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Farhi arithmetic functions associated to Lucas sequences

by

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1. Introduction. Throughout this paper, let $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^*$ denote the field of rational numbers, the ring of rational integers, the set of nonnegative integers and the set of positive integers. For any $a, b \in \mathbb{Z} \setminus \{0\}$, let $\gcd(a,b)$ (resp. $\operatorname{lcm}(a,b)$) denote the *positive greatest common divisor* (resp. the *positive least common multiple*) of a and b. Let $d \geq 2$ and m be any positive integers. Define $v_d(m) = \alpha$ if $d^\alpha \parallel m$. For example, $v_6(12) = 1,\ v_6(72) = 2$. If p is a prime, then v_p is the normalized p-adic valuation of $\mathbb{Q}$, i.e., $v_p(p^\alpha a/b) = \alpha$ if $\gcd(p, ab) = 1$, $\gcd(a,b) = 1$, $\alpha, a, b \in \mathbb{Z}$.

It is known that an equivalent version of the Prime Number Theorem states that $\log \operatorname{lcm}(1, 2, \ldots, n) \sim n$ as n tends to infinity (see e.g. [HW]). One thus expects that a better understanding of the function $\operatorname{lcm}(1, 2, \ldots, n)$ may entail a deeper understanding of the distribution of the prime numbers. Some progress has been made in this direction. Before we state our main theorems, let us first give a short account of the recent results on this subject.

In his pioneering paper [F], Farhi introduced the arithmetic functions

$$F_l(n) := \frac{n(n+1)\cdots(n+l)}{\operatorname{lcm}(n, n+1, \ldots, n+l)}, \qquad n \in \mathbb{N}^*. \tag{1}$$

He proved that the sequence $(F_l)_{l\in\mathbb{N}}$ satisfies the recursive relation

$$F_l(n) = \gcd(l!, (n+l)F_{l-1}(n)), \qquad n \in \mathbb{N}^*. \tag{2}$$

Using this relation, he proved

THEOREM 1.1 ([F]). *The function F_l $(l \in \mathbb{N})$ is $l!$-periodic.*

An interesting problem is to determine the least period of F_l (see [F]). In [HY], by using (2) and $F_l(1) \mid F_l(n)$ for any positive integer n, Hong and Yang gave a partial answer. A complete solution was given by Farhi and Kane [FK].

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The Least Period Theorem (Farhi, Kane). *The least period T_l of F_l is given by*

$$T_l = \prod_{p\,prime} p^{\delta_p(l)} = \prod_{p\,prime,\,p\le l} p^{\delta_p(l)}, \tag{3}$$

where

$$\delta_p(l) = \begin{cases} 0 & \textit{if } v_p(l+1) \ge \max\limits_{1\le i\le l} v_p(i), \\ \max\limits_{1\le i\le l} v_p(i) & \textit{otherwise.} \end{cases} \tag{4}$$

In [JJ], Q. Ji and C. Ji introduce an extension of the above arithmetic function $F_l(n)$. Precisely, they study the function $F_{(l,f)} : \mathbb{N} \to \mathbb{N}$ (with $l \in \mathbb{N}$ and $f \in \mathbb{Z}[x]$) defined by

$$F_{(l,f)}(n) = \frac{|f(n)f(n+1)\cdots f(n+l)|}{\operatorname{lcm}(f(n), f(n+1), \dots, f(n+l))}, \qquad n \in \mathbb{N}. \tag{5}$$

Note that for $f(x) = x$, $F_{(l,f)}(n)$ is Farhi's original function $F_l(n)$. Ji and Ji prove that $F_{(l,f)}$ is periodic for any $l \in \mathbb{N}$ and any $f \in \mathbb{Z}[x]$ with

$$\gcd(f(x), f(x+1)f(x+2)\cdots f(x+l)) = 1 \quad \text{ in } \mathbb{Q}[x].$$

They give, in addition, a multiple (effectively calculable) of the period of $F_{(l,f)}$ (for a fixed l and a fixed f). In the case when f is affine, they obtain an explicit formula for the exact period of $F_{(l,f)}$, looking like the Farhi–Kane formula (3).

In this paper, we shall generalize the sequence of natural numbers $\{n\}_{n\ge 0}$ to the Lucas sequence $\{L_n\}_{n\ge 0}$ which is defined as follows. Let P, Q be non-zero integers such that $\gcd(P, Q) = 1$. For each $n \ge 0$, define $L_n = L_{(P,Q)}(n)$ as follows:

$$L_0 = 0, \qquad L_1 = 1, \qquad L_{n+2} = PL_{n+1} + QL_n, \qquad n \ge 0. \tag{6}$$

The sequence $L = \{L_{(P,Q)}(n)\}_{n\ge 0}$ is called a *Lucas sequence with parameters* (P, Q). It is well-known that $L_n \ne 0$ for all $n \ge 1$ if and only if $(P, Q) \ne (\pm 1, -1)$.

Fix $l \in \mathbb{N}$. If $(P, Q) \ne (\pm 1, -1)$, we call

$$F_{(l,P,Q)}(n) := \frac{|L_n L_{n+1} \cdots L_{n+l}|}{\operatorname{lcm}(L_n, L_{n+1}, \dots, L_{n+l})}, \qquad n \ge 1, \tag{7}$$

the *Farhi arithmetic function associated to the Lucas sequence* $L = \{L_{(P,Q)}(n)\}_{n\ge 0}$.

Let p be a prime and $l \in \mathbb{N}$. Define $\varepsilon_p(l)$ as follows: If $v_p(l+1) < \max_{1\le i\le l} v_p(i)$, then $\varepsilon_p(l) := \max_{1\le i\le l} v_p(i)$.

Now assume that $v_p(l+1) \ge \max_{1\le i\le l} v_p(i)$. Put

$$T'_p = \prod_{\substack{q\,\text{prime},\,q\le l \\ q\ne p}} q^{\varepsilon_q(l)}.$$

Then $\varepsilon_p(l)$ is defined to be the least non-negative integer e such that, for any positive integers $d \geq 2$ and $n \geq 1$,

$$\{v_d(n), v_d(n+1), \ldots, v_d(n+l)\} \setminus \{\max_{0\leq i\leq l} v_d(n+i)\}$$
$$= \{v_d(n+p^eT'_p), v_d(n+1+p^eT'_p), \ldots, v_d(n+l+p^eT'_p)\} \setminus \{\max_{0\leq i\leq l} v_d(n+i+p^eT'_p)\}.$$

Remark ([FK, Proposition 3.3]). Fix $l \geq 1$. There is at most one prime p such that $v_p(l+1) \geq \max_{1\leq i\leq l} v_p(i)$. Hence T'_p is well-defined.

