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Metric inhomogeneous Diophantine approximation on the field of formal Laurent series

by

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1. Introduction. Inhomogeneous Diophantine approximation in the field of formal Laurent series has recently been studied by many researchers (e.g., [5], [11]). In this paper we discuss the inhomogeneous approximation for the rotation by a fixed irrational, which is closely related to the dynamical Borel–Cantelli lemma for irrational rotations ([7], [8]) and we will establish results analogous to the real number case.

Let q be a power of a prime p and $\mathbb{F}_q$ be the finite field with q elements. Denote by $\mathbb{F}_q[X]$ and $\mathbb{F}_q(X)$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and the quotient field of $\mathbb{F}_q[X]$, respectively. For each $P/Q \in \mathbb{F}_q(X)$, define $|P/Q| = q^{\deg(P)-\deg(Q)}$. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal Laurent series

$$\mathbb{F}_q((X^{-1})) = \{f = a_nX^n + \cdots + a_1X + a_0 + a_{-1}X^{-1} + \cdots : a_i \in \mathbb{F}_q\}.$$

Then $\mathbb{F}_q((X^{-1}))$ is the completion of $\mathbb{F}_q(X)$ with respect to the valuation $|P/Q|$. For each $f \in \mathbb{F}_q((X^{-1}))$ we have $|f| = q^{\deg(f)}$. Note that this field is nonarchimedean since $|f + g| \le \max(|f|, |g|)$. Let

$$\mathbb{L} = \{f \in \mathbb{F}_q((X^{-1})) : f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots,\ a_i \in \mathbb{F}_q\}.$$

For $f \in \mathbb{L}$ and a polynomial Q there exists a unique polynomial P such that $\deg(Qf - P) < 0$. We put $\{Qf\} = Qf - P$. A power series f is said to be *irrational* if it is not a rational function.

Let μ be the probability measure on $\mathbb{L}$ defined by

$$\mu(\{g = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots \in \mathbb{L} : a_{-1} = \alpha_1, \ldots, a_{-n} = \alpha_n\}) = \frac{1}{q^n}$$

for any $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$. In what follows, "almost every" means "μ-almost every."

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We consider the inhomogeneous Diophantine approximation

$$|\{Qf\} - g| < \frac{1}{q^{n+\ell_n}}, \qquad \deg(Q) = n. \tag{1.1}$$

We will look for a condition on $f \in \mathbb{L}$ so that the following statement holds: for any sequence $\{\ell_n\}$ of nonnegative integers, (1.1) has infinitely many solutions Q for almost every $g \in \mathbb{L}$ whenever $\sum 1/q^{\ell_n}$ diverges. For the related results, both randomly chosen f and g in (1.1) are considered in [11]. The inhomogeneous approximation for a fixed g was investigated by Fuchs [5]. He also mentioned the existence of $\{\ell_n\}$ such that (1.1) has infinitely many solutions for almost every $(f, g) \in \mathbb{L}^2$ but there exists $f \in \mathbb{L}$ for which (1.1) has finitely many solutions for almost every $g \in \mathbb{L}$. We also refer to [9] for this type of question. The metric theory of homogeneous approximation has been discussed in [2], [4], [6], and [12].

In the field of real numbers, for an irrational number θ, the inhomogeneous Diophantine approximation theorem states (see e.g. [13]) that

$$\liminf_{n\to\infty} n\|n\theta - s\| \le \frac{1}{\sqrt{5}} \qquad \text{for every } s \in \mathbb{R},$$

where $\|\cdot\|$ is the distance to the nearest integer. An irrational θ is said to be *of bounded type* if there exists a $C > 0$ such that $n\|n\theta\| > C$ for all positive integers n. In [10, Theorem 1], Kurzweil showed that an irrational θ is of bounded type if and only if every decreasing positive function ψ with $\sum \psi(n) = \infty$ satisfies the condition that for almost every s,

$$\|n\theta - s\| < \psi(n) \qquad \text{for infinitely many } n \in \mathbb{N}$$

(see also [3]). For any irrational θ, by the first Borel–Cantelli lemma, if $\sum \psi(n) < \infty$, then for almost every s, the inequality $\|n\theta - s\| < \psi(n)$ holds for only finitely many n's. In [7], it is shown that for any irrational θ,

$$\liminf_{n\to\infty} n\|n\theta - s\| = 0 \qquad \text{for almost every } s \in \mathbb{R}. \tag{1.2}$$

See also [15] for a related result.

We will prove an analogue of (1.2) and a Kurzweil type theorem for formal Laurent series.

THEOREM 1. *Let f be an irrational in $\mathbb{L}$. For almost every $g \in \mathbb{L}$,*

$$\liminf_{n\to\infty} \Big(q^n \min_{\deg(Q)=n} |\{Qf\} - g| \Big) = 0.$$

By the first Borel–Cantelli lemma, if $\sum 1/q^{\ell_n} < \infty$, then for μ-almost every g there are at most finitely many solutions $Q \in \mathbb{F}_q[X]$ of

$$|\{Qf\} - g| < \frac{1}{q^{n+\ell_n}}, \quad \deg(Q) = n.$$

Our result is the following.

