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Distribution of zeros of Dirichlet L-functions and an explicit formula for $\psi(t,\chi)$

by

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1. Introduction. Numerical results on (i) bounds for zeros of the Dirichlet L-functions $L(s,\chi)$, (ii) the zero-density estimates for $L(s,\chi)$ and (iii) the asymptotic formula for the generalized Chebyshev function $\psi(t,\chi)$ are indispensable in the investigation of many problems involving prime numbers if all the relevant constants are required to be explicit. Many authors obtained results of general interest in these three topics (e.g., [RS], [M], [C], etc.).

In this paper we shall concentrate on some numerical problems related to these results. As an interesting application of the results of this paper, it is proved in a forthcoming paper [LW] that every odd integer exceeding $\exp(3100)$ $(= 10^{1346.3\ldots})$ is a sum of three odd primes, which improves the best known numerical result $\exp(\exp(11.503))$ $(= 10^{43000.5\ldots})$ in [CW1] on the three primes Goldbach conjecture.

As a continuation of [M, Theorems 1 and 2], our Theorems 1 and 2 give bounds for zeros of $L(s,\chi)$ with the help of the ideas in [G]; Theorem 3 gives an explicit numerical bound for the Siegel zero $\widetilde{\beta}$ defined as in Lemma 2.1; Theorem 4 uses some ideas of Heath-Brown [HB] to give explicit numerical bounds for the zero-density of $L(s,\chi)$ near the vertical line $\alpha = 1$; Theorems 5 and 6 give explicit numerical zero-density estimates for $L(s,\chi)$ in the strip $0 < \alpha < 1$, while Theorem 7 gives explicit numerical zero-density estimates for $L(s,\chi)$ taking care of the strip $1/2 < \alpha < 1$; Theorem 8 gives an explicit formula for $\psi(t,\chi)$ with a numerical value for the constant in the error term.

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As far as the authors are aware, there are no previous records on the numerical value of the constants in the bounds of Theorems 3 and 8.

An expanded version of this paper, giving full details of the numerical calculations, is available on request from the second author.

2. Bounds for zeros of Dirichlet L-functions. In this section we give some explicit upper bounds for zeros of Dirichlet L-functions: we first present a zero-free region due to McCurley [M], and then give explicit regions in which the function defined by (2.2) below has at most four zeros or two zeros respectively in Theorems 1 and 2, and lastly give an upper bound for the possible exceptional zero $\widetilde{\beta}$ in Theorem 3. Suppose q is any integer satisfying

$$1 \le q \le x. \tag{2.1}$$

Let $\chi \pmod q$ be any Dirichlet character to modulus q, and $L(s,\chi)$ be the corresponding L-function. Define

$$\Pi(s) := \prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi). \tag{2.2}$$

LEMMA 2.1. *Let $c_1 := 9.645908801$ (so $1/c_1 = 0.10367089\ldots$) and $x \ge 10$. Then the function $\Pi(s)$ defined by* (2.2) *has at most one zero $\widetilde{\beta}$, called the exceptional zero or the Siegel zero, in the region*

$$\sigma > 1 - 1/(c_1 \log x), \qquad |t| \le x/q.$$

Such a zero, if it exists, is real and simple, and corresponds to a nonprincipal real character $\chi \pmod q$ induced by the unique nonprincipal real primitive character $\widetilde{\chi} \pmod{\widetilde{r}}$ with $\widetilde{r} \le x$, for which $L(\widetilde{\beta}, \widetilde{\chi}) = 0$. Moreover, we have $\widetilde{r} \ge 987$.

Proof. The first part is [M, Theorem 1], upon noting that the M there is less than x. The uniqueness for $\widetilde{\chi}$ (so for $\widetilde{r}$) is due to [M, Theorem 2], since the M_1 there is now less than x^2 and the $2R_1$ there is less than c_1. Finally, $\widetilde{r} \ge 987$ comes from [M, p. 9, before §2]. ∎

Lemmas 2.2 to 2.5 below are used to prove Theorems 1 and 2. To state them, we first give some notations. Let $\sigma > 1$ and set $\sigma_1 := (1+\sqrt{1+4\sigma^2})/2$. For any real t and any $\chi \pmod q$ with $q \ge 1$ define

$$f(\sigma,t,\chi) := \mathrm{Re}\left\{\frac{1}{\sqrt{5}} \cdot \frac{L'}{L}(\sigma_1 + it, \chi) - \frac{L'}{L}(\sigma + it, \chi)\right\}. \tag{2.3}$$

Throughout the paper, the letter p, with or without subscripts, always denotes a prime number.

LEMMA 2.2. *Let χ_0 be the principal character modulo q. Then for* $1 < \sigma < 1.15$,

$$f(\sigma, 0, \chi_0) < 1/(\sigma - 1) - 0.8973 - s(q),$$

where

$$s(q) := \sum_{p|q} (\log p)\left(\frac{1}{p^\sigma - 1} - \frac{1}{\sqrt{5}\,(p^{\sigma_1} - 1)}\right).$$

Proof. This is [M, Lemma 3]. ∎

LEMMA 2.3. *Let $\kappa := (5 - \sqrt{5})/10$ and $s(q)$ be defined as in Lemma 2.2. Then for any $q \ge 1$ we have*

$$s(q) \le \kappa \log q + 0.4977 \quad \textit{and} \quad s(q) \le 0.1 \log q + 1.0886.$$

Proof. For any $a > 1$, $a^{\sigma_1 - \sigma}$ is decreasing for $\sigma > 0$. Hence for $\sigma > 1$ and prime p,

$$\frac{1}{p^\sigma - 1} - \frac{1}{\sqrt{5}\,(p^{\sigma_1} - 1)} \le \sum_{n=1}^{\infty} \left(\frac{1}{p^n}\right)^{(1+\sqrt{5})/2} \left(p^{n((1+\sqrt{5})/2 - 1)} - \frac{1}{\sqrt{5}}\right)$$
$$\le \begin{cases} 0.784 & \text{if } p = 2, \\ 0.4091 & \text{if } p = 3, \\ 0.2143 & \text{if } p = 5, \\ 0.1467 & \text{if } p = 7, \\ 1/(p-1) & \text{if } p \ge 11. \end{cases}$$

Thus

$$s(q) - \kappa \log q \le (0.784 - \kappa)\log 2 + (0.4091 - \kappa)\log 3 \le 0.4977,$$

which proves the first inequality in the lemma. The second inequality follows from

$$\begin{aligned} s(q) - 0.1 \log q &\le (0.784 - 0.1)\log 2 + (0.4091 - 0.1)\log 3 \\ &\quad + (0.2143 - 0.1)\log 5 + (0.1467 - 0.1)\log 7 \le 1.0886. \end{aligned}$$ ∎

LEMMA 2.4. *Let χ_0 be the principal character modulo q, and let t be any real number. Then for $1 < \sigma < 1.15$ we have*

$$f(\sigma, t, \chi_0) < \begin{cases} \operatorname{Re}((\sigma - 1 + it)^{-1}) - \kappa \log \pi + 0.0615 + s(q) - Z(\sigma, t) & \textit{if } |t| < 1, \\ \kappa \log |t| - \kappa \log \pi + 0.3316 + s(q) - Z(\sigma, t) & \textit{if } |t| \ge 1, \end{cases}$$

where κ and $s(q)$ are defined in Lemmas 2.3 and 2.2 respectively, and $Z(\sigma, t)$ is defined by

$$Z(\sigma, t) := \sum_{\varrho} \left(\operatorname{Re}\left(\frac{1}{\sigma + it - \varrho}\right) - \frac{1}{\sqrt{5}} \operatorname{Re}\left(\frac{1}{\sigma_1 + it - \varrho}\right)\right) \tag{2.4}$$

with the sum over all nontrivial zeros ϱ of $\zeta(s)$.

Proof. By [D, §14, before (1)], and the definition of $f(\sigma,t,\chi_0)$ in (2.3) we get

$$f(\sigma,t,\chi_0) = -\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma+it)\right)+\frac{1}{\sqrt5}\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma_1+it)\right) -\mathrm{Re}\Bigg(\sum_{\substack{n=1\\(n,q)>1}}^{\infty}\left(\frac{\Lambda(n)}{n^{\sigma+it}}-\frac{1}{\sqrt5}\cdot\frac{\Lambda(n)}{n^{\sigma_1+it}}\right)\Bigg). \tag{2.5}$$

Note that the sum on the right hand side of (2.5) has absolute value $\le s(q)$. From [D, §12, (8) and (11)], we see that the sum of the first two terms on the right hand side of (2.5) is

$$\begin{aligned} &= \left\{\mathrm{Re}\left(\frac{1}{\sigma-1+it}\right)-\frac{1}{\sqrt5}\mathrm{Re}\left(\frac{1}{\sigma_1-1+it}\right)\right\}-\frac12\left(1-\frac{1}{\sqrt5}\right)\log\pi \\ &+\frac12\left\{\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+it}{2}+1\right)\right)-\frac{1}{\sqrt5}\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{\sigma_1+it}{2}+1\right)\right)\right\} \\ &-\sum_{\varrho}\mathrm{Re}\left(\frac{1}{\sigma+it-\varrho}\right)+\frac{1}{\sqrt5}\sum_{\varrho}\mathrm{Re}\left(\frac{1}{\sigma_1+it-\varrho}\right). \end{aligned} \tag{2.6}$$

The expression in the first curly brackets in (2.6) is <0 by [M, (20) with $m=1$], if $|t|\ge1$, and is trivially $\le \mathrm{Re}((\sigma-1+it)^{-1})$ for all t. Noting $1<\sigma<1.15$, by [M, Lemmas 1 and 2 with $m=1, a=2$], the expression in the second curly brackets together with the factor $\frac12$ in (2.6) is

$$< \begin{cases} 0.0615 & \text{if } |t|<1,\\ \kappa\log|t|+0.3316 & \text{if } |t|\ge 1.\end{cases}$$

Gathering together the above completes the proof. ■

LEMMA 2.5. *Let χ be a nonprincipal character to modulus q and $1<\sigma<1.15$. Suppose that t is any real number. Then, if χ is primitive, we have*

$$f(\sigma,t,\chi)\le\kappa\log(q\max\{1,\,|t|\})-\kappa\log\pi+0.3918-Z(\sigma,t,\chi); \tag{2.7}$$

if χ is imprimitive and is induced by primitive $\chi_1 \pmod{q_1}$, we have

$$f(\sigma,t,\chi)\le\kappa\log(q_1\max\{1,\,|t|\})-\kappa\log\pi+0.3918 - Z(\sigma,t,\chi)+s(q_1,q), \tag{2.8}$$

where κ is defined in Lemma 2.3, $Z(\sigma,t,\chi)$ is defined as in (2.4) with the sum $\sum_\varrho$ over all nontrivial zeros ϱ of $L(s,\chi)$, and

$$s(q_1,q):=\sum_{p\mid q,\ p\nmid q_1}(\log p)\left(\frac{1}{p^\sigma-1}-\frac{1}{\sqrt5}\cdot\frac{1}{p^{\sigma_1}-1}\right). \tag{2.9}$$

Proof. By [D, §12, (17) and (18)], and (2.3) we get

$$(2.10)\quad f(\sigma,t,\chi)=\kappa\log\frac{q}{\pi}+\frac{1}{2}\operatorname{Re}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+it+\delta}{2}\right)-\frac{1}{\sqrt{5}}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{\sigma_1+it+\delta}{2}\right)\right\} - Z(\sigma,t,\chi),$$

where $\delta=(1-\chi(-1))/2$. Using [M, Lemmas 1 and 2 with $m=1$] to estimate the term $\frac{1}{2}\operatorname{Re}\{\ldots\}$ on the right hand side of (2.10) we get (2.7). To prove (2.8), we note first that if $\chi \pmod q$ is induced by the primitive character $\chi_1 \pmod{q_1}$ then by the definition of $f(\sigma,t,\chi)$ in (2.3),

$$|f(\sigma,t,\chi)-f(\sigma,t,\chi_1)|\le\sum_{p\mid q,\,p\nmid q_1}(\log p)\sum_{k=1}^{\infty}\left(\frac{1}{p^{k\sigma}}-\frac{1}{\sqrt{5}\,p^{k\sigma_1}}\right)=s(q_1,q).$$

Then using (2.7) to estimate $f(\sigma,t,\chi_1)$, we get (2.8). ■

THEOREM 1. *Let x be a real number satisfying $x\ge 8\cdot10^9$ and q be as in* (2.1). *Then the function $\Pi(s)$ defined by* (2.2) *has at most four zeros in the region*

$$1-0.26213/\log x<\operatorname{Re}(s)<1,\qquad |\operatorname{Im}(s)|\le x/q.$$