In this paper, we prove the following theorems.

Theorem 1.2. *Let the notation be as above. Then the Farhi arithmetic function $F_{(l,P,Q)}$ is* $\mathrm{lcm}(1,2,\ldots,l)$*-periodic.*

Theorem 1.3. *The notation being as above, let $T_{(l,P,Q)}$ be the least period of the Farhi arithmetic function $F_{(l,P,Q)}$. Then*

(1) $T_{(1,P,Q)} = 1,\ T_{(2,P,Q)} = \begin{cases} 1 & \text{if } |P| = 1, \\ 2 & \text{otherwise.} \end{cases}$

(2) *Assume that $l \geq 3$. If $P^2 + 4Q \neq 0$ and the map $L : \mathbb{N} \to \mathbb{Z}$, $n \mapsto |L_n|$, is injective, then*

- *$T_{(l,P,Q)}$ is a multiple of T_l and a divisor of* $\mathrm{lcm}(1,2,\ldots,l)$.
- *Assume that $v_p(l+1) < \max_{1\leq i\leq l} v_p(i)$ for every prime $p \leq l$. Then $T_{(l,P,Q)} = \mathrm{lcm}(1,2,\ldots,l)$. In addition, if $l+1 \geq 5$ is a prime, then also $T_{(l,P,Q)} = \mathrm{lcm}(1,2,\ldots,l)$.*
- *The least period $T_{(l,P,Q)}$ is*

$$T_{(l,P,Q)} = \prod_{p\,prime,\, p\leq l} p^{\varepsilon_p(l)}. \tag{8}$$

Theorem 1.4. *Fix a positive integer l. Let $p \leq l$ be a prime such that $\varepsilon_p(l) = \max_{1\leq i\leq l} v_p(i)$. Let T be a positive multiple of $p^{\varepsilon_p(l)}$. Then, for any integer $n \geq 1$,*

$$\{v_p(n), v_p(n+1), \ldots, v_p(n+l)\} \setminus \{\max_{1\leq i\leq l} v_p(n+i)\}$$
$$= \{v_p(n+T), v_p(n+1+T), \ldots, v_p(n+l+T)\} \setminus \{\max_{1\leq i\leq l} v_p(n+i+T)\}.$$

This paper is organized as follows: In §2, we prove the periodicity of the $F_{(l,P,Q)}$. In §3, we consider the least period of $F_{(l,P,Q)}$. In §4, we apply Theorems 1.2, 1.3 to special Lucas sequences. For example, the sequences of Fibonacci numbers and of Pell numbers are Lucas sequences. If $f(z) = \sum_{n=1}^{\infty} c_n q^n$ is the q-expansion of some normalized cusp eigenform of even weight k with respect to $\Gamma_0(N)$, then, for every prime p with $p \nmid Nc_p$, the sequence $\{c_{p^n}\}_{n\geq 0}$ is a Lucas sequence.

2. Periodicity of $F_{(l,P,Q)}$. In this section, let $L = \{L_{(P,Q)}(n)\}_{n\geq 0}$ be the Lucas sequence with parameters (P,Q) and $(P,Q) \neq (\pm 1, -1)$.

For a fixed $l \in \mathbb{N}$, in order to prove the periodicity of the Farhi arithmetic function $F_{(l,P,Q)}$ defined by (7), we will use the following key lemma which is easy to prove.

Lemma 2.1 ([JJ]). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be any $2n$ positive integers. If $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leq i < j \leq n$, then*

$$\frac{a_1 \cdots a_n}{\operatorname{lcm}(a_1, \ldots, a_n)} = \frac{b_1 \cdots b_n}{\operatorname{lcm}(b_1, \ldots, b_n)}. \tag{9}$$

It is clear that the Lucas sequence $L_n = L_{(P,Q)}(n)$ defines two sequences of integer numbers, $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$, as follows: given $m \geq 1$, put

$$P_1 = P, \quad Q_1 = Q, \quad L_{m+n} = P_n L_m + Q_n L_{m-1}, \quad n \geq 1.$$

It is easy to see that the sequences $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ are well-defined and do not depend on the choice of m. In fact,

$$P_n = L_{n+1} \quad \text{and} \quad Q_n = QL_n \tag{10}$$

by the well-known formula $L_{m+n} = L_{n+1}L_m + QL_nL_{m-1}$.

Lemma 2.2. *For $n \geq 1$, we have the recursive formulae*

$$P_{n+1} = PP_n + Q_n, \quad Q_{n+1} = QP_n, \tag{11}$$

and

$$\gcd(P_n, Q) = 1, \quad \gcd(P_n, Q_n) = 1. \tag{12}$$

Proof. The formulas (11) are obvious from (10) and the definition of the Lucas sequence; and (12) is easy to prove by induction on n. ∎

Lemma 2.3. *Fix $l \geq 1$. Then the function $h_l(n) = \gcd(L_n, Q_l)$ for $n \geq 1$ is l-periodic, i.e.,*

$$\gcd(L_n, Q_l) = \gcd(L_{n+l}, Q_l), \quad n \geq 1. \tag{13}$$

Proof. For $n \geq 1$ and $l \geq 1$, by (12), we have

$$\gcd(L_{n+l}, Q_l) = \gcd(P_l L_n + Q_l L_{n-1}, Q_l) = \gcd(P_l L_n, Q_l) = \gcd(L_n, Q_l). \ ∎$$

Lemma 2.4. *Fix $l \geq 1$. Then the function $g_l(n) = \gcd(L_n, L_{n+l})$ for $n \geq 1$ is l-periodic, i.e.,*

$$g_l(n+l) = g_l(n), \quad n \geq 1.$$

Proof. By Lemma 3.1(i) below, we have $\gcd(L_n, L_{n-1}) = 1$. Hence, for all $n \geq 2$,

$$\begin{aligned} g_l(n) = \gcd(L_n, L_{n+l}) &= \gcd(L_n, P_l L_n + Q_l L_{n-1}) \\ &= \gcd(L_n, Q_l L_{n-1}) = \gcd(L_n, Q_l). \end{aligned}$$