THEOREM 2. *The irrational f is of bounded type if and only if the inequality $|\{Qf\}-g| < 1/q^{n+\ell_n}$ with* $\deg(Q) = n$ *has infinitely many solutions for almost all g for any sequence $\{l_n\}$ with $\sum 1/q^{\ell_n} = \infty$.*

In Section 2 we give some lemmas and then prove the first main theorem (Theorem 1). In Section 3, we prove the second main theorem (Theorem 2). It is natural to ask whether the statement following (1.1) holds for an unbounded $f \in \mathbb{L}$ if $\{\ell_n\}$ is restricted to certain classes of sequences. From this point of view, in Section 4, we discuss some monotonicity conditions on $\{\ell_n\}$. The first claim is that even if $\{n + \ell_n\}$ is monotone, the statement is still false for every unbounded f. On the other hand if $\{\ell_n\}$ is monotone, then the statement can be either true or false. For the former we give an example and for the latter a sufficient condition on f is presented.

2. Geometry of $\mathbb{L}$ and uniform approximation. Fix an irrational $f \in \mathbb{L}$. Let A_k be the partial quotients of f and

$$\frac{P_k}{Q_k} = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots + 1/A_k}}}, \quad (P_k, Q_k) = 1 \text{ with } P_0 = 0 \text{ and } Q_0 = 1,$$

be the principal convergent of f. Denote $\deg(Q_k)$ by n_k. We have the recurrence relation $Q_{k+1} = A_{k+1}Q_k + Q_{k-1}$ and $n_{k+1} - n_k = \deg(A_{k+1})$ for $k \geq 0$ with $Q_0 = 1$.

For any $k \geq 0$ we have

$$|\{Q_k f\}| = \frac{1}{|Q_{k+1}|} \tag{2.1}$$

(see [1]).

LEMMA 1.

(i) *For each $Q \in \mathbb{F}_q[X]$ with* $\deg(Q) < n_{k+1}$, *there is a unique decomposition*

$$Q = B_1Q_0 + B_2Q_1 + \cdots + B_{k+1}Q_k, \quad B_i \in \mathbb{F}_q[X],\ \deg(B_i) < \deg(A_i).$$

(ii) *For each nonzero $Q \in \mathbb{F}_q[X]$ with* $\deg(Q) < n_{k+1}$, *we have $|\{Qf\}| \geq q^{-n_{k+1}}$. Moreover,*

$$|\{Qf\}| = \frac{1}{q^s}, \quad 0 < s \leq n_{k+1},$$

if and only if in the decomposition of Q from (i), *$B_i = 0$ for all $1 \leq i \leq m$ with $n_m < s \leq n_{m+1}$ and* $\deg(B_{m+1}) = n_{m+1} - s$.

Proof. (i) There exists a unique B_{k+1} such that $Q = B_{k+1}Q_k + Q'$ with $\deg(Q') < \deg(Q_k)$. Since $\deg(Q) < \deg(Q_{k+1}) = \deg(Q_k) + \deg(A_{k+1})$, we have $\deg(B_{k+1}) < \deg(A_{k+1})$. Then (i) follows by induction.

(ii) For each $\deg(Q) < n_{k+1}$, let $Q = B_1Q_0 + B_2Q_1 + \cdots + B_{k+1}Q_k$ with $B_i \in \mathbb{F}_q[X]$ and $\deg(B_i) < \deg(A_i)$. If $B_i = 0$ for $i \le m$ and $\deg(B_{m+1}) = r$, then

$$|\{Qf\}| = |\{(Q_mB_{m+1} + \cdots + Q_kB_{k+1})f\}| = \max_{m\le j\le k} |\{B_{j+1}Q_jf\}|.$$

From (2.1), we see that

$$|\{B_{j+1}Q_jf\}| = \frac{q^{\deg(B_{j+1})}}{q^{n_{j+1}}} < \frac{1}{q^{n_j}}.$$

Thus

$$|\{Qf\}| = |\{B_{m+1}Q_mf\}| = \frac{|B_{m+1}|}{|Q_{m+1}|} = \frac{1}{q^{n_{m+1}-r}}.$$

The uniqueness of the decomposition yields (ii). ∎

We are going to find a condition on f and $\{\ell_n\}$ such that there are infinitely many $Q \in \mathbb{F}_q[X]$ satisfying (1.1) for μ-almost every $g \in \mathbb{L}$.

Let $B(x, r)$ be the open ball centered at x with radius r, i.e., $B(x, r) = \{y : |x - y| < r\}$. Note that $\mu(B(x, q^{-n})) = q^{-n}$ for each integer $n \ge 0$. Put

$$E_n = \bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-n-\ell_n}).$$

Note that any two balls in E_n are either disjoint or coincide.