Proof. Let $\varrho_j=\beta_j+i\gamma_j$ $(1\le j\le 3)$ with $\beta_j\ge 1/2$ be any three given nontrivial zeros of the function defined by (2.2), and let $L(s,\chi_j')$ be the corresponding L-functions, with $\chi_j' \pmod q$ induced by primitive characters $\chi_j \pmod{q_j}$. Then $L(\varrho_j,\chi_j)=0$ for $1\le j\le 3$. Suppose ϱ_j $(1\le j\le 3)$ satisfies

$$(2.11)\qquad \begin{cases}\varrho_j\ne\varrho_k,\overline{\varrho}_k & \text{for } 1\le j\ne k\le 3,\\ \max\limits_{1\le j\le 3}\{1-\beta_j\}\le\sigma-1, & \\ q|\gamma_j|\le x\quad \text{for } 1\le j\le 3 & \text{and}\quad x\ge 8\cdot 10^9,\end{cases}$$

where $1<\sigma<1.15$ is a parameter to be specified later. Put

$$(2.12)\quad g(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3):=f(\sigma,0,\chi_0 \ (\operatorname{mod} 1))+\sum_{j=1}^{3}f(\sigma,\gamma_j,\chi_j) + \sum_{1\le j<k\le 3}f(\sigma,\gamma_j+\gamma_k,\chi_j\chi_k) + f(\sigma,\gamma_1+\gamma_2+\gamma_3,\chi_1\chi_2\chi_3).$$

Then as in [G, (16)],

$$(2.13)\quad g(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3)+g(\chi_1,\overline{\chi}_2,\chi_3;\varrho_1,\overline{\varrho}_2,\varrho_3) + g(\chi_1,\chi_2,\overline{\chi}_3;\varrho_1,\varrho_2,\overline{\varrho}_3)+g(\chi_1,\overline{\chi}_2,\overline{\chi}_3;\varrho_1,\overline{\varrho}_2,\overline{\varrho}_3)\ge 0.$$

Now we give an upper bound for g. We start by giving lower bounds for $Z(\sigma,t,\chi)$, defined as in Lemma 2.5, in (2.14) and (2.16) below. Note that these bounds hold for $Z(\sigma, t)$ since $Z(\sigma, t) = Z(\sigma, t, \chi_0)$. In view of the definition of $Z(\sigma, t, \chi)$, if there exist m (≥ 1) zeros $\varrho_j^* = \beta_j^* + i\gamma_j^*$ ($1 \leq j \leq m$) of $L(s, \chi)$ with $\gamma_j^* = t$, $\beta_j^* \geq 1/2$, then by [M, Lemma 4] we get

$$Z(\sigma, t, \chi) \geq \sum_{j=1}^{m} \frac{1}{\sigma - \beta_j^*};$$

if they do not exist, then by [M, Lemma 4] we have the bound $Z(\sigma, t, \chi) \geq 0$. Thus for $m \geq 0$,

$$Z(\sigma, t, \chi) \geq \Sigma(m), \tag{2.14}$$

where $\Sigma(m) = 0$ if $m = 0$, and $\Sigma(m) = \sum_{1 \leq j \leq m} 1/(\sigma - \beta_j^*)$ if $m \geq 1$ (so $\Sigma(m) \geq 0$). Now suppose further that $\varrho_j^* = \beta_j^* + i\gamma_j^*$ ($m+1 \leq j \leq m+n$, n is an integer ≥ 1) are nontrivial zeros of $L(s, \chi)$ with $\beta_j^* \geq 1/2$ but not necessarily $\gamma_j^* = t$. Then by [M, Lemma 4] we have

$$\begin{aligned} Z(\sigma, t, \chi) \geq \Sigma(m) &+ \sum_{j=m+1}^{m+n} \mathrm{Re}\left(\frac{1}{\sigma + it - \varrho_j^*}\right) \\ &+ \sum_{j=m+1}^{m+n} \left\{ \mathrm{Re}\left(\frac{1}{\sigma + it - 1 + \overline{\varrho}_j^*}\right) - \frac{1}{\sqrt{5}} \mathrm{Re}\left(\frac{1}{\sigma_1 + it - 1 + \overline{\varrho}_j^*}\right) \right. \\ &\left. - \frac{1}{\sqrt{5}} \mathrm{Re}\left(\frac{1}{\sigma_1 + it - \varrho_j^*}\right) \right\}. \end{aligned} \tag{2.15}$$

Note that $1 < \sigma < 1.15$, $\sigma_1 \geq (1+\sqrt{5})/2$ and $1/2 \leq \beta_j^* < 1$. Thus the second term in curly brackets in (2.15) is $\leq 1/(\sqrt{5}\,(\sigma_1 - 1 + \beta_j^*)) \leq 2/5$, and the last term is $\leq 1/(\sqrt{5}\,(\sigma_1 - \beta_j^*)) \leq (5 + \sqrt{5})/10$. Also, if $|t - \gamma_j^*| \leq 1$, the first term in curly brackets is $\geq (\sigma - 1 + \beta_j^*)/((\sigma - 1 + \beta_j^*)^2 + 1) \geq 0.4$. Altogether we see that, for $m \geq 0$ and $n \geq 1$, if $|t - \gamma_j^*| \leq 1$ for all $m + 1 \leq j \leq m + n$, then by (2.15),

$$Z(\sigma, t, \chi) \geq \Sigma(m) + \sum_{j=m+1}^{m+n} \mathrm{Re}\left(\frac{1}{\sigma + it - \varrho_j^*}\right) - \frac{5 + \sqrt{5}}{10} n. \tag{2.16}$$

To estimate g we need to consider the following 8 cases according as χ_1, χ_2, χ_3 and the products of two or three of them are principal or not:

(i) All χ_1, χ_2 and χ_3 are principal.
(ii) Exactly two of χ_j are principal.
(iii) Exactly one of χ_j is principal.
(iv) None of χ_j is principal, and all $\chi_1\chi_2, \chi_2\chi_3, \chi_3\chi_1$ and $\chi_1\chi_2\chi_3$ are nonprincipal.
(v) None of χ_j is principal, and exactly one of $\chi_j\chi_k$ with $1 \leq j < k \leq 3$ is principal.

(vi) None of χ_j is principal, and exactly two of $\chi_j\chi_k$ are principal.
(vii) None of χ_j is principal, whereas all $\chi_j\chi_k$ are principal.
(viii) None of χ_j is principal, whereas $\chi_1\chi_2\chi_3 = \chi_0 \pmod q$.

The arguments are very similar and the worst case is (iv). So we only give the details in Case (iv) and in a subcase of Case (ii), namely, $\chi_1 = \chi_2 = \chi_0 \pmod 1$ and $|\gamma_1 + \gamma_2| < 1$, to illustrate the methods; the latter case is used to demonstrate the influence of the principal character.

The estimate for g in the above subcase of Case (*ii*). Note that χ_j $(1 \le j \le 2)$ is primitive. Thus $L(s,\chi_j) = \zeta(s)$ for $1 \le j \le 2$. So we may assume that $|\gamma_j| \ge 1894438 =: A$, $1 \le j \le 2$; for otherwise, if $|\gamma_{j_0}| < A$ for some $1 \le j_0 \le 2$, then [RS, §0, (0.1)] yields $\beta_{j_0} = 1/2$, and the desired result in Theorem 1 follows at once since $1 - 0.26213/\log x \ge 1 - 0.26213/\log(8\cdot 10^9) > 1/2$. Thus by the second inequality in Lemma 2.4, for $1 \le j \le 2$ we get

$$f(\sigma,\gamma_j,\chi_j \pmod 1) \le \kappa\log|\gamma_j| - \kappa\log\pi + 0.3316 - Z(\sigma,\gamma_j).$$

This together with (2.11), (2.16) with $m = n = 1$ and $\varrho_1^* = \varrho_1, \varrho_2^* = \overline{\varrho}_2$ (if $j = 1$) or $\varrho_1^* = \varrho_2, \varrho_2^* = \overline{\varrho}_1$ (if $j = 2$) and $\Sigma(m) \ge 0$ gives for $1 \le j \ne k \le 2$,

$$\begin{aligned}(2.17)\quad f(\sigma,\gamma_j,\chi_j \pmod 1) &\le \kappa\log|\gamma_j| - \kappa\log\pi + 0.3316 \\ &\quad - \frac{1}{\sigma-\beta_j} - \operatorname{Re}\left(\frac{1}{\sigma-\beta_k+i(\gamma_1+\gamma_2)}\right) + \frac{5+\sqrt5}{10}.\end{aligned}$$

Using [M, Lemma 4] and (2.14) with $m = 1$ and $\varrho_1^* = \varrho_3$, we get $Z(\sigma,\gamma_3,\chi_3) \ge (\sigma-\beta_3)^{-1}$. Thus by (2.7), $q|\gamma_3| \le x$ and $q \le x$ (see (2.11) and (2.1)) we get

$$(2.18)\qquad f(\sigma,\gamma_3,\chi_3) \le \kappa\log x - \kappa\log\pi + 0.3918 - (\sigma-\beta_3)^{-1}.$$

Using $Z(\sigma,t,\chi) \ge 0$, $q \le x$, (2.7) and (2.11), we have

$$(2.19)\qquad f(\sigma,\gamma_1+\gamma_2+\gamma_3,\chi_1\chi_2\chi_3) \le \kappa\log(3x) - \kappa\log\pi + 0.3918,$$

and for $(j,k) = (1,3)$ and $(2,3)$,

$$(2.20)\qquad f(\sigma,\gamma_j+\gamma_k,\chi_j\chi_k) \le \kappa\log(2x) - \kappa\log\pi + 0.3918.$$

Using the first inequality of Lemma 2.4 with $t = \gamma_1+\gamma_2$, and using $Z(\sigma,t) \ge 0$, we get

$$\begin{aligned}(2.21)\quad f(\sigma,\gamma_1+\gamma_2,\chi_1\chi_2 \pmod 1)& \\ &\le \operatorname{Re}\left(\frac{1}{\sigma-1+i(\gamma_1+\gamma_2)}\right) - \kappa\log\pi + 0.0615.\end{aligned}$$

Also Lemma 2.2 with $q = 1$ gives

$$(2.22)\qquad f(\sigma,0,\chi_0 \pmod 1) \le (\sigma-1)^{-1} - 0.8973.$$

By (2.12) and (2.17) to (2.22) we can summarize that as

$$
\begin{aligned}
(2.23)\quad & g(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3) \\
\le\ & \frac{1}{\sigma-1} - \sum_{j=1}^{3} \frac{1}{\sigma-\beta_j} \\
&+ \left\{ \operatorname{Re}\left(\frac{1}{\sigma-1+i(\gamma_1+\gamma_2)} \right) - \operatorname{Re}\left(\frac{1}{\sigma-\beta_1+i(\gamma_1+\gamma_2)} \right) \right. \\
&\qquad \left. - \operatorname{Re}\left(\frac{1}{\sigma-\beta_2+i(\gamma_1+\gamma_2)} \right) \right\} \\
&+ \kappa\{\log|\gamma_1| + \log|\gamma_2| + \log x + 2\log(2x) + \log(3x)\} \\
&+ \left\{ -0.8973 - 7\kappa\log\pi + 0.3316\cdot 2 + \frac{5+\sqrt{5}}{10}\cdot 2 + 0.3918\cdot 4 + 0.0615 \right\}.
\end{aligned}
$$

By [G, Lemma 2] and (2.11), the expression in the first curly brackets is ≤ 0, and the one in the second curly brackets is $\le 6\log x + 2\log 2 + \log 3$. The expression in the last curly brackets is ≤ 0.6271. Thus (2.23) is

$$
\begin{aligned}
&\le \frac{1}{\sigma-1} - \sum_{j=1}^{3} \frac{1}{\sigma-\beta_j} + 6\kappa\log x + (2\log 2 + \log 3)\kappa + 0.6271 \\
&\le \frac{1}{\sigma-1} - \sum_{j=1}^{3} \frac{1}{\sigma-\beta_j} + 7\kappa\log x.
\end{aligned}
$$

The estimate for g in Case (*iv*). Similarly to (2.18), by (2.7) and (2.11) we have for $1 \le j \le 2$,

$$
(2.24)\qquad f(\sigma, \gamma_j, \chi_j) \le \kappa\log x - \kappa\log\pi + 0.3918 - (\sigma-\beta_j)^{-1}.
$$

Similarly to the case of $s(q)$ in Lemma 2.3, by (2.9) we have

$$
s(q_1, q) \le \kappa\log(q/q_1) + 0.4977.
$$

Thus for any $1 \le j < k \le 3$, if we suppose $\chi_j\chi_k$ is induced by the primitive character χ^* $(\operatorname{mod} q_1)$, then by (2.8) and using the bound $Z(\sigma, t, \chi) \ge 0$, (2.11) and $q \le x$ in (2.1), we get

$$
\begin{aligned}
(2.25)\qquad f(\sigma, \gamma_j+\gamma_k, \chi_j\chi_k) &\le \kappa\log(q_1\max\{1, |\gamma_j+\gamma_k|\}) - \kappa\log\pi + 0.3918 \\
&\qquad - Z(\sigma, \gamma_j+\gamma_k, \chi_j\chi_k) + s(q_1, q) \\
&\le \kappa\log(2x) - \kappa\log\pi + 0.3918 + 0.4977.
\end{aligned}
$$