Moreover, $g_l(1) = \gcd(L_1, Q_l)$. Hence, Lemma 2.3 applies. ∎

Proof of Theorem 1.2. If $l = 0$, then $F_{(l,P,Q)}(n) = 1$ is a constant function, hence it is periodic. Fix $l \geq 1$. For any pair (i, j) such that $0 \leq i < j \leq l$, by Lemma 2.4, the function $g_{(i,j)}(n) = \gcd(L_{n+i}, L_{n+j})$ is $(j - i)$-periodic. Put $T = \operatorname{lcm}(1, 2, \ldots, l)$. Then T is also a period of $\gcd(L_{n+i}, L_{n+j})$, i.e., $\gcd(L_{n+i}, L_{n+j}) = \gcd(L_{n+i+T}, L_{n+j+T})$ for all $n \geq 1$. By Lemma 2.1, we obtain

$$\frac{|L_n L_{n+1} \cdots L_{n+l}|}{\operatorname{lcm}(L_n, L_{n+1}, \ldots, L_{n+l})} = \frac{|L_{n+T} L_{n+1+T} \cdots L_{n+l+T}|}{\operatorname{lcm}(L_{n+T}, L_{n+1+T}, \ldots, L_{n+l+T})},$$

i.e. $F_{(l,P,Q)}(n) = F_{(l,P,Q)}(n + T)$ for all $n \geq 1$. ∎

3. The least period of $F_{(l,P,Q)}$. Let the notation be as in §2. First we recall some well-known properties of the Lucas sequence.

LEMMA 3.1. *Let the notation be as above. We have the following properties:*

(i) ([R, p. 9, (2.11)]) $\gcd(L_n, L_m) = |L_{\gcd(n,m)}|$.
(ii) ([R, p. 10, (2.14)]) $\gcd(L_n, Q) = 1$ *for* $n \geq 1$.
(iii) ([R, p. 5, (2.1)], Binet's formula) *Let* α, β *be the roots of the polynomial* $X^2 - PX - Q$. *Then*

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

By Theorem 1.2, we know that the Farhi arithmetic function $F_{(l,P,Q)}$ is periodic. Let $T_{(l,P,Q)}$ be its least period.

PROPOSITION 3.2. $T_{(1,P,Q)} = 1$, $T_{(2,P,Q)} = \begin{cases} 1 & \textit{if } |P| = 1, \\ 2 & \textit{otherwise.} \end{cases}$

Proof. By Lemma 3.1(i), we have $\gcd(L_n, L_{n+1}) = 1$. Hence

$$F_{(1,P,Q)} = \frac{|L_n L_{n+1}|}{\operatorname{lcm}(L_n, L_{n+1})} = 1.$$

Therefore $T_{(1,P,Q)} = 1$.

If $n = 2k$, then

$$\gcd(L_n, L_{n+2}) = |L_2| = |P|, \qquad \gcd(L_n, L_{n+1}) = \gcd(L_{n+1}, L_{n+2}) = 1.$$

If $n = 2k + 1$, then

$$\gcd(L_n, L_{n+2}) = \gcd(L_n, L_{n+1}) = \gcd(L_{n+1}, L_{n+2}) = 1.$$

Hence

$$F_{(2,P,Q)} = \frac{|L_n L_{n+1} L_{n+2}|}{\operatorname{lcm}(L_n, L_{n+1}, L_{n+2})} = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ |P| & \text{if } n \text{ is even}. \end{cases}$$

Therefore $T_{(2,P,Q)} = \begin{cases} 1 & \text{if } |P| = 1, \\ 2 & \text{otherwise.} \end{cases}$ ∎

For every positive integer n, denote by $\Phi_n(X)$ the nth cyclotomic polynomial. We have a result analogous to [W, Lemma 2.9].

LEMMA 3.3. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ be a quadratic field. Let p be a prime and $\mathfrak{p}$ an ideal over p. Suppose $p \nmid n$ and a is an integer in $\mathbb{K}$.*

(i) *$\mathfrak{p} \mid \Phi_n(a)$ if and only if the multiplicative order of a modulo $\mathfrak{p}$ is n (i.e., $a^n \equiv 1 \pmod{\mathfrak{p}}$ and n is minimal).*
(ii) *Assume $\mathfrak{p} \mid \Phi_n(a)$. Then we have the following statements:*
  (a) *If $\left(\frac{d}{p}\right) = 1$, then $p \equiv 1 \pmod{n}$.*
  (b) *If $\left(\frac{d}{p}\right) = -1$, then $p^2 \equiv 1 \pmod{n}$.*

PROPOSITION 3.4. *Let p be a prime such that the map $\mathbb{N} \to \mathbb{Z}$, $n \mapsto |L_{p^n}|$, is injective. Let $a \in \mathbb{N}$. Then there exists a prime q such that $q \mid L_{p^{a+1}}$ and $q \nmid L_{p^a}$.*

Proof. By Lemma 3.1(iii), we have

$$(\alpha - \beta)L_n = \beta^n \prod_{d \mid n} \Phi_d(\alpha/\beta).$$

Let p be a prime and $a \in \mathbb{N}$. Then

$$L_{p^{a+1}} = \beta^{p^a(p-1)} \Phi_{p^{a+1}}(\alpha/\beta) L_{p^a}. \tag{14}$$

Let $q \mid L_{p^{a+1}}$ be a prime. Denote by $\mathfrak{q}$ an ideal over q in $K = \mathbb{Q}(\sqrt{P^2 + 4Q})$. Since $N_{K/\mathbb{Q}}(\beta) = \alpha\beta = -Q$ and $\gcd(L_n, Q) = 1$ for any integer n, we deduce that $q \mid \beta^{p^a(p-1)} \Phi_{p^{a+1}}(\alpha/\beta)$ if and only if the multiplicative order of α/β modulo $\mathfrak{q}$ is p^{a+1} by Lemma 3.3(i) (or [W, Lemma 2.9]). By assumption, $|L_{p^{a+1}}| \neq |L_{p^a}|$, hence there exists a prime q such that $q \mid \beta^{p^a(p-1)} \Phi_{p^{a+1}}(\alpha/\beta)$, so $q \mid L_{p^{a+1}}$, but $q \nmid L_{p^a}$. ■