Let ξ_n $(= \xi_n(Q))$ be the number of different polynomials Q' with $\deg(Q')$ $= n$ which satisfy $B(\{Qf\}, q^{-n-\ell_n}) = B(\{Q'f\}, q^{-n-\ell_n})$ for a fixed Q with $\deg(Q) = n$. Here, $B(\{Qf\}, q^{-n-\ell_n}) = B(\{Q'f\}, q^{-n-\ell_n})$ if and only if $|\{(Q - Q')f\}| < 1/q^{n+\ell_n}$. Thus ξ_n is independent of Q with $\deg(Q) = n$. Since $\mu(B(\{Qf\}, q^{-n-\ell_n})) = q^{-n-\ell_n}$ and the number of Q with $\deg(Q) = n$ is $(q - 1)q^n$, we have

$$\mu(E_n) = \frac{(q-1)q^n}{\xi_n} \cdot \frac{1}{q^{n+\ell_n}} = \frac{q-1}{\xi_n q^{\ell_n}}.$$

Lemma 2. *For each $n_k \le n < n_{k+1}$, we have*

$$\xi_n = \begin{cases} (q-1)q^{n-n_k} & \text{if } n_k \le n < (n_{k+1}+n_k)/2 - \ell_n/2, \\ q^{n_{k+1}-n-\ell_n} & \text{if } (n_{k+1}+n_k)/2 - \ell_n/2 \le n < n_{k+1} - \ell_n, \\ 1 & \text{if } n \ge n_{k+1} - \ell_n. \end{cases}$$

Therefore,

$$\mu(E_n) = \begin{cases} \dfrac{1}{q^{n-n_k+\ell_n}} & \textit{if } n_k \le n < (n_{k+1}+n_k)/2 - \ell_n/2, \\ \dfrac{q-1}{q^{n_{k+1}-n}} & \textit{if } (n_{k+1}+n_k)/2 - \ell_n/2 \le n < n_{k+1} - \ell_n, \\ \dfrac{q-1}{q^{\ell_n}} & \textit{if } n \ge n_{k+1} - \ell_n. \end{cases}$$

Proof. By Lemma 1, for a fixed $Q \in \mathbb{F}_q[X]$ with $\deg(Q) = n$ and $0 \le r < n_{k+1} - n_k$,

$$|\{Qf - Q'f\}| = \frac{q^r}{q^{n_{k+1}}} \quad \text{for some } Q' \text{ with } \deg(Q') < n_{k+1}$$

if and only if $Q' = Q + PQ_k$ with $\deg(P) = r$.

Therefore, for $0 \le r < n - n_k$, we have

$$|\{Qf\} - \{Q'f\}| = \frac{q^r}{q^{n_{k+1}}} \quad \text{and} \quad \deg(Q') = n$$

if and only if $Q' = Q + PQ_k$ with $\deg(P) = r$. The number of such Q' is the number of degree r polynomials, $(q-1)q^r$.

When $r = n - n_k$, we have

$$|\{Qf\} - \{Q'f\}| = \frac{q^r}{q^{n_{k+1}}} \quad \text{and} \quad \deg(Q') = n$$

if and only if $Q' = Q + PQ_k$ with $\deg(P) = r$ and the leading coefficient of PQ_k plus the leading coefficient of Q is not zero. Thus $q-2$ elements of $\mathbb{F}_q$ are allowed at the leading coefficient of P and the number of such P or Q' is $(q-2)q^r$.

If $n - n_k < r < n_{k+1} - n_k$, then

$$|\{Qf\} - \{Q'f\}| = \frac{q^r}{q^{n_{k+1}}}$$

yields $Q' = Q + PQ_k$, $\deg(P) = r$ and $\deg(Q') = n_k + r > n$, so there is no such Q' with $\deg(Q') = n$.

If $q^{n-n_k}/q^{n_{k+1}} < 1/q^{n+\ell_n}$ (equivalently, $2n < n_{k+1} + n_k - \ell_n$), then the number of $Q' \ne Q$ with $\deg(Q') = n$ such that

$$|\{Qf - Q'f\}| < \frac{1}{q^{n+\ell_n}} \quad (\text{or } B(\{Q'f\}, q^{-n-\ell_n}) = B(\{Qf\}, q^{-n-\ell_n}))$$

is

$$\begin{aligned} \sum_{r=0}^{n-n_k-1} (q-1)q^r + (q-2)q^{n-n_k} &= (q-1)\frac{q^{n-n_k}-1}{q-1} + (q-2)q^{n-n_k} \\ &= (q-1)q^{n-n_k} - 1. \end{aligned}$$

Therefore, there are $(q-1)q^{n-n_k}$ identical balls in $\bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-n-\ell_n})$, or $\xi_n = (q-1)q^{n-n_k}$.

Suppose that $1/q^{n_{k+1}} < 1/q^{n+\ell_n} \le q^{n-n_k}/q^{n_{k+1}}$ (equivalently, $(n_{k+1} + n_k - \ell_n)/2 \le n < n_{k+1} - \ell_n$). Then the number of $Q' \ne Q$ with $\deg(Q') = n$ such that

$$|\{Qf - Q'f\}| < \frac{1}{q^{n+\ell_n}} \quad (\text{or } B(\{Q'f\}, q^{-n-\ell_n}) = B(\{Qf\}, q^{-n-\ell_n}))$$

is

$$(q-1) + (q-1)q + \cdots + (q-1)q^{d-1} = (q-1)\frac{q^d - 1}{q-1} = q^d - 1,$$

where d is the integer satisfying $1/q^{n+\ell_n} = q^d/q^{n_{k+1}}$. Thus $\xi_n = q^d = q^{n_{k+1}-n-\ell_n}$.