Similarly,

$$
(2.26)\qquad f(\sigma, \gamma_1+\gamma_2+\gamma_3, \chi_1\chi_2\chi_3) \le \kappa\log(3x) - \kappa\log\pi + 0.3918 + 0.4977.
$$

By (2.12), (2.22), (2.18), (2.24)–(2.26), and then by (2.11) we get

$$
\begin{aligned}
(2.27)\qquad & g(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3)\\
&\le (\sigma-1)^{-1} - 0.8973 + (\kappa\log x - \kappa\log\pi + 0.3918)\cdot 3\\
&\quad - \sum_{j=1}^{3}\frac{1}{\sigma-\beta_j} + (\kappa\log(2x) - \kappa\log\pi + 0.3918 + 0.4977)\cdot 3\\
&\quad + \kappa\log(3x) - \kappa\log\pi + 0.3918 + 0.4977\\
&\le \frac{1}{\sigma-1} - \sum_{j=1}^{3}\frac{1}{\sigma-\beta_j} + 7\kappa\log x + 2.4998.
\end{aligned}
$$

For the other 7 cases, except for the constant term, we can obtain the same estimate. The constant 2.4998 in (2.27) can be replaced by: (i) 0.0876, (ii) 0.3284, (iii) 1.3238, (v) 2.4396, (vi) 2.3795, (vii) 2.3193 and (viii) 2.4397. Therefore, as claimed, (iv) is the worst case. The other three g on the left hand side of (2.13) can be estimated in completely the same way, and have the bound given in (2.27). Using this and (2.13) we get

$$
\frac{1}{\sigma-1} - \sum_{j=1}^{3}\frac{1}{\sigma-\beta_j} + 7\kappa\log x + 2.4998 \ge 0. \tag{2.28}
$$

Now we let $\sigma = 1 + a/\log x$, $\beta_j = 1 - b_j/\log x$, with a to be chosen later. Then (2.28) yields

$$
\max_{1\le j\le 3}\{b_j\} \le \frac{3}{1/a + 7\kappa + 2.4998/\log x} - a. \tag{2.29}
$$

The optimal choice for a is

$$
a = \frac{\sqrt{3}-1}{7\kappa + 2.4998/\log x} \qquad (\le 0.3784),
$$

which yields $1 < \sigma < 1.15$ in Lemmas 2.2 to 2.5. With this choice of a, by (2.29) we get $\max_{1\le j\le 3}\{b_j\} > 0.26213$. This together with (2.11) completes the proof of Theorem 1. ■

THEOREM 2. *Let x be a real number satisfying $x \ge 8\cdot 10^9$ and q be as in* (2.1). *Then the function $\Pi(s)$ defined by* (2.2) *has at most two zeros in the region*

$$
1 - 0.2067/\log x < \mathrm{Re}(s) < 1, \qquad |\mathrm{Im}(s)| \le x/q.
$$

Moreover, if ϱ_j, $1\le j\le 2$, are two zeros of $\Pi(s)$ with real part $1-\lambda_j/\log x$, imaginary part $\le x/q$ and with $\varrho_2 \ne \varrho_1, \overline{\varrho}_1$, then we have the following table, where a is a parameter used in the proof:

Table 1

$\lambda_1 \le$	$\lambda_2 >$	$a =$	$\lambda_1 \le$	$\lambda_2 >$	$a =$
0.10367089	0.3534	0.473	0.17	0.2477	0.496
0.12	0.3221	0.482	0.18	0.2356	0.498
0.13	0.3050	0.486	0.19	0.2242	0.499
0.14	0.2891	0.489	0.20	0.2135	0.499
0.15	0.2743	0.492	0.206	0.2074	0.499
0.16	0.2605	0.495	0.2067	0.2067	0.499

Proof. Let ϱ_1, ϱ_2 be two zeros of (2.2), and $L(\varrho_1, \chi_1') = L(\varrho_2, \chi_2') = 0$. Suppose that χ_1' and χ_2' are induced by primitive χ_1 and χ_2 respectively. Then $L(\varrho_1, \chi_1) = L(\varrho_2, \chi_2) = 0$. We always suppose that

$$\varrho_2 \ne \varrho_1, \overline{\varrho}_1. \tag{2.30}$$

Let $f(\sigma, t, \chi)$ be defined as in (2.3), and define

$$\begin{aligned} & g_1(\chi_1, \chi_2; \varrho_1, \varrho_2) \\ & \quad = f(\sigma, 0, \chi_0 \ (\mathrm{mod}\, 1)) + \sum_{j=1}^{2} f(\sigma, \gamma_j, \chi_j) + f(\sigma, \gamma_1 + \gamma_2, \chi_1\chi_2). \end{aligned} \tag{2.31}$$

Then similarly to (2.13) we can obtain

$$g_1(\chi_1, \chi_2; \varrho_1, \varrho_2) + g_1(\chi_1, \overline{\chi}_2; \varrho_1, \overline{\varrho}_2) \ge 0. \tag{2.32}$$

We first give an upper bound for $g_1(\chi_1, \chi_2; \varrho_1, \varrho_2)$. There are four cases to be considered according as χ_1, χ_2 or $\chi_1\chi_2$ are principal or not:

(i) $\chi_1 = \chi_2 = \chi_0 \ (\mathrm{mod}\, 1)$.
(ii) Exactly one of χ_j is principal.
(iii) No χ_j $(1 \le j \le 2)$ is principal, whereas $\chi_1\chi_2$ is principal.
(iv) No χ_j is principal, and $\chi_1\chi_2$ is nonprincipal.

The worst bound for g_1 is in Case (iv), and in this case we can use (2.31), (2.22), (2.24), (2.25) with $(j, k) = (1, 2)$ to get

$$\begin{aligned} g_1(\chi_1, \chi_2; \varrho_1, \varrho_2) &\le (\sigma - 1)^{-1} - 0.8973 + (\kappa \log x - \kappa \log \pi + 0.3918) \cdot 2 \\ &\qquad - \sum_{j=1}^{2} \frac{1}{\sigma - \beta_j} + \kappa \log(2x) - \kappa \log \pi + 0.3918 + 0.4977 \\ &\le \frac{1}{\sigma - 1} - \sum_{j=1}^{2} \frac{1}{\sigma - \beta_j} + 3\kappa \log x + 0.0182, \end{aligned}$$

where $1 < \sigma < 1.15$. (For Cases (i)–(iii) we can replace the constant 0.0182 by 0.) The same bound can be derived for the other g_1 on the left hand side of (2.32). Therefore, under (2.30) and the second and the third inequalities in (2.11) with $1 \le j \le 2$, we have

$$\frac{1}{\sigma-1} - \sum_{j=1}^{2} \frac{1}{\sigma-\beta_j} + 3\kappa \log x + 0.0182 \ge 0.$$

Letting $\sigma = 1 + a/\log x$ and $\beta_j = 1 - b_j/\log x$ we get

$$\frac{1}{a} - \sum_{j=1}^{2} \frac{1}{a+b_j} + 3\kappa + \frac{0.0182}{\log x} \ge 0, \tag{2.33}$$

so

$$\max_{1\le j\le 2}\{b_j\} \ge \frac{2}{1/a + 3\kappa + 0.0182/\log x} - a. \tag{2.34}$$

Choose

$$a = \frac{\sqrt{2}-1}{3\kappa + 0.0182/\log x} \qquad (\le 0.4996),$$

which yields $1 < \sigma < 1.15$ since $x \ge 8\cdot 10^9$. With this choice of a, (2.34) becomes

$$\max_{1\le j\le 2}\{b_j\} \ge \frac{3-2\sqrt{2}}{3\kappa + 0.0182/\log x} > 0.2067.$$

Also by (2.33) with $x \ge 8\cdot 10^9$, we can derive Table 1. ■

THEOREM 3. *Let $\widetilde{\beta}$ and $\widetilde{r}$ be as in Lemma 2.1. Then*

$$\widetilde{\beta} \le 1 - \pi/(0.4923\widetilde{r}^{1/2}\log^2 \widetilde{r}).$$

Proof. The proof uses the class number formula of Dirichlet. Given any integer $d \ne 0$, we define the number of classes of quadratic forms by $h(d)$ as in [D, §6, before (2)] (if $d < 0$, only the positive definite forms are counted). Then $h(d) \ge 1$. If $d > 0$, let υ_0, u_0 be the integer solution to the equation

$$\upsilon^2 - du^2 = 4$$

with both υ_0 and u_0 positive and u_0 being the least one as in [D, §6, after (5)]. Put $\varepsilon = (\upsilon_0 + u_0\sqrt{d})/2$ temporarily as in [D, §6, p. 46, before (10)]. If $d < 0$, we use the notation w in [D, §6, (3)], so that $w = 2$ if $d < -4$, $w = 4$ if $d = -4$ and $w = 6$ if $d = -3$. Then [D, §6, (15) and (16)] asserts that

$$h(d) = \begin{cases} \dfrac{w|d|^{1/2}}{2\pi} L(1,\chi) & \text{for } d < 0, \\ \dfrac{d^{1/2}}{\log \varepsilon} L(1,\chi) & \text{for } d > 0, \end{cases}$$

where χ is a primitive character to modulus $|d|$. It turns out that (with the above d taken to be $\widetilde{r}$ or $-\widetilde{r}$)

$$1 \le h(-\widetilde{r}) = \frac{w\widetilde{r}^{1/2}}{2\pi} L(1, \widetilde{\chi}) \quad \text{if } \widetilde{\chi}(-1) = -1,$$
$$1 \le h(\widetilde{r}) = \frac{\widetilde{r}^{1/2}}{\log \varepsilon} L(1, \widetilde{\chi}) \quad \text{if } \widetilde{\chi}(-1) = 1;$$

whence always

$$L(1, \widetilde{\chi}) \ge \min\left\{ \frac{2\pi}{w\widetilde{r}^{1/2}}, \frac{\log \varepsilon}{\widetilde{r}^{1/2}} \right\}, \tag{2.35}$$

whether $\widetilde{\chi}(-1) = 1$ or not. Note that $\varepsilon = \frac{1}{2}(v_0 + u_0\sqrt{\widetilde{r}}) \ge \frac{1}{2}(\sqrt{4 + \widetilde{r}} + \sqrt{\widetilde{r}})$, and $\widetilde{r} \ge 987$ (see Lemma 2.1), which clearly implies that the w in (2.35) is 2. So by (2.35) we get

$$L(1, \widetilde{\chi}) \ge \min\left\{ \frac{\pi}{\widetilde{r}^{1/2}}, \frac{\log\left(\frac{1}{2}(\sqrt{4 + \widetilde{r}} + \sqrt{\widetilde{r}})\right)}{\widetilde{r}^{1/2}} \right\} \ge \frac{\pi}{\widetilde{r}^{1/2}}. \tag{2.36}$$

For any $\sigma \ge \widetilde{\beta} \ge 1 - 1/(c_1 \log x)$, we have

$$\frac{d}{d\sigma} L(\sigma, \widetilde{\chi}) = -\sum_{n=1}^{\infty} (\log n)\widetilde{\chi}(n)n^{-\sigma}.$$

So for any $y \ge 10$ to be chosen later, by [D, p. 135, (2)] and since $\sigma \ge 1 - 1/(c_1 \log x)$, we have

$$\begin{aligned} |L'(\sigma, \widetilde{\chi})| &\le \sum_{n=2}^{y} (\log n)n^{-\sigma} + \Big| \sum_{n>y} \widetilde{\chi}(n)(\log n)n^{-\sigma} \Big| \\ &\le \left(\frac{1}{2}\log^2 y - \frac{1}{2}\log^2 3 + \frac{\log 3}{3} + \frac{\log 2}{2} + \widetilde{r}^{1/2}(\log \widetilde{r})y^{-1}\log y \right) \\ &\quad \times e^{(\log y)/(c_1 \log x)}. \end{aligned}$$

Take $y = 13\widetilde{r}^{1/2}$, then $\log y \le 0.8721 \log \widetilde{r}$. So the above is $\le 0.4923 \log^2 \widetilde{r}$. Also, the mean value theorem gives

$$L(1, \widetilde{\chi}) = L(1, \widetilde{\chi}) - L(\widetilde{\beta}, \widetilde{\chi}) = L'(\sigma, \widetilde{\chi})(1 - \widetilde{\beta}) \quad \text{where } \sigma \in (\widetilde{\beta}, 1).$$

These together with (2.36) yield

$$1 - \widetilde{\beta} = \frac{L(1, \widetilde{\chi})}{|L'(\sigma, \widetilde{\chi})|} \ge \frac{\pi}{\widetilde{r}^{1/2}} \cdot \frac{1}{0.4923 \log^2 \widetilde{r}}. \ \blacksquare$$

3. The zero-density estimates for Dirichlet L-functions. Throughout this section we suppose that $x \ge 1$, $y \ge 0$, and

$$z \ge \max\{xy, 10^{11}\}. \tag{3.1}$$

For any character χ to modulus $q \le x$, let $N(\alpha, \chi, y)$ denote the number of zeros of $L(s, \chi)$ in the region

$$\alpha \le \mathrm{Re}(s) < 1, \qquad |\mathrm{Im}(s)| \le y. \tag{3.2}$$

Put

$$N(\alpha, q, y) := \sum_{\chi \,(\mathrm{mod}\, q)} N(\alpha, \chi, y). \tag{3.3}$$

In this section, we first give explicit upper bounds for $N(\alpha, q, y)$ when α is very near to the line $\mathrm{Re}(s) = 1$ based on the results obtained in Section 2, then give explicit upper bounds for individual $N(\alpha, \chi, y)$ for any $\alpha \in (0, 1)$ by the classical method as in [D, §§15–16], and finally give a revised form of [C, Theorem], which takes care of α lying in the "middle range". For convenience, if α is very near to the line $\mathrm{Re}(s) = 1$, we put $\alpha = 1 - \lambda/\log z$ with $\lambda > 0$. We always assume $0.262132 \le \lambda \le 0.5$.