Let d be a positive integer. Define the *local* Farhi arithmetic function $F_{(l,L,d)}$ as follows:

$$F_{(l,L,d)}(n) = \frac{|L_{\gcd(n,d)} L_{\gcd(n+1,d)} \cdots L_{\gcd(n+l,d)}|}{\operatorname{lcm}(L_{\gcd(n,d)}, L_{\gcd(n+1,d)}, \ldots, L_{\gcd(n+l,d)})}. \tag{15}$$

PROPOSITION 3.5. *The local Farhi arithmetic function $F_{(l,L,d)}$ is $\gcd(d, \operatorname{lcm}(1, 2, \ldots, l))$-periodic.*

Proof. On the one hand, it is easy to see that $F_{(l,L,d)}$ is $\operatorname{lcm}(1, 2, \ldots, l)$-periodic. On the other hand, d is clearly a period of $F_{(l,L,d)}$, since

$$\gcd(\gcd(n + i, d), \gcd(n + j, d)) = \gcd(\gcd(n + i + d, d), \gcd(n + j + d, d))$$

for all $0 \le i, j \le l$. Hence $\gcd(d, \operatorname{lcm}(1, 2, \ldots, l))$ is a period of $F_{(l,L,d)}$. ■

Let $d \geq 2$. Put

$$e_d(l) = \max_{1 \leq i \leq l} v_d(i). \tag{16}$$

Define

$$\begin{aligned} g_{(l,L,d)}(n) &= F_{(l,L,d^{e_d(l)})}(n) \\ &= \frac{|L_{\gcd(n,d^{e_d(l)})} L_{\gcd(n+1,d^{e_d(l)})} \cdots L_{\gcd(n+l,d^{e_d(l)})}|}{\operatorname{lcm}(L_{\gcd(n,d^{e_d(l)})}, L_{\gcd(n+1,d^{e_d(l)})}, \ldots, L_{\gcd(n+l,d^{e_d(l)})})}. \end{aligned}$$

Proposition 3.6. *We have*

$$g_{(l,L,d)}(n) = \frac{|L_{d^{v_d(n)}} L_{d^{v_d(n+1)}} \cdots L_{d^{v_d(n+l)}}|}{\operatorname{lcm}(L_{d^{v_d(n)}}, L_{d^{v_d(n+1)}}, \ldots, L_{d^{v_d(n+l)}})}. \tag{17}$$

Hence $d^{e_d(l)}$ is a period of $g_{(l,L,d)}$.

Proof. On the one hand,

$$\frac{|L_{d^{v_d(n)}} L_{d^{v_d(n+1)}} \cdots L_{d^{v_d(n+l)}}|}{\operatorname{lcm}(L_{d^{v_d(n)}}, L_{d^{v_d(n+1)}}, \ldots, L_{d^{v_d(n+l)}})} = \frac{\prod_{i=0}^{l} |L_{d^{v_d(n+i)}}|}{|L_{\max_{0 \leq i \leq l} d^{v_d(n+i)}}|}$$

and

$$\begin{aligned} &\frac{|L_{\gcd(n,d^{e_d(l)})} L_{\gcd(n+1,d^{e_d(l)})} \cdots L_{\gcd(n+l,d^{e_d(l)})}|}{\operatorname{lcm}(L_{\gcd(n,d^{e_d(l)})}, L_{\gcd(n+1,d^{e_d(l)})}, \ldots, L_{\gcd(n+l,d^{e_d(l)})})} \\ &\qquad = \frac{\prod_{i=0}^{l} |L_{\gcd(n+i,d^{e_d(l)})}|}{|L_{\max_{0 \leq i \leq l} \gcd(n+i,d^{e_d(l)})}|}. \end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned} &\{d^{v_d(n)}, d^{v_d(n+1)}, \ldots, d^{v_d(n+l)}\} \setminus \{\max_{0 \leq i \leq l} d^{v_d(n+i)}\} \\ &= \{\gcd(n, d^{e_d(l)}), \gcd(n+1, d^{e_d(l)}), \ldots, \gcd(n+l, d^{e_d(l)})\} \setminus \{\max_{1 \leq i \leq l} \gcd(n+i, d^{e_d(l)})\}. \end{aligned}$$

Hence (17) holds. By Proposition 3.5, $d^{e_d(l)} = \gcd(d^{e_d(l)}, \operatorname{lcm}(1, 2, \ldots, l))$ is a period of $g_{(l,L,d)}$. ∎

Lemma 3.7. *Let p be a prime. Assume that the map $\mathbb{N} \to \mathbb{Z}$, $n \mapsto |L_{p^n}|$, is injective.*

(i) *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{N}$ and*

$$a_1 \leq \cdots \leq a_n, \quad b_1 \leq \cdots \leq b_n.$$

Then $L_{p^{a_1}} \cdots L_{p^{a_n}} = L_{p^{b_1}} \cdots L_{p^{b_n}}$ if and only if $a_i = b_i$, $1 \leq i \leq n$.

(ii) p^e *is the least period of* $g_{(l,L,p)}$ *if and only if* e *is the least non-negative integer such that, for any* $n \geq 1$,

$$\begin{aligned}&\{v_p(n), v_p(n+1), \ldots, v_p(n+l)\} \setminus \{\max_{1\leq i\leq l} v_p(i)\}\\ &= \{v_p(n+p^e), v_p(n+1+p^e), \ldots, v_p(n+l+p^e)\} \setminus \{\max_{1\leq i\leq l} v_p(n+i+p^e)\}.\end{aligned}$$

(iii) *If* p^e *is a period of* $g_{(l,L,p)}$, *then it is also a period of*

$$K_{(l,p)}(n) = v_p(F_l(n)) = v_p\left(\frac{n(n+1)\cdots(n+l)}{\operatorname{lcm}(n, n+1, \ldots, n+l)}\right).$$

(iv) $p^{\delta_p(l)}$ *is the least period of* $g_{(l,L,p)}$, *where* $\delta_p(l)$ *is defined by* (4).

Proof. (i) We argue by induction on n. The statement is obvious for $n = 1$. Suppose that it holds for $n-1$. If $a_n < b_n$, then there exists a prime q such that $q \mid L_{p^{b_n}}$ and $q \nmid L_{p^{a_n}}$ by Proposition 3.4. Hence $q \mid L_{p^{b_1}} \cdots L_{p^{b_n}}$ and $q \nmid L_{p^{a_1}} \cdots L_{p^{a_n}}$. This is a contradiction. Hence $a_n \geq b_n$. Similarly, we get $a_n \leq b_n$. Therefore $a_n = b_n$. Hence

$$L_{p^{a_1}} \cdots L_{p^{a_{n-1}}} = L_{p^{b_1}} \cdots L_{p^{b_{n-1}}}.$$

By induction, we get $a_i = b_i$, $1 \leq i \leq n-1$.