Finally, if $1/q^{n+\ell_n} \le 1/q^{n_{k+1}}$, then there is no intersection among the balls in E_n and $\xi_n = 1$. ∎

As a corollary, we have

$$\xi_n \le \begin{cases} (q-1)q^{(n_{k+1}-n_k-\ell_n)/2} & \text{for } \ell_n \le n_{k+1} - n_k, \\ 1 & \text{for } \ell_n > n_{k+1} - n_k. \end{cases} \tag{2.2}$$

Lemma 3. *If a measurable set E in $\mathbb{L}$ is invariant under the action $\{\cdot + \{Qf\} : Q \in \mathbb{F}_q[X]\}$, then $\mu(E) = 0$ or 1.*

Proof. Suppose that $\mu(E) > 0$. Then there exists $(\alpha_1, \ldots, \alpha_\ell)$ such that

$$\frac{\mu(E \cap \langle \alpha_1, \ldots, \alpha_\ell \rangle)}{\mu(\langle \alpha_1, \ldots, \alpha_\ell \rangle)} > 1 - \varepsilon,$$

$$\langle \alpha_1, \ldots, \alpha_\ell \rangle = \{a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots \in \mathbb{L} : a_{-1} = \alpha_1, \ldots, a_{-\ell} = \alpha_\ell\}.$$

Since $\{\{Qf\} : Q \in \mathbb{F}_q[X]\}$ is dense in $\mathbb{L}$, we have $\mu(E) > 1 - \varepsilon$ for all $\varepsilon > 0$. ∎

Remark 1 (Rényi–Lamperti type lemma; e.g. [14, p. 17]). If $\sum \mu(E_n) = \infty$, then

$$\mu\Big(\bigcap_{N\ge1}\bigcup_{n\ge N} E_n\Big) \ge \limsup_{N\to\infty} \frac{(\sum_{n=1}^N \mu(E_n))^2}{\sum_{n=1}^N \sum_{m=1}^N \mu(E_n \cap E_m)}.$$

If there is $K > 0$ such that $\mu(E_k \cap E_m) \le K\mu(E_k)\mu(E_m)$, then the right hand side of the above inequality is positive.

Proof of Theorem 1. For a fixed $\ell \ge 0$ let

$$\begin{aligned} F_n &= \{g \in \mathbb{L} : |\{Qf\} - g| < 1/q^{n+\ell}, \deg(Q) = n\} \\ &= \bigcup_{\deg(Q)=n} B(\{Qf\}, 1/q^{n+\ell}). \end{aligned}$$

We only consider the case when $n = n_k$. By Lemma 2, there are at most $q-1$ polynomials Q of degree n_k for which $B(\{Qf\}, 1/q^{n_k+\ell})$ are the same ball. Thus

$$\mu(F_{n_k}) \geq \frac{(q-1)q^{n_k}}{q-1} \cdot \frac{1}{q^{n_k+\ell}} = \frac{1}{q^\ell}.$$

By the Rényi–Lamperti type lemma (Remark 1), we see that $\#\{k \geq 1 : g \in F_{n_k}\} = \infty$ for a μ-positive subset of $\mathbb{L}$. From Lemma 3, this holds for μ-a.e. g. Because $\ell \geq 0$ is arbitrary, we have the assertion of the theorem. ∎

3. Bounded type irrationals. In this section we assume that f is an irrational of bounded type. We denote by c the maximum of $\deg(A_k) = n_{k+1} - n_k$; then for any positive integer $n \geq c$, there exists a principal convergent P_k/Q_k of f with $\deg(Q_k) \in (n-c, n]$ and $1/q^{n+c} \leq |\{Q_k f\}| < 1/q^n$.

Proof of Theorem 2. For the sufficiency part, it is enough to show that there exists a constant K such that

$$\mu(E_n \cap E_m) \leq K\mu(E_n)\mu(E_m) \quad \text{ for } n < m.$$

Now we consider $E_n \cap E_m$ with $n < m$. For Q, $\deg(Q) = n$ and Q', $\deg(Q') = m$, we consider

$$B(\{Qf\}, q^{-n-\ell_n}) \cap B(\{Q'f\}, q^{-m-\ell_m}).$$

Choose k with $n_k \leq m < n_{k+1}$. There are two cases: (i) $n + \ell_n \geq n_k$ and (ii) $n + \ell_n < n_k$.