LEMMA 3.1. *Let χ be any character to modulus q, and t any real number. Let n be the number of zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ in* (3.2) *such that*

$$|t - \gamma| \le b/\log z, \tag{3.4}$$

where b is a positive parameter as in Table 2 *below. Then we have the following table, where $a > b$ is a parameter used in the proof*:

Table 2

$\lambda \le$	0.27	0.28	0.29	0.3	0.31	0.32
$b =$	0.738	0.7262	0.7142	0.7019	0.6897	0.677
$n \le$	1	1	1	1	1	1
$a =$	1.73	1.72	1.7	1.69	1.68	1.67
$b =$	2.0387	2.0327	2.0267	2.0206	2.0153	2.009
$n \le$	2	2	2	2	2	2
$a =$	3.33	3.32	3.31	3.3	3.3	3.29

$\lambda \le$	0.33	0.36	0.39	0.42	0.45	0.48
$b =$	0.6633	0.6219	0.577	0.5276	0.4725	0.4092
$n \le$	1	1	1	1	1	1
$a =$	1.66	1.62	1.58	1.55	1.51	1.48
$b =$	2.002	1.9832	1.964	1.9444	1.9244	1.9039
$n \le$	2	2	2	2	2	2
$a =$	3.27	3.25	3.22	3.19	3.16	3.12

Proof. Suppose that $\chi \,(\mathrm{mod}\, q)$ is induced by the primitive character $\chi_1 \,(\mathrm{mod}\, q_1)$. Then $q_1 \mid q$ and $L(s, \chi)$ and $L(s, \chi_1)$ have the same set of nontrivial zeros. Consider two cases according as $q_1 = 1$ or not to estimate $f(\sigma, t, \chi_1)$

defined by (2.3). When $q_1 \neq 1$, the worse case, we may use (2.7) with $\chi = \chi_1$ (so q there equals q_1). First by (3.4) and (2.16) with $m = 0$ and noting $a > b$ (in Table 2) we have

$$Z(\sigma, t, \chi_1) \geq \frac{(a+\lambda)n}{(a+\lambda)^2 + b^2} \log z + \frac{5+\sqrt{5}}{10} n.$$

Secondly, from $b \leq 2.0387$ and $z \geq 10^{11}$ one can derive readily that

$$\log(q_1 \max\{1, |t|\}) \leq \log z + 0.0775.$$

Thus by (2.7),

$$\begin{aligned} (3.5) \quad & f(\sigma, t, \chi_1) \\ & \leq \kappa \log z + 0.0775\kappa - \kappa \log \pi + 0.3918 + \frac{5+\sqrt{5}}{10} n - \frac{(a+\lambda)n}{(a+\lambda)^2 + b^2} \log z. \end{aligned}$$

When $q_1 = 1$, it is clear that we can assume $|t| \geq 1$. So using the second inequality for $f(\sigma, t, \chi_0)$ in Lemma 2.4 to replace (2.7) above we find easily that (3.5) holds with 0.0775 replaced by $8.05 \cdot 10^{-13}$. Plainly by (2.3) we have $f(\sigma, 0, \chi_0 \pmod 1) + f(\sigma, t, \chi_1) \geq 0$; thus by (3.5) and Lemma 2.2 with $q = 1$ and $\sigma = 1 + a/\log z$ we get

$$\begin{aligned} \frac{\log z}{a} - 0.8973 + \kappa \log z - \frac{(a+\lambda)n}{(a+\lambda)^2 + b^2} \log z + 0.0775\kappa \\ -\kappa \log \pi + 0.3918 + \frac{5+\sqrt{5}}{10} n \geq 0; \end{aligned}$$

and consequently

$$(3.6) \qquad n \leq \left[\frac{\kappa + 1/a - (0.8973 - 0.3918 - 0.0775\kappa + \kappa \log \pi)/\log z}{(a+\lambda)/((a+\lambda)^2 + b^2) - (5+\sqrt{5})/(10 \log z)} \right],$$

where $[u]$ denotes the greatest integer not exceeding u for any real u. Note that the right hand side of (3.6) is nondecreasing with respect to $\lambda > 0$ since $a > b$. By (3.6), Table 2 is established: For instance, if $\lambda \leq 0.27$, then we may replace the λ, a and b on the right hand side of (3.6) by 0.27, 1.73 and 0.738 respectively, and then it can be observed that the right hand side of (3.6) is nonincreasing with respect to z, so we can replace the z by 10^{11} to obtain

$$n \leq [1.99999] = 1$$

as stated in Table 2. ■

Now we turn to estimating $N(\alpha, q, y)$ defined by (3.3). We also use the notation $\sigma = 1 + a/\log z$. For any zero $\varrho = \beta + i\gamma$ of (2.2) inside the region given by (3.2), let $\chi' \pmod q$ be a corresponding character, which is assumed to be induced by the primitive character $\chi \pmod{q_1}$. Then $L(\varrho, \chi) = 0$.

Noting (3.1), (3.2) and $\alpha = 1-\lambda/\log z$, and considering two cases according as $q_1 = 1$ or not (in fact, $q_1 \neq 1$ is the worse case), by Lemma 2.4, (2.7) and (2.14) with $m = 1$, we get

$$f(\sigma,\gamma,\chi) \le \kappa\log z - \kappa\log\pi + 0.3918 - \frac{\log z}{a+\lambda}. \tag{3.7}$$

Summing up (3.7) with respect to all the zeros ϱ considered, on noting (3.3) we get

$$\sum_{(\varrho)} f(\sigma,\gamma,\chi) \le \left(\kappa\log z - \kappa\log\pi + 0.3918 - \frac{\log z}{a+\lambda}\right) N(\alpha,q,y), \tag{3.8}$$

where the sum is over the above mentioned zeros ϱ, i.e. all the zeros $\varrho = \beta+i\gamma$ of $\Pi(s)$ inside the region given by (3.2). Now to estimate $N(\alpha,q,y)$, we need an upper bound for the squared absolute value of the left hand side of (3.8). By (2.3) and Hölder's inequality we get

$$\begin{aligned} \Big|\sum_{(\varrho)} f(\sigma,\gamma,\chi)\Big|^2 &= \Big|\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\left(1-\frac{1}{\sqrt{5}\,n^{\sigma_1-\sigma}}\right)\sum_{(\varrho)} \mathrm{Re}\left(\frac{\chi(n)}{n^{i\gamma}}\right)\Big|^2 \\ &\le \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\left(1-\frac{1}{\sqrt{5}\,n^{\sigma_1-\sigma}}\right) \\ &\quad \times \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\left(1-\frac{1}{\sqrt{5}\,n^{\sigma_1-\sigma}}\right)\Big|\sum_{(\varrho)} \mathrm{Re}\left(\frac{\chi(n)}{n^{i\gamma}}\right)\Big|^2, \end{aligned} \tag{3.9}$$

where σ_1 is defined in the paragraph preceding (2.3). By Lemma 2.2 with $q = 1$, the first sum over n on the right hand side of (3.9) is

$$\le 1/(\sigma-1) - 0.8973 = (\log z)/a - 0.8973. \tag{3.10}$$

If we denote the ϱ in (3.9) by $\varrho_j = \beta_j + i\gamma_j$, and the corresponding primitive character $\chi \pmod{q_1}$ by χ_j, and if we temporarily write N for $N(\alpha,q,y)$ (see (3.3)), then the last term on the right hand side of (3.9) is

$$\Big|\sum_{j=1}^{N} \mathrm{Re}\left(\frac{\chi_j(n)}{n^{i\gamma_j}}\right)\Big|^2 \le \Big|\sum_{j=1}^{N} \frac{\chi_j(n)}{n^{i\gamma_j}}\Big|^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} \mathrm{Re}\left(\frac{\chi_j\overline{\chi}_k(n)}{n^{i(\gamma_j-\gamma_k)}}\right).$$

Thus the second sum over n on the right hand side of (3.9) is

$$\begin{aligned} &\le \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}}\left(1-\frac{1}{\sqrt{5}\,n^{\sigma_1-\sigma}}\right)\mathrm{Re}\left(\frac{\chi_j\overline{\chi}_k(n)}{n^{i(\gamma_j-\gamma_k)}}\right) \\ &= \sum_{j=1}^{N}\sum_{k=1}^{N} f(\sigma,\gamma_j-\gamma_k,\chi_j\overline{\chi}_k). \end{aligned} \tag{3.11}$$

By Lemma 2.2 with $q = 1$, the total contribution to (3.11) from the terms with $j = k$ is

$$\leq \sum_{j=1}^{N} f(\sigma, 0, \chi_0 \ (\mathrm{mod}\, 1)) \leq ((\log z)/a - 0.8973)N. \tag{3.12}$$

For $j \neq k$ and $\chi_j \overline{\chi}_k \neq \chi_0$, by (2.8), (2.14) with $m = 0$ and $s(q_1, q) \leq \kappa \log(q/q_1) + 0.4977$ (see after (2.24)) we get

$$\begin{aligned} f(\sigma, \gamma_j - \gamma_k, \chi_j \overline{\chi}_k) &\leq \kappa \log(2z) - \kappa \log \pi + 0.3918 + 0.4977 \\ &\leq \kappa \log z + 0.7647. \end{aligned} \tag{3.13}$$

If $j \neq k$ and $\chi_j \overline{\chi}_k = \chi_0$ (so $\chi_j = \chi_k$), by Lemma 2.4 and (2.14) with $m = 0$ we get

$$\begin{aligned} &f(\sigma, \gamma_j - \gamma_k, \chi_j \overline{\chi}_k) \\ &\quad \leq \max \Big\{ \mathrm{Re}\left(\frac{1}{\sigma - 1 + i(\gamma_j - \gamma_k)}\right) - \kappa \log \pi + 0.0615 + s(q), \\ &\qquad\qquad \kappa \log |\gamma_j - \gamma_k| - \kappa \log \pi + 0.3316 + s(q) \Big\}. \end{aligned} \tag{3.14}$$

Now we use Lemma 3.1 with $n \leq 1$, and denote the parameter b in Table 2 corresponding to $n \leq 1$ by $b(\lambda)$. In view of $\chi_j = \chi_k$ we have $|\gamma_j - \gamma_k| \geq 2b(\lambda)/\log z$, so that by Lemma 2.3 the first term in the curly brackets in (3.14) is

$$\begin{aligned} &\leq \frac{a}{a^2 + 4b(\lambda)^2} \log z - \kappa \log \pi + 0.0615 + s(q) \\ &\leq \left(\frac{a}{a^2 + 4b(\lambda)^2} + 0.1\right) \log z + 0.8338, \end{aligned} \tag{3.15}$$

which can be dominated by the right hand side of (3.13) if a satisfies (3.16) below. Also by Lemma 2.3 and (3.1), the second term in curly brackets in (3.14) is $\leq \kappa \log(2z) - \kappa \log \pi + 0.3316 + 0.4977$, which can clearly be dominated by the right hand side of (3.13). Hence by (3.12)–(3.14) one can estimate (3.11) further by $((\log z)/a - 0.8973)N + (\kappa \log z + 0.7647)(N^2 - N)$, with

$$a \leq \begin{cases} 0.365 & \text{if } 0.27 \leq \lambda \leq 0.3, \\ 0.3519 & \text{if } 0.3 < \lambda \leq 0.31, \\ 0.3382 & \text{if } 0.31 < \lambda \leq 0.32, \\ 0.3238 & \text{if } 0.32 < \lambda \leq 0.33, \\ 0.2825 & \text{if } 0.33 < \lambda \leq 0.36, \\ 0.2413 & \text{if } 0.36 < \lambda \leq 0.39, \\ 0.2003 & \text{if } 0.39 < \lambda \leq 0.42. \end{cases} \tag{3.16}$$