(ii) This is clear from (i).

(iii) For $0 \leq i \leq l$, set $\alpha_i = v_p(n+i)$ and $\beta_i = v_p(n+i+p^e)$. Then

$$\frac{|L_{p^{\alpha_0}} L_{p^{\alpha_1}} \cdots L_{p^{\alpha_l}}|}{\operatorname{lcm}(L_{p^{\alpha_0}}, L_{p^{\alpha_1}}, \ldots, L_{p^{\alpha_l}})} = \frac{|L_{p^{\beta_0}} L_{p^{\beta_1}} \cdots L_{p^{\beta_l}}|}{\operatorname{lcm}(L_{p^{\beta_0}}, L_{p^{\beta_1}}, \ldots, L_{p^{\beta_l}})}. \tag{18}$$

By Lemma 3.1(i), we obtain

$$\begin{aligned}\operatorname{lcm}(L_{p^{\alpha_0}}, L_{p^{\alpha_1}}, \ldots, L_{p^{\alpha_l}}) &= |L_{p^{\max(\alpha_0,\ldots,\alpha_l)}}|,\\ \operatorname{lcm}(L_{p^{\beta_0}}, L_{p^{\beta_1}}, \ldots, L_{p^{\beta_l}}) &= |L_{p^{\max(\beta_0,\ldots,\beta_l)}}|.\end{aligned}$$

By (18) and (i), we get

$$\sum_{i=0}^{l} \alpha_i - \max_{1\leq i\leq l} \alpha_i = \sum_{i=0}^{l} \beta_i - \max_{1\leq i\leq l} \beta_i.$$

Hence $v_p(F_l(n)) = v_p(F_l(n+p^e))$, i.e., $K_{(l,p)}(n) = K_{(l,p)}(n+p^e)$. Therefore p^e is also a period of $K_{(l,p)}(n)$.

(iv) If $v_p(l+1) < e_p(l) = \max_{1\leq i\leq l} v_p(i)$, then from the Least Period Theorem and Proposition 3.6, it is easy to see that $p^{e_p(l)}$ is the least period of $g_{(l,L,p)}$.

If $v_p(l+1) \geq e_p(l)$, then $p^{\delta_p(l)} = 1$. It is easy to show that

$$\{v_p(n), v_p(n+1), \ldots, v_p(n+l)\} \setminus \{\max_{1\leq i\leq l} v_p(i)\}$$
$$= \{v_p(n+1), v_p(n+1+1), \ldots, v_p(n+l+1)\} \setminus \{\max_{1\leq i\leq l} v_p(n+i+1)\}.$$

Hence, from (ii), we know that $p^{\delta_p(l)} = 1$ is the least period of $g_{(l,L,p)}$. ∎

THEOREM 3.8. *Let the notation be as above. Assume that the map* $\mathbb{N} \to \mathbb{Z}$, $n \mapsto |L_n|$, *is injective. Fix* $l \in \mathbb{N}$.

(i) *If a positive integer* T *is a period of* $F_{(l,P,Q)}$, *then it is also a period of* F_l.

(ii) *The least period* $T_{(l,P,Q)}$ *is a multiple of* T_l *and a divisor of* $\mathrm{lcm}(1, 2, \ldots, l)$.

(iii) *If for every prime* $p \leq l$, *we have* $v_p(l+1) < \max_{1\leq i\leq l} v_p(i)$, *then* $T_{(l,P,Q)} = T_l = \mathrm{lcm}(1, 2, \ldots, l)$. *In addition, if* $l+1 \geq 5$ *is a prime, then* $T_{(l,P,Q)} = \mathrm{lcm}(1, 2, \ldots, l)$.

(iv) *The least period* $T_{(l,P,Q)}$ *is given by*

$$T_{(l,P,Q)} = \prod_{p\,prime,\,p\leq l} p^{\varepsilon_p(l)}, \tag{19}$$

where $\varepsilon_p(l)$ *is defined as follows: If* $v_p(l+1) < \max_{1\leq i\leq l} v_p(i)$, *then* $\varepsilon_p(l) := e_p(l) = \max_{1\leq i\leq l} v_p(i)$. *If* $v_p(l+1) \geq \max_{1\leq i\leq l} v_p(i)$, *then put*

$$T'_p = \prod_{\substack{q\,prime,\,q\leq l\\ q\neq p}} q^{e_q(l)}$$

and define $\varepsilon_p(l)$ *to be the least non-negative integer* e *such that, for any positive integers* $d \geq 2$ *and* $n \geq 1$, $T = p^e T'_p$ *is a period of* $g_{(l,L,d)}$.

Proof. (i) Let T be a period of $F_{(l,P,Q)}$. Let p be any prime. Then T is also a period of $g_{(l,L,p)}$. By Lemma 3.7(iv), we obtain $p^{\delta_p(l)} \mid T$. Hence $T_l \mid T$, i.e., T is a period of $F_l(n)$.

(ii) This is obvious by (i) and Theorem 1.2.

(iii) This is clear from (ii).

(iv) By Theorem 1.2,

$$T_{(l,P,Q)} = \prod_{p\,\mathrm{prime},\,p\leq l} p^{\varepsilon_p(l)},$$

where $0 \leq \varepsilon_p(l) \leq e_p(l)$. By the Least Period Theorem and (i), we have $\varepsilon_p(l) = e_p(l) = \max_{1\leq i\leq l} v_p(i)$ if $v_p(l+1) < \max_{1\leq i\leq l} v_p(i)$.

Note that there is at most one prime p such that $v_p(l+1) \geq \max_{1\leq i\leq l} v_p(i)$. Assume now that there exists one prime $p \leq l$ such that this inequality holds.