Case (i): $n + \ell_n \geq n_k$. If $B(\{Qf\}, q^{-n-\ell_n}) \cap B(\{Q'f\}, q^{-m-\ell_m}) \neq \emptyset$, then

$$|\{Qf\} - \{Q'f\}| < \max(q^{-n-\ell_n}, q^{-m-\ell_m}) \leq q^{-n_k}.$$

Let $r = m - n_k$ $(0 \leq r < n_{k+1} - n_k \leq c)$. Since $\deg(Q - Q') = m$, we deduce from Lemma 1 that

$$Q - Q' = PQ_k \quad \text{ for some } P \text{ with } \deg(P) = r. \tag{3.1}$$

For each Q there are $(q-1)q^r$ polynomials Q' satisfying (3.1). Since the number of $B(\{Qf\}, q^{-n-\ell_n})$ with $\deg(Q) = n$ is $(q-1)q^n/\xi_n$, we have

$$\mu(E_n \cap E_m) \leq \begin{cases} \mu(E_n), & n + \ell_n \geq m + \ell_m, \\ \dfrac{(q-1)q^n}{\xi_n} \cdot \dfrac{(q-1)q^r}{q^{m+\ell_m}}, & n + \ell_n < m + \ell_m. \end{cases} \tag{3.2}$$

If $n + \ell_n \geq m + \ell_m$, then $\ell_m \geq c$ implies

$$\max(q^{-n-\ell_n}, q^{-m-\ell_m}) \leq q^{-m-c} \leq q^{-n_{k+1}} \leq |\{Qf\} - \{Q'f\}|,$$

which yields $B(\{Qf\}, q^{-n-\ell_n}) \cap B(\{Q'f\}, q^{-m-\ell_m}) = \emptyset$. Therefore, we can assume $\ell_m < c$. From Lemma 2, we see that

$$\mu(E_m) \geq 1/q^c.$$

Thus we have

$$\mu(E_n \cap E_m) \le \mu(E_n) \le q^c \mu(E_n)\mu(E_m).$$

If $n+\ell_n < m+\ell_m$, then we claim $n+\ell_n \le m+c$, because if $n+\ell_n > m+c$, then

$$\max(q^{-n-\ell_n}, q^{-m-\ell_m}) < q^{-m-c} \le q^{-n_{k+1}} \le |\{Qf\} - \{Q'f\}|,$$

which implies $E_n \cap E_m = \emptyset$. Hence, from $n+\ell_n \le m+c$, $r < c$, and (2.2) we have

$$\begin{aligned}\mu(E_n \cap E_m) &\le \frac{(q-1)q^n}{\xi_n} \cdot \frac{(q-1)q^r}{q^{m+\ell_m}} = \frac{\xi_m q^{n+\ell_n+r}}{q^m} \cdot \frac{q-1}{\xi_n q^{\ell_n}} \cdot \frac{q-1}{\xi_m q^{\ell_m}} \\ &\le \xi_m q^{c+r} \mu(E_n)\mu(E_m) < (q-1)q^{3c}\mu(E_n)\mu(E_m).\end{aligned}$$

Case (ii): $n+\ell_n < n_k$. We have to count the number of Q' with $\deg(Q') = m$ such that

$$B(\{Q'f\}, q^{-m-\ell_m}) \subset B(\{Qf\}, q^{-n-\ell_n}).$$

Fix $b_1, \dots, b_{n+\ell_n} \in \mathbb{F}_q$ and put

$$f = f_1X^{-1} + f_2X^{-2} + \cdots, \quad Q' = a_mX^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0.$$

We consider

$$\mathbf{b} = M\mathbf{a};\ \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+\ell_n}\end{bmatrix},\ \mathbf{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_m\end{bmatrix},\ M = \begin{bmatrix} f_1 & f_2 & \cdots & f_{m+1} \\ f_2 & f_3 & \cdots & f_{m+2} \\ \cdots & \cdots & & \cdots \\ f_{n+\ell_n} & f_{n+\ell_n+1} & \cdots & f_{n+\ell_n+m}\end{bmatrix}$$

and estimate the dimension of the kernel of M as a linear map. To do this, we claim that

$$\operatorname{rank}(M) = n + \ell_n.$$

Suppose

$$\begin{aligned}&\alpha_1(f_1, \dots, f_{m+1}) + \alpha_2(f_2, \dots, f_{m+2}) + \cdots \\ &\qquad + \alpha_{n+\ell_n}(f_{n+\ell_n}, \dots, f_{n+\ell_n+m}) = (0, \dots, 0).\end{aligned}$$

This means $|\{Pf\}| < 1/q^{m+1}$ with

$$P = \alpha_{n+\ell_n}X^{n+\ell_n-1} + \cdots + \alpha_2X + \alpha_1.$$

Since $\deg(P) = n+\ell_n-1 < n_k \le m$, by Lemma 1 we have $|\{Pf\}| \ge 1/q^{n_k}$ and so there is no such $P \in \mathbb{F}_q[X]$. Hence we have $\operatorname{rank}(M) = n+\ell_n$ and by the dimension formula of linear algebra,

$$\dim(\operatorname{Ker}(M)) = m + 1 - \operatorname{rank}(M) = m + 1 - (n + \ell_n).$$

Let

$$M_0 = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ f_2 & f_3 & \cdots & f_{m+1} \\ \cdots & \cdots & & \cdots \\ f_{n+\ell_n} & f_{n+\ell_n+1} & \cdots & f_{n+\ell_n+m-1} \end{bmatrix}.$$