This together with (3.10) enables one to estimate (3.9) by

$$((\log z)/a - 0.8973)^2 N + ((\log z)/a - 0.8973)(\kappa \log z + 0.7647)(N^2 - N),$$

under (3.16). This in combination with (3.8) gives

$$\begin{aligned}((\log z)/(a+\lambda) - \kappa \log z + \kappa \log \pi - 0.3918)^2 N^2 \\ \le ((\log z)/a - 0.8973)^2 N \\ + ((\log z)/a - 0.8973)(\kappa \log z + 0.7647)(N^2 - N)\end{aligned}$$

if $1/(a+\lambda) - \kappa + (\kappa \log \pi - 0.3918)/\log z \ge 0$. Noting $\kappa \log \pi - 0.3918 \ge -0.0755$, we can conclude that

$$(3.17) \quad N \le \left[\frac{(1/a - 0.8973/\log z)^2 - (1/a - 0.8973/\log z)(\kappa + 0.7647/\log z)}{(1/(a+\lambda) - \kappa - 0.0755/\log z)^2 - (1/a - 0.8973/\log z)(\kappa + 0.7647/\log z)}\right],$$

under (3.16) and

$$(3.18) \quad \begin{cases} 1/(a+\lambda) - \kappa - 0.0755/\log z \ge 0, \\ (1/(a+\lambda) - \kappa - 0.0755/\log z)^2 \\ \qquad - (1/a - 0.8973/\log z)(\kappa + 0.7647/\log z) > 0. \end{cases}$$

Now we use (3.17) to give upper bounds for $N = N(\alpha, q, y)$ when α is very near the line $\mathrm{Re}(s) = 1$. If $0.27 \le \lambda \le 0.3$, then numerical experiments show that the optimal choice for a in (3.17) satisfying (3.16) is approximately $a = 0.365$, which clearly satisfies (3.18) since $z \ge 10^{11}$. With this choice of a, and noting $z \ge 10^{11}$, we may deduce from (3.17) the following table:

Table 3

$\lambda \le$	0.27	0.28	0.3
$N \le$	7	8	9

If λ has upper bounds at 0.31, 0.32, 0.33, 0.36, 0.39 and 0.42, we take a to be 0.3519, 0.3382, 0.3238, 0.2825, 0.2413 and 0.2003 respectively. With these choices of a, by (3.17) we may deduce the following table:

Table 4

$\lambda \le$	0.31	0.32	0.33	0.36	0.39	0.42
$N \le$	10	11	13	20	35	89

Now we turn to an alternative method of estimating $N(\alpha, q, y)$ for $0.45 \le \lambda \le 0.48$. For each χ label the zeros counted by $N(\alpha, \chi, y)$ as $\varrho_j = \beta_j + i\gamma_j$, with $\gamma_j \le \gamma_{j+1}$. Then Lemma 3.1 shows that $\gamma_{j+2} - \gamma_j \ge 2(1.9039/\log z)$.

Thus, if we allow N_1 to count $\gamma_1, \gamma_3, \gamma_5, \ldots$, we will have $[(N(\alpha,\chi,y)+1)/2]$ zeros, with imaginary parts separated by at least $2(1.9039/\log z)$. Therefore, we can find

$$N_1 := \sum_{\chi \,(\mathrm{mod}\, q)} [(N(\alpha,\chi,y)+1)/2] \tag{3.19}$$

zeros of the function $\mathit{\Pi}(s)$ defined by (2.2), having the property that if there are two or more zeros corresponding to a single L-function inside (3.2) then the differences between their imaginary parts will be at least $2(1.9039/\log z)$. Also it can be derived from (3.19) that

$$N_1 \geq N(\alpha,q,y)/2. \tag{3.20}$$

Now summing up (3.7) with respect to only the N_1 zeros from (3.19), we get, instead of (3.8),

$$\sum_{(\varrho)}{}_1\, f(\sigma,\gamma,\chi) \leq (\kappa \log z - \kappa \log \pi + 0.3918 - (\log z)/(a+\lambda))N_1, \tag{3.21}$$

where the sum is over all the N_1 zeros from (3.19). Now we can repeat the arguments from (3.9) to (3.17), and then give an upper bound for N_1. The only difference is that the first term in curly brackets on the right hand side of (3.14) is now

$$\leq \frac{a}{a^2+(1.9039\cdot 2)^2}\log z - \kappa \log \pi + 0.0615 + 1.0886 + 0.1 \log z$$

which can always be dominated by the right hand side of (3.13) without any further constraints on a. As in (3.17), by (3.21) we get for $0 < a \leq 0.4$, which satisfies (3.18) since $\lambda \leq 0.48$ and $z \geq 10^{11}$,

$$N_1 \leq \left[\frac{(1/a-0.8973/\log z)^2-(1/a-0.8973/\log z)(\kappa+0.7647/\log z)}{(1/(a+\lambda)-\kappa-0.0755/\log z)^2-(1/a-0.8973/\log z)(\kappa+0.7647/\log z)}\right].$$

Therefore by (3.20) we can conclude that under (3.18),

$$\begin{aligned} &N(\alpha,q,y) \\ &\leq 2\left[\frac{(1/a-0.8973/\log z)^2-(1/a-0.8973/\log z)(\kappa+0.7647/\log z)}{(1/(a+\lambda)-\kappa-0.0755/\log z)^2-(1/a-0.8973/\log z)(\kappa+0.7647/\log z)}\right]. \end{aligned} \tag{3.22}$$

Now if $\lambda \leq 0.45$, then numerical experiments show that the optimal choice for a in (3.22) is approximately 0.34. With this choice, by (3.22) and using $z \geq 10^{11}$ we can deduce $N(\alpha,q,y) \leq 4\cdot 91 = 364$. In this way we can establish the following table:

Table 5

$\lambda \le$	0.45	0.46	0.47	0.475	0.478
$a =$	0.34	0.33	0.32	0.315	0.311
$N \le$	182	292	664	$834 \cdot 2$	$7000 \cdot 2$

Summarizing, we can conclude the following

THEOREM 4. *Let $N(\alpha, q, y)$ be defined as in* (3.3) *with $\alpha = 1 - \lambda/\log z$. Then the bounds for $N = N(\alpha,\, q,\, y)$ in Tables* 3 *to* 5 *hold under* (3.1).

Now we estimate individual $N(\alpha, \chi, y)$ for $\alpha \in (0,1)$. First, consider the case $\chi = \chi_0$. Note that $L(s,\chi)$ and $\zeta(s)$ have the same set of nontrivial zeros, so the results in [T1, p. 389, §15.2] ensure that $N(\alpha,\chi_0,y) = 0$ for $y \le 14$; and thus we may assume that $y \ge 14$ in this case. Also, we may assume that y does not coincide with the ordinate of any nontrivial zero of $\zeta(s)$, otherwise we may use $N(\alpha,\chi_0,y+0)$ instead of $N(\alpha,\chi_0,y)$, and then take limits to deduce the required result. Let $\boldsymbol{L}$ denote the line from 2 to $2+iy$ and then to $1/2+iy$. Since the zeros of $\zeta(s)$ are symmetric with respect to the line $\sigma = 1/2$ and the real axis, by [D, p. 97, line -7] we have

$$N(\alpha,\chi_0,y) \le (2/\pi)\Delta_{\boldsymbol{L}} \arg \xi(s), \tag{3.23}$$

where $\xi(s) = (s-1)\pi^{-s/2}\Gamma(s/2+1)\zeta(s)$, $s = u+iv$ and $\Delta_{\boldsymbol{L}}$ denotes the continuous variation of the argument of $\xi(s)$ along $\boldsymbol{L}$. By the definition of $\boldsymbol{L}$ we have $\pi/2 \le \Delta_{\boldsymbol{L}} \arg(s-1) \le 0.5114\pi$, and $\Delta_{\boldsymbol{L}} \arg \pi^{-s/2} = \Delta_{\boldsymbol{L}}\bigl(-\frac{1}{2}v\log\pi\bigr) = -\frac{1}{2}y\log\pi$. By [T2, p. 151 (2)] with $w = \sigma+it \ne 0$ and $\sigma \ge 0$ we have

$$\begin{aligned}\log\Gamma(w) &= (\sigma-1/2)\log|w| - t\arg w - \sigma + (1/2)\log(2\pi)\\ &\quad + i(t\log|w| + (\sigma-1/2)\arg w - t) + \int_0^\infty \frac{[u]-u+1/2}{u+w}\,du.\end{aligned} \tag{3.24}$$

The absolute value of the last integral is $\le \pi/(16|w|)$. Thus

$$\begin{aligned}&\Delta_{\boldsymbol{L}} \arg\Gamma(s/2+1)\\ &\quad = \frac{y}{2}\log\sqrt{(5/4)^2+(y/2)^2} + (5/4-1/2)\arctan(2y/5) - y/2 + R_1,\end{aligned}$$

where $|R_1| \le \pi/(4\sqrt{25+4y^2})$. So (3.23) can be rewritten as

$$\begin{aligned}\frac{\pi}{2}N(\alpha,\chi_0,y) &\le -\frac{y}{2}\log\pi + \frac{y}{2}\log\sqrt{(5/4)^2+(y/2)^2} + \frac{3}{4}\arctan\frac{2y}{5} - \frac{y}{2}\\ &\quad + \Delta_{\boldsymbol{L}}\arg\zeta(u+iv) + 0.5114\pi + \pi/(4\sqrt{25+4y^2}).\end{aligned} \tag{3.25}$$

The continuous variation of $\arg\zeta(u+iv)$ along the straight line from 2 to $2+iy$ is $\arg\zeta(2+iy)$, which satisfies $0 > \arg\zeta(2+iy) \ge -0.34\pi$. Again, considering the integral of $\frac{\zeta'}{\zeta}(w)$ along the line from $1/2+iy$ to $2+iy$, we get

$$\int_{1/2+iy}^{2+iy} \frac{\zeta'}{\zeta}(w)\,dw = \log\zeta(2+iy) - \log\zeta(1/2+iy).$$

So the continuous variation of $\arg\zeta(u+iv)$ along the straight line from $2+iy$ to $1/2+iy$ is

$$\arg\log\zeta(1/2+iy) - \arg\log\zeta(2+iy) = -\operatorname{Im}\biggl(\int_{1/2+iy}^{2+iy} \frac{\zeta'}{\zeta}(w)\,dw\biggr). \tag{3.26}$$

Now we need an estimate for $\frac{\zeta'}{\zeta}(w)$. Consider [D, §12, (8)] with s there to be $w = u+iv$ and $2+iv$ respectively, and then use [D, §12, (9)]. For $w \neq 1$ we get

$$\begin{aligned} \frac{\zeta'}{\zeta}(w) &= \frac{\zeta'}{\zeta}(2+iv) + \frac{1}{1+iv} - \frac{1}{w-1} + \sum_{\varrho}\biggl(\frac{1}{w-\varrho} - \frac{1}{2+iv-\varrho}\biggr) \\ &\quad + \sum_{n=1}^{\infty}\biggl(\frac{1}{w+2n} - \frac{1}{2+iv+2n}\biggr). \end{aligned} \tag{3.27}$$

The total contribution from the first three terms on the right hand side of (3.27) has absolute value at most $0.6105 + 1/|1+iv| + 1/|u-1+iv|$, by the use of [RS, (1.17)]. The last summation on the right hand side of (3.27) has absolute value at most $|2-u|/4$ for $u \geq 0$. For the sum over ϱ in (3.27), the contribution from the terms with $|\gamma - v| \geq 1$ has absolute value

$$\leq 5.25|2-u| \sum_{|\gamma-v|\geq 1} (1.25 + 4(\gamma-v)^2)^{-1}.$$

Also we have

$$\Bigl|\sum_{|\gamma-v|<1} (2+iv-\varrho)^{-1}\Bigr| \leq 5.25 \sum_{|\gamma-v|<1} (1.25 + 4(\gamma-v)^2)^{-1}.$$

Thus the sum over ϱ in (3.27) is $\sum_{|\gamma-v|<1}(w-\varrho)^{-1} + R_2$, where by [CW2, Lemma 2], with $x = v$,

$$|R_2| \leq 5.25\max\{1, |2-u|\}(0.5\log(2+|v|) + 0.25/(0.0625+v^2) + 2.6459).$$

Altogether, from (3.27) we get

$$\frac{\zeta'}{\zeta}(w) = \sum_{|\gamma-v|<1} (w-\varrho)^{-1} + R_3, \tag{3.28}$$

where

$$|R_3| \le 0.6105 + |1+iv|^{-1} + |u-1+iv|^{-1} + |2-u|/4 \\ + 5.25\max\{1, |2-u|\}(0.5\log(2+|v|) + 0.25/(0.0625+v^2) + 2.6459).$$

Note that we are going to give an estimate for (3.26) and so we consider the integral $\mathrm{Im}(\int_{1/2+iy}^{2+iy}(w-\varrho)^{-1}\,dw)$, which is clearly equal to

$$\Delta\,\mathrm{Im}(\log(w-\varrho)) = \Delta\arg(w-\varrho), \tag{3.29}$$

where Δ indicates the continuous variation of $\arg(w-\varrho)$ along the straight line from $1/2+iy$ to $2+iy$. Hence (3.29) clearly has absolute value at most π. Now substituting (3.28) with $v=y$ into (3.26) we see that (3.26) can be dominated by

$$5.25(1.5^2+\pi)(0.5\log(2+y) + 0.25/(0.0625+y^2) + 2.6459) \\ + 1.5(0.6105 + |1+iy|^{-1} + y^{-1} + 1.5/4).$$

This in combination with (3.25) yields the following

THEOREM 5. *For any $y>0$ and $\alpha\in(0,1)$ we have*

$$\begin{aligned} N(\alpha,\chi_0,y) \le{} & \frac{y}{\pi}\log\sqrt{(5/4)^2+(y/2)^2} + \frac{3}{2\pi}\arctan\frac{2y}{5} - \frac{1+\log\pi}{\pi}y \\ & + 0.5114\cdot 2 + \frac{1}{2\sqrt{25+4y^2}} \\ & + \frac{10.5(2.25+\pi)}{\pi}(0.5\log(2+y) + 0.25/(0.0625+y^2) + 2.6459) \\ & + \frac{3}{\pi}(0.6105 + |1+iy|^{-1} + y^{-1} + 1.5/4), \end{aligned}$$

where $N(\alpha,\chi_0,y)$ is defined after (3.1).

