riemann–Roch Theorem (see, e.g. [Box]). Having done so directly, Goren gave a moduli-theoretic proof of Theorem 2 up to an unspecified constant [Go]. Likewise, we can see that there is no analogue of Theorem 2 for $m = 5$, because on the curve $C : y^2 = x^5 + 1$, the divisor of $y - 1$ is $5((0, 1) - \infty)$, where $\infty$ denotes the Weierstrass point at infinity (see [BG]). See [BGL] for a complete description of the moduli space of curves of genus 2 such that there is a $P \in C$, $P \neq \infty$, such that $5(P - \infty)$ is the divisor of a function.

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