Then as for M, we can show that $\operatorname{rank}(M_0) = n + \ell_n$ and the dimension of the kernel of M_0 is $m - (n + \ell_n)$. Since

$$[c_0, \dots, c_{m-1}]^T \in \operatorname{Ker}(M_0) \ \Leftrightarrow\ [c_0, \dots, c_{m-1}, 0]^T \in \operatorname{Ker}(M),$$

the number of vectors $\mathbf{a} = [a_0, \dots, a_m]^T \in \operatorname{Ker}(M)$ with $a_m \neq 0$ is $q^{m-(n+\ell_n)+1} - q^{m-(n+\ell_n)} = (q-1)q^{m-(n+\ell_n)}$. This shows that for each $B(\{Qf\}, q^{-(n+\ell_n)})$ there are $(q-1)q^{m-(n+\ell_n)}$ polynomials Q' with $\deg(Q') = m$ such that $B(\{Q'f\}, q^{-(m+\ell_m)}) \subset B(\{Qf\}, q^{-(n+\ell_n)})$. Thus

$$\mu(E_n \cap E_m) = \frac{1}{q^{m+\ell_m}} \cdot \frac{(q-1)q^{m-(n+\ell_n)}}{\xi_m} \cdot \frac{(q-1)q^n}{\xi_n} = \mu(E_n)\mu(E_m).$$

By the Rényi–Lamperti type (Borel–Cantelli) lemma (Remark 1), we see $\mu(\bigcap_N \bigcup_{m \geq N} E_m) > 0$ and then $\mu(\bigcap_N \bigcup_{m \geq N} E_m) = 1$ by Lemma 3, which proves the assertion of the theorem.

Now we show the necessity part. For this, we construct a "bad" sequence $\{\ell_n\}$. Suppose that $f \in \mathbb{L}$ is not of bounded type, i.e., $\sup_k (n_{k+1} - n_k) = \infty$. In this case, there exists $\{k_i : i \geq 1\}$ such that

$$n_{k_i+1} - n_{k_i} > 2i + 1.$$

Choose $\ell_n > 0$ so that

$$\sum_{\substack{n \neq n_{k_i}+i \\ 1 \leq i < \infty}} 1/q^{\ell_n} < \infty.$$

Then, by the Borel–Cantelli lemma, for almost every $g \in \mathbb{L}$, there exist at most finitely many $Q \in \mathbb{F}_q[X]$ such that

$$|\{Qf\} - g| < 1/q^{n+\ell_n}, \qquad \deg(Q) = n,\ n \neq n_{k_i} + i,\ \forall i \geq 1.$$

Write $v_i = n_{k_i} + i$ for $i \geq 1$ and put $\ell_{v_i} = 1$. By Lemma 2,

$$\mu(\{g \in \mathbb{L} : |\{Qf\} - g| < 1/q^{v_i + \ell_{v_i}},\ \deg(Q) = v_i\}) \leq 1/q^{i+1}.$$

Thus, by the Borel–Cantelli lemma again, for almost every $g \in \mathbb{L}$, there exist at most finitely many $Q \in \mathbb{F}_q[X]$ such that

$$|\{Qf\} - g| < 1/q^{v_i + \ell_{v_i}}, \qquad \deg(Q) = v_i.$$

We have proved the following: for $f \in \mathbb{L}$ of unbounded type, there exists $\{\ell_n\}_{n \geq 1}$, $\ell_n \geq 1$, such that $\sum 1/q^{\ell_n} = \infty$ and for almost every $g \in \mathbb{L}$, there exist at most finitely many $Q \in \mathbb{F}_q[X]$ such that

$$|\{Qf\} - g| < 1/q^{n+\ell_n}, \qquad \deg(Q) = n. \ \blacksquare$$

4. Unbounded type irrationals. In this section we discuss f of unbounded type under some conditions on $\{\ell_n\}$. First we consider the case of $\{n + \ell_n\}$ monotone. Proposition 1 below states that the monotonicity of $\{n + \ell_n\}$ is not enough for the existence of infinitely many solutions for (1.1) for a.e. g if f is of unbounded type. On the other hand, there exists f of unbounded type such that (1.1) has infinitely many solutions for a.e. g whenever $\{\ell_n\}$ is monotone and $\sum_n 1/q^{\ell_n} = \infty$ (see Example 1). Finally we give a condition on f such that the monotonicity of $\{\ell_n\}$ and $\sum_n 1/q^{\ell_n} = \infty$ do not imply the existence of infinitely many solutions (see Theorem 3).

Lemma 4. *Let $n_k \le n < n_{k+1}$. If $n - n_k < r$, then*

$$\bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-n_{k+1}+r}) \subset \bigcup_{\deg(Q)<n_k} B(\{Qf\}, q^{-n_{k+1}+r}).$$

Proof. For each Q with $\deg(Q) = n$, by Lemma 1(i) we have $Q = B_1Q_0 + \cdots + B_{k+1}Q_k$ with $\deg(B_i) < \deg(A_i)$. Put $Q' = B_1Q_0 + \cdots + B_kQ_{k-1}$. Then from Lemma 1(ii) we have

$$|\{Qf\} - \{Q'f\}| = |\{B_{k+1}Q_kf\}| = 1/q^{n_{k+1}-\deg(B_{k+1})}.$$

If $\deg(B_{k+1}) = n - n_k < r$, then

$$|\{Qf\} - \{Q'f\}| < 1/q^{n_{k+1}-r}$$

and

$$B(\{Qf\}, q^{-n_{k+1}+r}) = B(\{Q'f\}, q^{-n_{k+1}+r}). \ \blacksquare$$

By taking the union of all balls with $\deg(Q) < n$ we have the following consequence:

Lemma 5. *Let $n_k \le n < n_{k+1}$. Then for $n - n_k < r$,*

$$\mu\Big(\bigcup_{\deg(Q)\le n} B(\{Qf\}, q^{-n_{k+1}+r})\Big) \le \frac{q^{n_k}}{q^{n_{k+1}-r}}.$$

The following proposition states that monotonicity of $n + \ell_n$ is not sufficient for infinitely many solutions Q:

Proposition 1. *If f is not of bounded type, then we can choose $\{l_n\}$ with $\{n + \ell_n\}$ increasing such that $\sum_{n=1}^{\infty} 1/q^{\ell_n} = \infty$ but for almost every $g \in \mathbb{L}$ there are finitely many Q's satisfying*

$$|\{Qf\} - g| < 1/q^{n+\ell_n}, \qquad \deg(Q) = n.$$

Proof. Since f is not of bounded type, we have $\limsup(n_{k+1}-n_k)=\infty$. Choose an increasing subsequence $\{k_i\}$ satisfying $n_{k_i+1}-n_{k_i}>2i$. Let

$$t_i = n_{k_i} + \lfloor (n_{k_i+1}-n_{k_i})/2 \rfloor$$

and define

$$\ell_n = \begin{cases} t_i - n, & t_{i-1} \le n < t_i, \\ t_1 - n, & 0 \le n < t_1. \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{q^{\ell_n}} \ge \sum_{i=1}^{\infty} \frac{1}{q^{\ell_{t_i-1}}} = \sum_{i=1}^{\infty} \frac{1}{q^{t_i-(t_i-1)}} = \sum_{i=1}^{\infty} \frac{1}{q} = \infty.$$

On the other hand,

$$\begin{aligned}\bigcup_{t_{i-1}\le n<t_i} E_n &= \bigcup_{t_{i-1}\le n<t_i} \Big(\bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-n-\ell_n}) \Big) \\ &= \bigcup_{t_{i-1}\le n<t_i} \Big(\bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-t_i}) \Big) \subset \bigcup_{\deg(Q)<t_i} B(\{Qf\}, q^{-t_i}).\end{aligned}$$

Since $t_i - 1 - n_{k_i} < n_{k_i+1} - t_i$, by Lemma 5 we have

$$\mu\Big(\bigcup_{\deg(Q)\le t_i-1} B(\{Qf\}, q^{-t_i}) \Big) \le \frac{q^{n_{k_i}}}{q^{t_i}} = \frac{1}{q^{\lfloor (n_{k_i+1}-n_{k_i})/2 \rfloor}} \le \frac{1}{q^i}.$$

Therefore,

$$\sum_i \mu\Big(\bigcup_{t_{i-1}\le n<t_i} E_n \Big) \le \sum_i \frac{1}{q^i} < \infty$$

and by the Borel–Cantelli lemma for almost every $g \in \mathbb{L}$ there are at most finitely many n's such that $g \in E_n$. ∎

In the rest of the section we consider the case of a nondecreasing sequence $\{\ell_n\}$. For some kind of irrationals of unbounded type, $\sum_n 1/q^{\ell_n} = \infty$ with monotone $\{\ell_n\}$ guarantees infinitely many solutions of (1.1) for almost every g.

Example 1. Let f be an irrational of unbounded type with $\{n_k\}$ such that

$$n_{k+1} = \begin{cases} 2n_k & \text{if } n_k = 2\cdot 4^j,\ j \ge 0, \\ n_k + 1 & \text{otherwise.} \end{cases}$$

Let

$$\Lambda = \{m \in \mathbb{N} : 4^j \le m < 2\cdot 4^j \text{ for some } j \ge 0\}.$$

Then for each $m \in \Lambda$, there is an integer k such that $m = n_k$ and $n_{k+1} = m+1$, so $\xi_m = 1$ for $m \in \Lambda$. Now we consider $\mu(E_n \cap E_m)$ for $n < m$ with $m \in \Lambda$. Put $n_k = m$.