Now we give an explicit upper bound for $N(\alpha,\chi,y)$ when $\chi\ne\chi_0$. We may assume that $\chi \pmod q$ is primitive and nonprincipal since $L(s,\chi)$ and $L(s,\chi_1)$ have the same set of nontrivial zeros if χ is imprimitive and is induced by the primitive character $\chi_1 \pmod{q_1}$. Let $\boldsymbol{L}_1$ be the line from $1/2-iy$ to $5/2-iy$, then to $5/2+iy$ and to $1/2+iy$. Then by the arguments in [D, p. 101, the last two equalities], we get

$$N(\alpha,\chi,y) \le \frac{1}{\pi}\Delta_{\boldsymbol{L}_1}\arg\xi(s,\chi), \tag{3.30}$$

where $\xi(s,\chi) = (q/\pi)^{(s+\delta)/2}\Gamma((s+\delta)/2)L(s,\chi)$ with $\delta=0$ if $\chi(-1)=1$, $\delta=1$ if $\chi(-1)=-1$, and $\Delta_{\boldsymbol{L}_1}$ denotes the continuous variation along $\boldsymbol{L}_1$ starting from the point $1/2-iy$. We have

$$\Delta_{\boldsymbol{L}_1}\arg(q/\pi)^{(s+\delta)/2} = \Delta_{\boldsymbol{L}_1}((v/2)\log(q/\pi)) = y\log(q/\pi).$$

By (3.24) we get

$$(3.31)\quad \Delta_{\boldsymbol{L}_1} \arg \Gamma((s+\delta)/2)$$
$$= \operatorname{Im}(\log \Gamma((1/2+iy+\delta)/2)) - \operatorname{Im}(\log \Gamma((1/2+iy+\delta)/2))$$
$$\le y \log \left|\frac{1/2+\delta+iy}{2}\right| - y + (\delta - 1/2)\arctan \frac{y}{1/2+\delta} + \frac{\pi}{4|1/2+\delta+iy|}.$$

Now we estimate the continuous variation of $\arg L(s,\chi)$ along $\boldsymbol{L}_1$ starting from $1/2 - iy$. The continuous variation of $\arg L(s,\chi)$ along the line from $5/2 - iy$ to $5/2 + iy$ has absolute value at most π. Again similarly to (3.26) the continuous variation of $\arg L(s,\chi)$ along the straight line from $5/2+iy$ to $1/2+iy$ is

$$(3.32)\qquad -\operatorname{Im}\left(\int_{1/2+iy}^{5/2+iy} \frac{L'}{L}(w,\chi)\,dw\right).$$

Now similarly to (3.27), by [D, §12, (17)], for $w = u + iv$ we get

$$(3.33)\quad \frac{L'}{L}(w,\chi) = \frac{L'}{L}(2+iv,\chi) + \sum_{n=0}^{\infty}\left(\frac{1}{w+2n+\delta} - \frac{1}{2+iv+2n+\delta}\right)$$
$$+ \sum_{\varrho}\left(\frac{1}{w-\varrho} - \frac{1}{2+iv-\varrho}\right).$$

For $u \ge 0$, the sum over n has absolute value

$$\le \frac{|2-u|}{|u+iv|\cdot|2+iv|} + \frac{|2-u|}{4}.$$

For the sum over ϱ, the contribution from the terms with $|\gamma - v| \ge 1$ has absolute value

$$\le 3.5|2-u| \sum_{|\gamma-v|\ge 1} \frac{1}{1.5+2(\gamma-v)^2}.$$

Also

$$\left|\sum_{|\gamma-v|<1} \frac{1}{2+iv-\varrho}\right| \le \sum_{|\gamma-v|<1} 1 \le 3.5 \sum_{|\gamma-v|<1} \frac{1}{1.5+2(\gamma-v)^2},$$

and

$$\left|\frac{L'}{L}(2+iv,\chi)\right| \le \left|\frac{\zeta'}{\zeta}(2)\right| \le 0.6105.$$

Thus for $u \ge 0$, (3.33) can be rewritten as

$$(3.34)\qquad \frac{L'}{L}(w,\chi) = \sum_{|\gamma-v|<1} \frac{1}{w-\varrho} + R_4,$$

where, by the use of [CW2, Lemma 8],

$$|R_4| \le 3.5\max\{1,\ |2-u|\}(0.5\log q(2+|v|)+0.59773) \\ +0.6105+\frac{|2-u|}{|u+iv|\cdot|2+iv|}+\frac{|2-u|}{4}.$$

By (3.29), (3.34), [CW2, Lemma 8], and in view of $|\Delta\arg(w-\varrho)|\le\pi$ along the straight line from $5/2+iy$ to $1/2+iy$, we see that (3.32) has absolute value at most

$$3.5(\pi+3)(0.5\log q(2+y)+0.59773) \\ +2(0.6105+1.5(|0.5+iy|\cdot|2+iy|)^{-1}+1.5/4).$$

The variation of $\arg L(s,\chi)$ along the straight line from $1/2-iy$ to $5/2-iy$ can be estimated in exactly the same way. Thus from (3.30) to (3.32) one can conclude the following

THEOREM 6. *For any $y\ge 0$ and $\alpha\in(0,1)$ we have*

$$N(\alpha,\chi,y)\le\frac{y}{\pi}\log\frac{q|3/2+iy|}{2\pi}-\frac{y}{\pi}+\frac{1}{2\pi}\arctan\frac{2y}{3}+\frac{1}{4|1/2+iy|} \\ +1+7(1+3/\pi)(0.5\log q(2+y)+0.59773) \\ +\frac{4}{\pi}(0.6105+1.5(|0.5+iy|\cdot|2+iy|)^{-1}+1.5/4),$$

where $N(\alpha,\chi,y)$ is defined after (3.1).

The remainder of this section is devoted to proving the following Theorem 7, which is a revised form of [C, Theorem].

THEOREM 7. *For any integer $q\ge 1$ and any real number α with $1/2\le\alpha<1$, let $N(\alpha,q,y)$ be defined as in* (3.3). *Then for any y satisfying $y\ge\max\{10^5q^{-1},\ 10^4\log q\}$ we have*

$$N(\alpha,q,y)\le 16541(\log y)^6+(17102+254231/\log(qy))(q^3y^4)^{1-\alpha}(\log(qy))^{6\alpha}.$$

For the proof we need Lemmas 3.2–3.5 below. By (3.3),

$$N(\alpha,q,y)=\begin{cases} N(\alpha,\chi_0,y) & \text{for } q=1 \text{ or } 2,\\ N(\alpha,\chi_0,y)+\sum\limits_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\ne\chi_0}}N(\alpha,\chi,y) & \text{for } q\ge 3.\end{cases}$$

The last sum can be treated by [C, Theorem]. So we focus on the estimate of $N(\alpha,1,y)$ since for any $q\ge1$, $N(\alpha,\chi_0,y)=N(\alpha,1,y)$. Note that $y\ge\max\{10^5q^{-1},10^4\log q\}$ implies $y\ge 10^4\log 6$. Next, it is clear that we may always assume $1/2+2\delta\le\alpha<1$ since otherwise Theorem 7 follows from Theorems 5 and 6 by direct computation. Here and later on, $\delta=1/(1.5\log y)$ as in (3.35) below. We first give some notations. Let $\mu(n)$ denote the Möbius function. Put

$$
(3.35)\quad \begin{cases} Q_y(s) := \sum_{n<y} \mu(n) n^{-s}, \quad f_y(s) := \zeta(s) Q_y(s) - 1, \quad F_y(s) := |f_y(s)|^2, \\ H_y(s) := 1 - f_y^2(s), \quad M_y(\sigma, u) := \int_{-u}^{u} F_y(\sigma + it)\, dt, \quad \delta := \dfrac{1}{1.5 \log y}, \\ g_y(s) := \dfrac{s-1}{s \cos(s/(2y))} f_y(s), \quad K_y(\sigma, y) := \max_{|t-y| \le 1.5} F_y(\sigma + it). \end{cases}
$$

Lemma 3.2. *Let* $F_y(s)$ *be defined as in* (3.35), *and define*

$$
c(t) = \begin{cases} 32.0745 & \text{if } |t| < 0.5, \\ 18.0559 & \text{if } 0.5 \le |t| < 1, \\ 14.272 & \text{if } |t| \ge 1. \end{cases}
$$

Then

$$
F_y(1/2 + it) \le c(t)(1/2 + |t|)(2.583y + 0.608 \log y + 5.608).
$$

Proof. By [T1, p. 49, (3.5.3)] with $N = |1/2 + it|/2$ we get

$$
(3.36)\qquad |\zeta(1/2 + it)| \le 2^{0.5}(2 + (1/2 + |t|)^{-1})(1/2 + |t|)^{1/2}.
$$

By [P, p. 309, (1.27)] and the bound in [C, (13)] with $k = 1$, we have

$$
|Q_y(1/2 + it)|^2 \le 2.583y + 0.608 \log y + 5.608.
$$

Then in view of $1/2 + |t| \ge 1/2$ and $2.583y + 0.608 \log y + 5.608 \ge 46292$ (by $y \ge 10^4 \log 6$), Lemma 3.6 follows from

$$
F_y(1/2 + it) \le 1 + 2|\zeta(1/2 + it) Q_y(1/2 + it)| + |\zeta(1/2 + it) Q_y(1/2 + it)|^2. \ \blacksquare
$$

Lemma 3.3. *For any* σ *with* $1/2 \le \sigma \le 4$, *let* $K_y(\sigma, y)$ *be defined as in* (3.35). *Then*

$$
K_y(\sigma, y) \le \begin{cases} 1.74(\log y)^4 & \text{if } 1 + \delta \le \sigma \le 4, \\ 42.021 y^{4(1-\sigma)} (\log y)^{8\sigma - 4} & \text{if } 1/2 \le \sigma \le 1 + \delta. \end{cases}
$$

Proof. The first inequality follows from [P, p. 305, (1.20)] and [C, Lemma 3] with $k = 1$. To prove the second inequality, we use [P, p. 401, Theorem 9.3] with the function there equal to $g_y(s)$. By Lemma 3.6, [C, (14) and (15)] and

$$
\max_{0 \le t \le 1} \frac{1/2 + t}{\exp(t/y) + \exp(-t/y) + 2} \le \frac{1.5}{4},
$$

we get

$$
(3.37)\qquad \max_t |g_y(1/2 + it)|^2 \le 33.03 y^2.
$$

Also by [C, (14)],

$$
(3.38)\qquad \max_t |g_y(1 + \delta + it)|^2 \le 1.75 (\log y)^4.
$$

Then by (3.37), (3.38), [P, p. 401, Theorem 9.3], and $\delta = 1/(1.5\log y)$ in (3.35) we get for $1/2 \le \sigma \le 1+\delta$,

$$\max_t |g_y(\sigma+it)|^2 \le 33.03 y^{4(1-\sigma)} (\log y)^{8\sigma-4}.$$

This together with [C, (19)] proves the assertion. ∎

LEMMA 3.4. *Let $M_y(\sigma,u)$ be defined as in* (3.35) *for $y \ge 10^4 \log 6$. Then for any $u \ge 0$,*

$$M_y(1/2, yu) \le 12.508 y(2+yu)(2.19021u + 2.906)\log y.$$

Proof. By (3.35) we have

$$M_y(1/2,yu) \le 2yu + \big(\max_{|t|\le yu} |\zeta(1/2+it)|^2\big) \int_{-yu}^{yu} |Q_y(1/2+it)|^2\,dt + 2\Big(\int_{-yu}^{yu} |\zeta(1/2+it)|^2\,dt\Big)^{1/2}\Big(\int_{-yu}^{yu} |Q_y(1/2+it)|^2\,dt\Big)^{1/2}.$$