Put

$$T'_p = \prod_{\substack{q\,\text{prime},\, q\le l\\ q\ne p}} q^{e_q(l)} \quad \text{and} \quad T = p^e T'_p, \quad 0 \le e \le e_p(l).$$

Then it is clear that T is a period of $F_{(l,P,Q)}$ if and only if T is a period of $g_{(l,L,d)}$ for every positive integer $d > 1$. ∎

LEMMA 3.9. *Let $L_n = L_{(P,Q)}(n)$ be the Lucas sequence with parameters (P,Q) such that* $\gcd(P,Q) = 1$ *and $\Delta = P^2 + 4Q > 0$. Then $|L_n| = |L_m|$ if and only if either $m = n$, or $P = \pm 1$ and $\{n,m\} = \{1,2\}$.*

Proof. Let α, β be the roots of the polynomial $X^2 - PX - Q$. By Lemma 3.1(iii), $L_n = L_n(\alpha,\beta) = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, which implies

$$L_{2n}(-\alpha,-\beta) = -L_{2n}(\alpha,\beta), \qquad L_{2n+1}(-\alpha,-\beta) = L_{2n+1}(\alpha,\beta), \qquad n \in \mathbb{N}.$$

Hence we may assume that $P > 0$. Let $\alpha = (P+\sqrt{\Delta})/2$, $\beta = (P-\sqrt{\Delta})/2$.

If $Q < 0$, then $\alpha > \beta > 0$ and it is clear that $L_{n+1} > L_n > 0$ for all $n \in \mathbb{N}$.

If $Q > 0$, then $\alpha > |\beta| > 0$, $\beta < 0$ and

$$L_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{\Delta}}, \qquad L_{2n+1} = \frac{\alpha^{2n+1} + |\beta|^{2n+1}}{\sqrt{\Delta}}, \qquad n \in \mathbb{N}.$$

It is obvious that

$$L_{2(n+1)} > L_{2n}, \qquad L_{2(n+1)+1} > L_{2n+1}, \qquad n \in \mathbb{N}. \tag{20}$$

Assume that $L_{2n} = L_{2m+1}$ for some $n, m \in \mathbb{N}$. By Lemma 3.1(i), we get

$$P = L_2 = \gcd(L_2, L_{2n}) = \gcd(L_2, L_{2m+1}) = L_1 = 1.$$

Since $\alpha > |\beta| > 0$, we have $2m+1 < 2n$. By Lemma 3.1(i) and (20), it is easy to see that $n = 2m+1$. The equality $L_{2(2m+1)} = L_{2m+1}$ implies that $\alpha^{2m+1} - |\beta|^{2m+1} = 1$. Hence $m = 0$, since the function $f(x) = \alpha^x - |\beta|^x$ is strictly increasing for $x \ge 1$ and $f(1) = \alpha - |\beta| = P = 1$. ∎

COROLLARY 3.10. *With the same assumptions as in Lemma* 3.9, *let p be a prime and let $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{N}$ be such that*

$$a_1 \le \cdots \le a_n, \qquad b_1 \le \cdots \le b_n.$$

Then

$$\frac{|L_{p^{a_1}} \cdots L_{p^{a_n}}|}{\operatorname{lcm}(L_{p^{a_1}},\dots,L_{p^{a_n}})} = \frac{|L_{p^{b_1}} \cdots L_{p^{b_n}}|}{\operatorname{lcm}(L_{p^{b_1}},\dots,L_{p^{b_n}})} \tag{21}$$

if and only if one of the following conditions holds:

(i) $|P| > 1$ *and* $a_i = b_i$, $1 \le i \le n-1$.
(ii) $|P| = 1$, *p is an odd prime, and* $a_i = b_i$, $1 \le i \le n-1$.
(iii) $|P| = 1$, $p = 2$, *and if there exists an integer $1 \le m \le n-1$ such that $a_m \le 1$ and $a_{m+1} \ge 2$, then $b_m \le 1$ and $a_i = b_i$, $m+1 \le i \le n-1$.*

Proof. This is straightforward from Lemmas 3.9 and 3.7. ∎

Proof of Theorem 1.4. Let $p \le l$ be a prime such that $v_p(l+1) < \max_{1\le i\le l} v_p(i)$. By the Least Period Theorem, $p^{\varepsilon_p(l)}$ is the least period of the function

$$K_{(l,p)}(n) = v_p(F_l(n)) = v_p\biggl(\frac{n(n+1)\cdots(n+l)}{\operatorname{lcm}(n,n+1,\ldots,n+l)}\biggr).$$

Let T be any positive multiple of $p^{\varepsilon_p(l)}$. Then

$$K_{(l,p)}(n) = K_{(l,p)}(n+T), \qquad n \ge 1.$$

Let $P \ge 2, Q \ge 1$ and $\gcd(P,Q) = 1$. Let $L_n = L_{(P,Q)}(n)$ be the Lucas sequence with parameters (P,Q). By Lemma 3.9, the map $\mathbb{N} \to \mathbb{Z}$, $n \mapsto |L_n|$, is injective. By Lemma 3.7, $p^{\varepsilon_p(l)}$ is the least period of $g_{(l,L,p)}$. Hence, for any $n \ge 1$, we have $g_{(l,L,p)}(n) = g_{(l,L,p)}(n+T)$, i.e.,

$$\frac{|L_{p^{v_p(n)}} L_{p^{v_p(n+1)}} \cdots L_{p^{v_p(n+l)}}|}{\operatorname{lcm}(L_{p^{v_p(n)}}, L_{p^{v_p(n+1)}}, \ldots, L_{p^{v_p(n+l)}})} = \frac{|L_{p^{v_p(n+T)}} L_{p^{v_p(n+1+T)}} \cdots L_{p^{v_p(n+l+T)}}|}{\operatorname{lcm}(L_{p^{v_p(n+T)}}, L_{p^{v_p(n+1+T)}}, \ldots, L_{p^{v_p(n+l+T)}})}.$$

Let $a_0 \le a_1 \le \cdots \le a_l$ be a permutation of $v_p(n), v_p(n+1), \ldots, v_p(n+l)$ and let $b_0 \le b_1 \le \cdots \le b_l$ be a permutation of $v_p(n+T), v_p(n+1+T), \ldots, v_p(n+l+T)$. By Corollary 3.10(i), we have

$$a_i = b_i, \qquad 0 \le i \le l-1,$$

that is,

$$\begin{aligned}&\{v_p(n), v_p(n+1), \ldots, v_p(n+l)\} \setminus \{\max_{1\le i\le l} v_p(n+i)\}\\ &\quad = \{v_p(n+T), v_p(n+1+T), \ldots, v_p(n+l+T)\} \setminus \{\max_{1\le i\le l} v_p(n+i+T)\}.\end{aligned}$$ ∎