If $n+\ell_n < m = n_k$, then by the proof of Theorem 2, case (ii),

$$\mu(E_n \cap E_m) = \mu(E_n)\mu(E_m).$$

Consider the case of $n+\ell_n \ge m$. Then $n+\ell_n \le m+1 = n_{k+1}$, because if $n+\ell_n > n_{k+1}$, then

$$\max(q^{-n-\ell_n}, q^{-m-\ell_m}) = q^{-n-\ell_n} < q^{-n_{k+1}} \le |\{(Q-Q')f\}|,$$

which implies $E_n \cap E_m = \emptyset$. Hence, $n+\ell_n \le m+1$ and (3.2) imply

$$\mu(E_n \cap E_m) \le \frac{(q-1)q^n}{\xi_n} \cdot \frac{q-1}{q^{m+\ell_m}} = \frac{q^{n+\ell_n}}{q^m}\mu(E_n)\mu(E_m) \le q\mu(E_n)\mu(E_m).$$

Let $\{\ell_n\}$ be any increasing sequence of positive integers with $\sum 1/q^{\ell_n} = \infty$. Then

$$\sum_{n\in\mathbb{N}\setminus\Lambda} \frac{1}{q^{\ell_n}} = \sum_{j=0}^{\infty}\Bigg(\sum_{m=2\cdot 4^j}^{4^{j+1}-1} \frac{1}{q^{\ell_m}}\Bigg) \le \sum_{j=0}^{\infty} \frac{2\cdot 4^j}{q^{\ell_{2\cdot 4^j}}} \le \sum_{j=0}^{\infty} 2\Bigg(\sum_{m=4^j}^{2\cdot 4^j-1} \frac{1}{q^{\ell_m}}\Bigg) = 2\sum_{n\in\Lambda}\frac{1}{q^{\ell_n}}.$$

Therefore $\sum_{n\in\Lambda} 1/q^{\ell_n} = \infty$ and by the Rényi–Lamperti (Borel–Cantelli) lemma and Lemma 3, for μ-almost every $g \in \mathbb{L}$ there are infinitely many $Q \in \mathbb{F}_q[X]$ such that

$$|\{Qf\} - g| < \frac{1}{q^{m+\ell_m}}, \qquad \deg(Q) = m \in \Lambda.$$

Lemma 6. *If $\{\ell_n\}$ is increasing, then*

$$\mu\Big(\bigcup_{n_k \le n < n_{k+1}} E_n\Big) \le \frac{q\ell_{n_k}+1}{q^{\ell_{n_k}}}.$$

Proof. By Lemma 4 we have, for $0 \le n - n_k < n_{k+1} - n - \ell_n$,

$$E_n = \bigcup_{\deg(Q)=n} B(\{Qf\}, q^{-n-\ell_n}) \subset \bigcup_{\deg(Q)<n_k} B(\{Qf\}, q^{-n-\ell_n}).$$

Therefore,

$$\bigcup_{n_k \le n < (n_{k+1}+n_k-\ell_n)/2} E_n \subset \bigcup_{\deg(Q)<n_k} B(\{Qf\}, q^{-n_k-\ell_{n_k}}).$$

By Lemma 2,

$$\mu(E_n) = \begin{cases} \dfrac{q-1}{q^{n_{k+1}-n}} & \text{if } (n_{k+1}+n_k)/2 - \ell_n/2 \le n < n_{k+1} - \ell_n, \\ \dfrac{q-1}{q^{\ell_n}} & \text{if } n \ge n_{k+1} - \ell_n. \end{cases}$$

Therefore, if we put $\ell = \ell_{n_k}$, then

$$\begin{aligned}
\mu\Big(\bigcup_{n_k\le n<n_{k+1}} E_n\Big) \\
&\le \frac{1}{q^\ell} + \sum_{(n_{k+1}+n_k-\ell)/2\le n<n_{k+1}-\ell} \frac{q-1}{q^{n_{k+1}-n}} + \sum_{n_{k+1}-\ell\le n<n_{k+1}} \frac{q-1}{q^\ell} \\
&\le \frac{1}{q^\ell} + \frac{q-1}{q^\ell}\Big(\frac{1}{q}+\frac{1}{q^2}+\cdots\Big) + \frac{(q-1)\ell}{q^\ell} \\
&= \frac{1}{q^\ell}(1+1+(q-1)\ell) \le \frac{1}{q^\ell}(1+q\ell). \ \blacksquare
\end{aligned}$$

By the first Borel–Cantelli lemma we have the following proposition:

PROPOSITION 2. *If $\sum_{k=1}^{\infty} \ell_{n_k}/q^{\ell_{n_k}} < \infty$, then for almost every g, we have $g \in E_n$ for at most finitely many n's.*

We present a sufficient condition for the existence of an increasing $\{\ell_n\}$ with $\sum_n 1/q^{\ell_n} = \infty$ which does not allow $g \in E_n$ infinitely often for almost every $g \in \mathbb{L}$.

THEOREM 3. *If $\sum_k (\log n_k)/n_k < \infty$, then there is an increasing sequence $\{\ell_n\}$ with $\sum_n 1/q^{\ell_n} = \infty$ and for almost every $g \in \mathbb{L}$, we have $g \in E_n$ at most finitely many times.*

Proof. Let

$$\ell_n = \lfloor \log_q n_k \rfloor \quad \text{for } n_{k-1} < n \le n_k.$$

Then

$$\sum_n \frac{1}{q^{\ell_n}} \ge \sum_k \frac{n_k - n_{k-1}}{n_k} = \infty.$$

But

$$\sum_k \frac{\ell_{n_k}}{q^{\ell_{n_k}}} \le \sum_k \frac{\log n_k}{q \cdot n_k} < \infty.$$

By Proposition 2, for almost every $g \in \mathbb{L}$, we have $g \in E_n$ for finitely many n's. ∎

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