By (3.36), the above maximum is $\le 12.5(2+yu)$, and $\int_{-yu}^{yu} |\zeta(1/2+it)|^2\,dt \le 8(2+yu)^2$. Then the assertion follows from [C, (24)] with $k=1$. ∎

LEMMA 3.5. *Let $M_y(\sigma,u)$ be defined as in* (3.35) *for $y \ge 10^4 \log 6$. Then for $1/2 \le \sigma \le 1-\delta$,*

$$M_y(\sigma,y) \le 3318.39 y^{4(1-\sigma)} (\log y)^{6\sigma-1}.$$

Proof. We first give a bound for $M_y(\sigma,1)$. For $1/2 \le \sigma \le 1-\delta$ and $|t| \le 1$, by [T1, p. 49, (3.5.3)] with $N=1$, we get

$$|\zeta(\sigma+it)| \le \frac{1}{2}\Big(\frac{(\sigma+1)^2+t^2}{(\sigma-1)^2+t^2}\Big)^{1/2} + \frac{(\sigma^2+t^2)^{1/2}}{2\sigma} \le 1.5\log y. \tag{3.39}$$

For $Q_y(s)$ as in (3.35) by [C, (11) and (12)], for any real t we get

$$|Q_y(1/2+it)| \le 1.3071 y^{1/2} \quad \text{and} \quad |Q_y(1+it)| \le 1.19\log y.$$

Thus by [P, p. 401, Theorem 9.3], for $1/2 \le \sigma \le 1$ and for any real t we have

$$|Q_y(\sigma+it)| \le (1.3071 y^{1/2})^{2(1-\sigma)} (1.19\log y)^{2\sigma-1}. \tag{3.40}$$

From (3.39), (3.40) and $F_y(s)$ and $M_y(\sigma,u)$ as in (3.35), for $1/2 \le \sigma \le 1-\delta$ we get

$$M_y(\sigma,1) \le 2\cdot 4.255 y^{2(1-\sigma)} (\log y)^{4\sigma}. \tag{3.41}$$

Now we turn to estimating $M_y(\sigma,y)$. To simplify notation, for $1/2 \le \sigma \le 1+\delta$ we put

$$M_y(\sigma) := \int_{-\infty}^{\infty} |g_y(\sigma+it)|^2\,dt, \tag{3.42}$$

where $g_y(s)$ is defined as in (3.35). Note that

$$\left|\frac{\sigma+it-1}{\sigma+it}\right|^2 \le 1 \quad \text{for } 1/2 \le \sigma \le 1+\delta.$$

Thus similarly to [C, (29)], for $1/2 \le \sigma \le 1+\delta$ we have

$$M_y(\sigma) \le (4+10^{-7}) \int_0^\infty e^u(2+e^u)^{-2} M_y(\sigma, yu)\,du. \tag{3.43}$$

Note that (3.43) together with Lemma 3.4 and the relevant estimates in [C, between (29) and (30)] gives

$$\begin{aligned}M_y(1/2) \le (4+10^{-7}) \int_0^\infty e^u(2+e^u)^{-2}(12.508y)(\log y)(2\cdot 2.19021u \\ + 2\cdot 2.906 + 2.19021yu^2 + 2.906yu)\,du \le 254.13y^2\log y.\end{aligned} \tag{3.44}$$

Also, by the proof of [C, Lemma 14], $M_y(1+\delta, yu) \le (67.929+0.0003u)\log^5 y$; thus similarly to [C, (31)], we obtain

$$M_y(1+\delta) \le 90.5727\log^5 y. \tag{3.45}$$

By (3.44), (3.45) and [P, p. 404, Theorem 9.5], for $1/2 \le \sigma \le 1+\delta$ we get

$$\begin{aligned} M_y(\sigma) &\le (254.13y^2\log y)^{(1+\delta-\sigma)/(1/2+\delta)}(90.5727\log^5 y)^{(\sigma-1/2)/(1/2+\delta)} \\ &\le 90.5727e^{4/1.5}y^{4(1-\sigma)}(\log y)^{8\sigma-3}. \end{aligned} \tag{3.46}$$

Hence by [C, (33)], for $M_y(\sigma, u)$ and $M_y(\sigma)$ defined in (3.35) and (3.42) respectively we can conclude that for $1/2 \le \sigma \le 1-\delta$,

$$M_y(\sigma, y) \le M_y(\sigma, 1) + 1.272M_y(\sigma) \max_{1\le|t|\le y}\left|\frac{\sigma+it}{\sigma-1+it}\right|^2. \tag{3.47}$$

Note that the above maximum is ≤ 2. So the assertion follows from (3.41), (3.46) and (3.47). ∎

Proof of Theorem 7. By the arguments in [P, p. 300, lines −15 to −9], we may assume y does not coincide with the ordinate of any zero of $\zeta(s)$. By the same arguments as in [P, p. 304, lines 4 to 16], and by [P, Appendix, Theorem 8.1], for $1/2+2\delta \le \alpha < 1$ we get

$$\begin{aligned} N(\alpha, \chi_0, y) &\le \frac{1}{2\pi\delta}\Big\{\int_{-y}^{y} (\log|H_y(\alpha-\delta+it)| - \log|H_y(2+it)|)\,dt \\ &\quad + \int_{\alpha-\delta}^{2} (\arg H_y(\sigma+iy) - \arg H_y(\sigma-iy))\,d\sigma\Big\} + \delta^{-1}\int_{\alpha-\delta}^{2} d\sigma. \end{aligned} \tag{3.48}$$

Now we need to estimate the first two integrals on the right hand side. We first consider the second one. Clearly, we only need to consider the estimate for $\arg H_y(\sigma+iy)$, and the estimate for $\arg H_y(\sigma-iy)$ can be derived in exactly the same way. Let

$$R_y(s,y) := H_y(s+iy) + H_y(s-iy),$$

and let $n_y(\xi,y)$ denote the number of zeros of $R_y(s,y)$ in the region $|s-2| \le 2-\xi$ for any $\xi \in [1/2, 2)$. Then by [P, p. 301, (1.9)] and [P, p. 302, above line -9], for $1/2 \le \xi < 2$ we get

$$|\arg H_y(\xi+iy)| \le (n_y(\xi,y)+1)\pi; \tag{3.49}$$

and by [P, p. 303, (1.13) and (1.15)], for $1/2+2\delta \le \xi < 1$ we have

$$\left(\frac{2-\xi+1.5\delta}{2-\xi+\delta}\right)^{n_y(\xi-\delta,y)} \le \frac{2}{|R_y(2,y)|}\exp\Big(\max_{\xi-1.5\delta\le\sigma\le 4-\xi+1.5\delta} K_y(\sigma,y)\Big), \tag{3.50}$$

where $K_y(\sigma,y)$ is defined in (3.35). Note that by the inequality in [P, p. 302, lines 7 to 8] and $y \ge 10^4\log 6$ we have

$$|f_y(2+it)| \le \sum_{n\ge y} d(n)n^{-2} \le 4.00012y^{-0.5}.$$

Thus by [P, p. 303, (1.14)], we get

$$R_y(2,y) \ge 4 - 2\exp(4.00012^2y^{-1}) \ge 1.9982.$$

This together with (3.50) and [C, (42)] gives for $1/2+2\delta \le \xi < 1$,

$$n_y(\xi-\delta,y) \le 3\delta^{-1}\bigg(\log\frac{2}{1.9982} + \max_{\xi-1.5\delta\le\sigma\le 3.5} K_y(\sigma,y)\bigg). \tag{3.51}$$

Note that $n_y(\xi,y)$ is nonincreasing with respect to ξ by definition. So by (3.49) and (3.51) the second integral on the right hand side of (3.48) for $1/2+2\delta \le \alpha < 1$ is

$$\begin{aligned}(3.52)\quad &\le 2\pi\int_{1-\delta}^{2}(1+n_y(1-\delta,y))\,d\sigma + 2\pi\int_{\alpha-\delta}^{1-\delta}(1+n_y(\sigma,y))\,d\sigma\\ &\le 2\pi(1+\delta)\bigg(1+3\delta^{-1}\bigg(\log\frac{2}{1.9982} + \max_{1-1.5\delta\le\sigma\le 3.5} K_y(\sigma,y)\bigg)\bigg)\\ &\quad + 2\pi\int_{\alpha-\delta}^{1-\delta}\bigg(1+3\delta^{-1}\bigg(\log\frac{2}{1.9982} + \max_{\sigma+\delta-1.5\delta\le\xi\le 3.5} K_y(\xi,y)\bigg)\bigg)\,d\sigma.\end{aligned}$$

By Lemma 3.3 for any σ with $\alpha-\delta \le \sigma \le 1-\delta$ and $1/2+2\delta \le \alpha < 1$, we have

$$\max_{\sigma+\delta-1.5\delta\le\xi\le 3.5} K_y(\xi,y) \le 42.021y^{4(1-\sigma+0.5\delta)}(\log y)^{8(\sigma-0.5\delta)-4}.$$

Thus, on noting $\delta=(1.5\log y)^{-1}$ and $y\ge 10^4\log 6$, (3.52) is

$$\begin{aligned}(3.53)\quad &\le 2\pi(1+\delta)\biggl(1+3\delta^{-1}\biggl(\log\frac{2}{1.9982}+42.021y^{4(1.5\delta)}(\log y)^{8(1-1.5\delta)-4}\biggr)\biggr)\\ &\quad+2\pi(1-\alpha)\biggl(1+3\delta^{-1}\log\frac{2}{1.9982}\biggr)\\ &\quad+6(42.021)\pi\delta^{-1}\int_{\alpha-\delta}^{1-\delta} y^{4(1-\sigma+0.5\delta)}(\log y)^{8(\sigma-0.5\delta)-4}\,d\sigma\\ &\le 2(11027.02)\pi\log^5 y+94.54725\pi e^4(y^{4(1-\alpha)}(\log y)^{8\alpha-4}-\log^4 y).\end{aligned}$$

Now we turn to estimating the first integral on the right hand side of (3.48). Note that by the definitions in (3.35) we get $\log|H_y(s)|\le F_y(s)$. Hence by Lemma 3.5 and [C, (47)], the first integral on the right hand side of (3.48) for $1/2+2\delta\le\alpha<1$ is

$$\begin{aligned}(3.54)\quad &\le \int_{-y}^{y}(F_y(\alpha-\delta+it)+F_y(2+it))\,dt\\ &\le 2\pi/(1.5\cdot 10^4)+3318.39e^{4/1.5}y^{4(1-\alpha)}(\log y)^{6\alpha-1}.\end{aligned}$$

By (3.48), (3.53) and (3.54), for $1/2+2\delta\le\alpha<1$ we get

$$\begin{aligned}N(\alpha,\chi_0,y)&\le\frac{1.5\log y}{2\pi}\biggl\{\frac{2\pi}{1.5\cdot 10^4}+3318.39e^{4/1.5}y^{4(1-\alpha)}(\log y)^{6\alpha-1}\\ &\quad+2(11027.02)\pi\log^5 y\\ &\quad+94.54725\pi e^4(y^{4(1-\alpha)}(\log y)^{8\alpha-4}-\log^4 y)\biggr\}+(2-\alpha+\delta)\delta^{-1}\\ &\le 16541\log^6 y+11402y^{4(1-\alpha)}(\log y)^{6\alpha}\\ &\quad+3872y^{4(1-\alpha)}(\log y)^{8\alpha-3}.\end{aligned}$$

This together with [C, Theorem] completes the proof. ■

4. An explicit formula for $\psi(t,\chi)$. Throughout this section we suppose that N is an integer satisfying $N\ge\exp(2000)$, and t is any real number in the interval $[0.001N,N]$. Put from now on

$$(4.1)\qquad \mathcal{L}:=\log N,\quad P:=\mathcal{L}^3,\quad P_1:=\mathcal{L}^6,\quad T:=\mathcal{L}^{15}.$$

Let θ denote a complex number with $|\theta|\le 1$, not necessarily the same at different occurrences. Note that $T<t$ since $N\ge\exp(2000)$. For any $\chi\pmod q$ with $1\le q\le P_1$, define

$$(4.2)\qquad \psi(t,\chi):=\sum_{n\le t}\Lambda(n)\chi(n).$$

The purpose of this section is to prove the following theorem, which gives a formula for $\psi(t,\chi)$ with an explicit error term.