THEOREM 3.11. *Fix $l \in \mathbb{N}$ and let the assumptions be as in Lemma* 3.9. *Let T be a positive integer. If T is a period of the Farhi arithmetic function $F_{(l,P,Q)}$, then T is a period of the Farhi arithmetic function F_l.*

Proof. Let T be a period of $F_{(l,P,Q)}$. Then for every $n \ge 1$, we have $F_{(l,P,Q)}(n) = F_{(l,P,Q)}(n+T)$. Let $n \ge 1$. For any prime p, set $\alpha_i = v_p(n+i)$, $\beta_i = v_p(n+i+T)$, $0 \le i \le l$. Then

$$\frac{|L_{p^{\alpha_0}} L_{p^{\alpha_1}} \cdots L_{p^{\alpha_l}}|}{\operatorname{lcm}(L_{p^{\alpha_0}}, L_{p^{\alpha_1}}, \ldots, L_{p^{\alpha_l}})} = \frac{|L_{p^{\beta_0}} L_{p^{\beta_1}} \cdots L_{p^{\beta_l}}|}{\operatorname{lcm}(L_{p^{\beta_0}}, L_{p^{\beta_1}}, \ldots, L_{p^{\beta_l}})}. \tag{22}$$

(1) Suppose $|P| > 1$. By Corollary 3.10(i), the condition (22) implies that
$$\sum_{i=0}^{l} \alpha_i - \max_{1\le i\le l} \alpha_i = \sum_{i=0}^{l} \beta_i - \max_{1\le i\le l} \beta_i.$$
Hence $v_p(F_l(n)) = v_p(F_l(n+T))$. Since p is an arbitrary prime, we find that $F_l(n) = F_l(n+T)$, i.e., T is a period of $F_l(n)$.

(2) Suppose $|P| = 1$. Since $|L_n(P,Q)| = |L_n(-P,Q)|$, it is sufficient to consider $P = 1$. Let p be an odd prime. By Corollary 3.10(ii), the condition (22) implies that $v_p(F_l(n)) = v_p(F_l(n+T))$.

Assume that $p = 2$. Let $T_l = \prod_{p\,\mathrm{prime},\, p\le l} p^{\delta_p(l)}$ be the least period of $F_l(n)$ defined by (3). Let $l = 2^e + a_1 2^{e-1} + \cdots + a_e$ be the 2-adic expansion, where $a_i \in \{0,1\}$, $1 \le i \le e$.

(i) If $l+1 = 2^{e+1}$, then $\delta_2(l) = 0$, i.e., $v_2(T_l) = 0$. Hence $v_2(F_l(n)) = v_2(F_l(n+T))$ for all $n \ge 1$.

(ii) If $l+1 \ne 2^{e+1}$, then $\delta_2(l) = e$, i.e., $v_2(T_l) = e$. Hence the implication (22) $\Rightarrow$ $[v_2(F_l(n)) = v_2(F_l(n+T))$ for any $n \ge 1]$ is equivalent to saying that: if there exists an integer $n_0 \ge 1$ such that $v_2(F_l(n_0)) \ne v_2(F_l(n_0 + 2^{e-1}))$, then

$$\frac{|L_{2^{\alpha_0}} L_{2^{\alpha_1}} \cdots L_{2^{\alpha_l}}|}{\mathrm{lcm}(L_{2^{\alpha_0}}, L_{2^{\alpha_1}}, \ldots, L_{2^{\alpha_l}})} \ne \frac{|L_{2^{\beta_0}} L_{2^{\beta_1}} \cdots L_{2^{\beta_l}}|}{\mathrm{lcm}(L_{2^{\beta_0}}, L_{2^{\beta_1}}, \ldots, L_{2^{\beta_l}})}, \tag{23}$$

where $\alpha_i = v_2(n_0 + i)$, $\beta_i = v_2(n_0 + i + 2^{e-1})$, $0 \le i \le l$.

Case 1. If $l = 2^e + 2^{e-1} + a_2 2^{e-2} + a_3 2^{e-3} + \cdots + a_e$, let $n_0 = 1$. Then
$$n_0 + l = 2^e + 2^{e-1} + b_2 2^{e-2} + b_3 2^{e-3} + \cdots + b_e,$$
$$n_0 + l + 2^{e-1} = 2^{e+1} + b_2 2^{e-2} + b_3 2^{e-3} + \cdots + b_e,$$
where $b_2, \ldots, b_e \in \{0,1\}$ since $l+1 < 2^{e+1}$.

Case 2. If $l = 2^e + a_2 2^{e-2} + a_3 2^{e-3} + \cdots + a_e$, let $n_0 = 2^{e-1}$. Then
$$n_0 + l = 2^e + 2^{e-1} + a_2 2^{e-2} + a_3 2^{e-3} + \cdots + a_e,$$
$$n_0 + l + 2^{e-1} = 2^{e+1} + a_2 2^{e-2} + a_3 2^{e-3} + \cdots + a_e.$$

In each case, we always have
$$\max_{n_0 \le n_0+i \le n_0+l} v_2(n_0+i) = e,$$
$$\max_{n_0+2^{e-1} \le n_0+i+2^{e-1} \le n_0+l+2^{e-1}} v_2(n_0+i+2^{e-1}) = e+1,$$
$$\sharp\{n_0 \le n_0+i \le n_0+l \mid v_2(n_0+i) = e\} = 1,$$
$$\sharp\{n_0+2^{e-1} \le n_0+i+2^{e-1} \le n_0+l+2^{e-1} \mid v_2(n_0+i) = e+1\} = 1,$$
$$\sharp\{n_0+2^{e-1} \le n_0+i+2^{e-1} \le n_0+l+2^{e-1} \mid v_2(n_0+i) = e\} = 1.$$

By Corollary 3.10(iii), (23) holds. Hence (22) implies that $v_2(F_l(n)) = v_2(F_l(n+T))$ for any $n \geq 1$.