THEOREM 8. *Let* $\psi(t,\chi)$ *be defined as in* (4.2), *and* T, $\mathcal{L}$ *as in* (4.1). *Then*

$$\psi(t,\chi)=\delta(\chi)t-\sum_{\beta\ge 1/2,\,|\gamma|\le T}\varrho^{-1}t^{\varrho}+1.3804\theta tT^{-1}\mathcal{L}^2,$$

where $\varrho=\beta+i\gamma$ *is any nontrivial zero of* $L(s,\chi)$, $\delta(\chi)=1$ *if* $\chi=\chi_0 \pmod q$, *and* $\delta(\chi)=0$ *otherwise.*

The starting point of the proof is the following lemma, which is [CW2, Lemma 1].

LEMMA 4.1. *Suppose that* $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ *is an absolutely convergent series for* $\mathrm{Re}(s)=\sigma>1$ *with* a_n *satisfying* $|a_n|\le A(n)$, *where* $A(n)$ *is increasing in* n. *Let* $b>1$, $c\ge 1$, *and* $x=m+0.5$, *where* m *is a positive integer. Then*

$$\sum_{n\le x}a_n=\frac{1}{2\pi i}\int_{b-ic}^{b+ic}f(s)s^{-1}x^s\,ds+R_5,$$

where

$$|R_5|\le\frac{1}{\pi c\log 2}\Big(x^b\sum_{n=1}^{\infty}|a_n|n^{-b}+2^bA(x)(x\log x+1.5x-0.5)$$
$$+xA(2x)(\log x+\log 2+2)\Big).$$

Proof of Theorem 8. We consider two cases according as $\chi \pmod q$ is principal or not. The procedures are standard and very similar in the two cases. The latter is a little harder, so we only sketch the proof when $\chi \pmod q$ is nonprincipal. We suppose temporarily that $\chi \pmod q$ is primitive. By [CW2, Lemma 8] with $x=T$, and $q\ge 3$, $T\ge 2000^{15}$ we get

$$\sum_{|\gamma-T|\le 1}1\le 3.5(0.5\log q(T+2)+0.59773)\le 1.7769(\log qT)-1. \tag{4.3}$$

Hence there exists a real T'' such that $|T''-T|\le 1$ and

$$|\gamma-T''|\ge(1.7769\log qT)^{-1} \tag{4.4}$$

for any nontrivial zero $\varrho=\beta+i\gamma$ of $L(s,\chi)$. Using the notation $t_0=[t]+0.5$, we have

$$\psi(t,\chi)=\psi(t_0,\chi);$$

and Lemma 4.2 with $f(s)=-\frac{L'}{L}(s,\chi)$, $A(y)=\log y$, $b=1+\mathcal{L}^{-1}$, $c=T''$

and $x = t_0$ or 2.5 gives

$$\psi(t,\chi) = \frac{1}{2\pi i}\int_{b-iT''}^{b+iT''}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{t_0^s}{s}\,ds + R_6,$$

$$\psi(2.5,\chi) = \frac{1}{2\pi i}\int_{b-iT''}^{b+iT''}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{2.5^s}{s}\,ds + R_7,$$

where $|R_6| \le 1.3802tT^{-1}\mathcal{L}^2$ and $|R_7| \le 1.154(T-1)^{-1}\mathcal{L}$. Here for the estimates of R_6 and R_7, we use [RS, (1.17)] to estimate the sum $\sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\mathcal{L}^{-1})}$ and use direct computations with $t-0.5 \le t_0 \le t+0.5$, $T' \ge T-1$, $t \in [0.001N, N]$ and $\mathcal{L} \ge 2000$. Consider the difference between $\psi(t,\chi)$ and $\psi(2.5,\chi)$. Then

$$\psi(t,\chi) = \frac{1}{2\pi i}\int_{b-iT''}^{b+iT''}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{t_0^s - 2.5^s}{s}\,ds + R_8, \tag{4.5}$$

where $|R_8| \le |R_6| + |R_7| + |\psi(2.5,\chi)| \le 1.38021tT^{-1}\mathcal{L}^2$ since $\mathcal{L} \ge 2000$, $T = \mathcal{L}^{15}$, $t \in [0.001N, N]$ and $|\psi(2.5,\chi)| \le \log 2$. The difference between the main term on the right hand side of (4.5) and its analogue with t_0 replaced by t is

$$\le \frac{1}{2\pi}\int_{-T''}^{T''}\Big|\frac{L'}{L}(b+iu,\chi)\Big|\Big|\int_{t_0}^{t} x^{b+iu-1}\,dx\Big|\,du \le (2\pi)^{-1}e^{\mathcal{L}^{-1}\log(N+0.5)}T\mathcal{L}$$

on noting $b = 1 + \mathcal{L}^{-1}$ and $|T'' - T| \le 1$. Thus (4.5) can be rewritten as

$$\psi(t,\chi) = \frac{1}{2\pi i}\int_{b-iT''}^{b+iT''}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{t^s - 2.5^s}{s}\,ds + R_9, \tag{4.6}$$

where $|R_9| \le |R_8| + (2\pi)^{-1}e^{\mathcal{L}^{-1}\log(N+0.5)}T\mathcal{L} \le 1.38022tT^{-1}\mathcal{L}^2$. Now take Γ to be the rectangle with vertices at $b \pm iT''$ and $-0.5 \pm iT''$, and consider the integral

$$\int_{\Gamma}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{t^s - 2.5^s}{s}\,ds.$$

Then Cauchy's residue theorem gives

$$\frac{1}{2\pi i}\int_{\Gamma}\Big(-\frac{L'}{L}(s,\chi)\Big)\frac{t^s - 2.5^s}{s}\,ds = -\sum_{|\gamma|\le T''}\frac{t^{\varrho} - 2.5^{\varrho}}{\varrho} + \theta\log t. \tag{4.7}$$

By [CW2, Lemma 9] with $s=\sigma+iT''$, (4.3), (4.4) and in view of $b=1+\mathcal{L}^{-1}$, $t \in [0.001N, N]$ and $|T'' - T| \le 1$, the total contribution from the two horizontal lines of integration can be estimated further by $0.0014tT^{-1}\log^2 qT$.

By [CW2, Lemma 9′] with $x = u$, and [CW2, Lemma 8], the left vertical line of integration is $\le 1.632(\log qT)\log T$. Thus by (4.6) and (4.7) we get

$$\psi(t,\chi) = -\sum_{|\gamma|\le T''} \frac{t^{\varrho} - 2.5^{\varrho}}{\varrho} + R_{10}, \tag{4.8}$$

where, in view of $q \le P_1 = \mathcal{L}^6$ and $T = \mathcal{L}^{15}$,

$$\begin{aligned}|R_{10}| &\le 1.38022tT^{-1}\mathcal{L}^2 + \log t + 0.0014tT^{-1}\log^2 qT + 1.632(\log qT)\log T\\ &\le 1.38024tT^{-1}\mathcal{L}^2.\end{aligned}$$

The difference between $\sum_{|\gamma|\le T''}\varrho^{-1}t^{\varrho}$ and $\sum_{|\gamma|\le T}\varrho^{-1}t^{\varrho}$ is $\le 0.00015tT^{-1}\mathcal{L}^2$ by (4.3) and $\mathcal{L} \ge 2000$. Thus (4.8) can be rewritten as

$$\psi(t,\chi) = -\sum_{|\gamma|\le T} \frac{t^{\varrho}}{\varrho} + \sum_{|\gamma|\le T''} \frac{2.5^{\varrho}}{\varrho} + 1.38039\theta tT^{-1}\mathcal{L}^2. \tag{4.9}$$

Note that our χ is now primitive so that $L(s,\,\chi)$ has no zero on the imaginary axis, and then on noting $q \le \mathcal{L}^6$ we may use Theorem 6 for $0 < y \le 1$ to get

$$N(0,\chi,y) \le 46\log\mathcal{L}. \tag{4.10}$$

Similarly, for any $y > 1$, by Theorem 6 we get

$$N(0,\chi,y) \le \frac{y}{\pi}\log q|3/2 + iy| + 3.5(1 + 3/\pi)\log q + c(y)y, \tag{4.11}$$

where

$y \ge$	1	2	5	10	20	41
$c(y) :=$	18.13	9.3548	4.8605	1.8674	0.6887	0

We now return to (4.9). We have

$$\left|\sum_{|\gamma|\le T''} \frac{2.5^{\varrho}}{\varrho}\right| \le 2.5\sum_{|\gamma|\le T+1} \frac{1}{|\varrho|} \le 2.5\biggl(\sum_{|\gamma|\le 1} \frac{1}{\mathrm{Re}(\varrho)} + \sum_{1<|\gamma|\le T+1} \frac{1}{|\gamma|}\biggr). \tag{4.12}$$

Note that all the nontrivial zeros of $L(s,\chi)$ are symmetric with respect to the line $\sigma = 1/2$. Thus by Lemma 2.1 with $x = P_1 = \mathcal{L}^6$ we see that all zeros $\varrho = \beta + i\gamma$ of $L(s,\chi)$ with $|\gamma| \le 1$ satisfy

$$1 - \frac{1}{c_1\log P_1} \ge \beta \ge \frac{1}{6c_1\log\mathcal{L}},$$

except for at most two simple real zero $\widetilde{\beta}$ and $1 - \widetilde{\beta}$; also by Theorem 3 we have

$$1 - \widetilde{\beta} \ge \frac{\pi/0.4923}{q^{1/2}\log^2 q}.$$

Thus by (4.10) and noting $q \leq P_1 = \mathcal{L}^6$ and $\mathcal{L} \geq 2000$, the first sum on the right hand side of (4.12) is

$$(4.13) \qquad \leq \sum_{|\gamma|\leq 1,\, \varrho\neq 1-\widetilde{\beta}} \frac{1}{\mathrm{Re}(\varrho)} + \frac{1}{1-\widetilde{\beta}} \leq \frac{q^{1/2}\log^2 q}{\pi/0.4923} + 6c_1(\log\mathcal{L})\sum_{|\gamma|\leq 1} 1$$
$$\leq 5.65\mathcal{L}^3\log^2\mathcal{L}.$$

By (4.11) and $T = \mathcal{L}^{15} \geq 2000^{15}$, the last sum on the right hand side of (4.12) is

$$\leq \frac{N(0,\chi,T+1)}{T+1} + \int_1^{T+1} y^{-2}N(0,\chi,y)\,dy \leq 107\log^2\mathcal{L}.$$

This together with (4.13) enables us to estimate (4.12) by

$$(4.14) \qquad 2.5(5.65\mathcal{L}^3 + 107)\log^2\mathcal{L} \leq 14.13\mathcal{L}^3\log^2\mathcal{L}.$$

Now we bound

$$(4.15) \qquad \sum_{|\gamma|\leq T,\, \beta<1/2} \varrho^{-1}t^{\varrho} = \sum_{|\gamma|\leq 1,\, \beta<1/2} \varrho^{-1}t^{\varrho} + \sum_{1<|\gamma|\leq T,\, \beta<1/2} \varrho^{-1}t^{\varrho}.$$

Similarly to (4.13), the first sum on the right hand side has absolute value

$$\leq \frac{t^{1-\widetilde{\beta}}}{1-\widetilde{\beta}} + t^{1/2}\sum_{|\gamma|\leq 1,\, \varrho\neq 1-\widetilde{\beta}} \frac{1}{\mathrm{Re}(\varrho)} \leq 2663t^{1/2}\log^2\mathcal{L}.$$

By (4.11), the last sum on the right hand side of (4.15) has absolute value $\leq 6.7t^{1/2}\mathcal{L}^{15}\log\mathcal{L}$. Thus (4.15) is

$$(4.16) \qquad \leq (2663\mathcal{L}^{-15}\log\mathcal{L} + 6.7)t^{1/2}\mathcal{L}^{15}\log\mathcal{L} \leq 6.71t^{1/2}\mathcal{L}^{15}\log\mathcal{L}.$$

By (4.14) and (4.16) we can rewrite (4.9) as

$$\Bigg|\psi(t,\chi) + \sum_{|\gamma|\leq T,\, \beta\geq 1/2} \frac{t^{\varrho}}{\varrho}\Bigg| \leq 1.380391tT^{-1}\mathcal{L}^2.$$

Finally, note that if $\chi \pmod q$ is induced by a primitive $\chi_1 \pmod{q_1}$ then the difference between $\psi(t,\chi)$ and $\psi(t,\chi_1)$ is $\leq 6\mathcal{L}\log\mathcal{L}/\log 2$. When $\chi \pmod q$ is the principal χ_0, by similar arguments we can obtain

$$\psi(t,\chi_0) = t - \sum_{|\gamma|\leq T,\, \beta\geq 1/2} \frac{t^{\varrho}}{\varrho} + 1.38038\theta tT^{-1}\mathcal{L}^2. \ \blacksquare$$

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