Thus, for any prime p, the equality $v_p(F_l(n)) = v_p(F_l(n+T))$ holds for any $n \geq 1$. Therefore $F_l(n) = F_l(n+T)$, i.e., T is a period of $F_l(n)$. ∎

Proof of Theorem 1.3. The least periods $T_{(1,P,Q)}$ and $T_{(2,P,Q)}$ have been determined in Proposition 3.2. Let $l \geq 3$. Assume $v_p(l+1) \geq \max_{1\leq i\leq l} v_p(i)$. Then $p^e T'_p$ is a period of $g_{(l,L,d)}$ if and only if

$$\{v_d(n), v_d(n+1), \ldots, v_d(n+l)\} \setminus \{\max_{0\leq i\leq l} v_d(n+i)\}$$
$$= \{v_d(n+p^eT'_p), v_d(n+1+p^eT'_p), \ldots, v_d(n+l+p^eT'_p)\} \setminus \{\max_{0\leq i\leq l} v_d(n+i+p^eT'_p)\}$$

for any $d \geq 2$ and $n \geq 1$. Hence the case (2) is obvious from Theorem 3.8, Lemma 3.9, Theorem 3.11 and formula (3). ∎

4. Special Lucas sequences. In this section, we shall apply the results of §2 and §3 to special Lucas sequences, which are important historically and for their own sake. These are the sequences of Fibonacci numbers, of Lucas numbers, of Pell numbers, the sequence $L_n = \frac{a^n - b^n}{a-b}$ and the sequence $\{c_{p^n}\}_{n\geq 0}$ which are the coefficients of the q-expansion of some normalized cusp eigenform of even weight k with respect to $\Gamma_0(N)$.

4.1. Fibonacci numbers. Let $P = Q = 1$. The numbers $L_n = L_{(1,1)}(n)$ are called *Fibonacci numbers*. Here are the initial terms of this sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Fix $l \in \mathbb{N}$. Since $P^2 + 4Q = 5 > 0$, by Theorem 1.3, the Farhi arithmetic function

$$F_{(l,1,1)}(n) = \frac{|L_n L_{n+1} \cdots L_{n+l}|}{\operatorname{lcm}(L_n, L_{n+1}, \ldots, L_{n+l})}$$

with respect to the sequence of Fibonacci numbers is $\operatorname{lcm}(1, 2, \ldots, l)$-periodic. For example:

- $T_2 = 2$, $T_{(2,1,1)} = 1$, and $\operatorname{lcm}(1,2) = 2$;
- $T_3 = T_{(3,1,1)} = 3$ and $\operatorname{lcm}(1,2,3) = 6$;
- $T_7 = 105$, $T_{(7,2,1)} = 210$, and $\operatorname{lcm}(1,\ldots,7) = 420$.

4.2. Pell numbers. Let $P = 2, Q = 1$, so $P^2 + 4Q = 8 > 0$. The numbers $L_n = L_{(2,1)}(n)$ are called *Pell numbers*. The first few terms of this sequence are:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots$$

By Theorem 1.3, for every $l \in \mathbb{N}$, the Farhi arithmetic function $F_{(l,2,1)}$ associated to the sequence of Pell numbers is periodic and if $l+1 \geq 5$ is a prime, then $\mathrm{lcm}(1,2,\ldots,l)$ is the least period of $F_{(l,2,1)}$. For example:

- $T_2 = T_{(2,2,1)} = 2$, and $\mathrm{lcm}(1,2) = 2$;
- $T_3 = T_{(3,2,1)} = 3$ and $\mathrm{lcm}(1,2,3) = 6$;
- $T_7 = 105$, $T_{(7,2,1)} = 210$, and $\mathrm{lcm}(1,\ldots,7) = 420$.

4.3. Cusp forms and elliptic curves. For any integer $N \geq 1$ and any even integer k, let $S_k(\Gamma_0(N))$ denote the set of all cusp forms of weight k with respect to $\Gamma_0(N)$. Let $f \in S_k(\Gamma_0(N))$ denote the normalized cusp eigenform. If the q-expansion of f at ∞ is $f(z) = \sum_{n=1}^{\infty} c_n q^n$, then the coefficients $\{c_n\}_{n=1}^{\infty}$ have the following properties:

$$c_{p^r} c_p = c_{p^{r+1}} + p^{k-1} c_{p^{r-1}} \quad \text{for } p \text{ prime, } p \nmid N, \tag{24}$$

$$c_{p^r} = c_p^r \quad \text{for } p \text{ prime, } p \mid N, \tag{25}$$

$$c_{mn} = c_m c_n \quad \text{if } \gcd(m,n) = 1, \tag{26}$$

$$|c(n)| \leq \sigma_0(n) n^{(k-1)/2}, \tag{27}$$

where $\sigma_0(n)$ is the number of positive divisors of n.

Let p be a prime such that $\gcd(p, Nc_p) = 1$. Put

$$L_0 = 0, \qquad L_n = c_{p^{n-1}}, \qquad n \geq 1.$$

It is clear that the sequence L_n is the Lucas sequence $L_{(P,Q)}(n)$ with $P = c_p$ and $Q = -p^{k-1}$, by (24). Fix an integer $l \in \mathbb{N}$. We define a Farhi arithmetic function $F_{(l,f,p)}(n)$ attached to the normalized cusp eigenform f and the prime p with $\gcd(p, Nc_p) = 1$ as follows:

$$F_{(l,f,p)}(n) = \frac{|c_{p^n} c_{p^{n+1}} \cdots c_{p^{n+l}}|}{\mathrm{lcm}(c_{p^n}, c_{p^{n+1}}, \ldots, c_{p^{n+l}})}, \qquad n \geq 0. \tag{28}$$

From (27), we get $P^2 + 4Q = c_p^2 - 4p^{k-1} < 0$. By Theorem 1.3, for every $l \in \mathbb{N}$, $F_{(l,f,p)}$ is periodic and if $l + 1 \geq 5$ is a prime, then $\mathrm{lcm}(1,2,\ldots,l)$ is its least period.

Example 1. Let $\tau(n)$ be the Ramanujan τ-function defined by

$$\Delta(z) = q \prod_{n\geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n. \tag{29}$$

It is well-known that $\Delta(z)$ is a normalized cusp eigenform of weight 12 with respect to the modular group $\Gamma_0(1) = \mathrm{SL}(2,\mathbb{Z})$. Hence, for every prime p with $p \nmid \tau(p)$, the arithmetic function

$$F_{(l,\Delta,p)}(n) = \frac{|\tau(p^n)\tau(p^{n+1}) \cdots \tau(p^{n+l})|}{\mathrm{lcm}(\tau(p^n), \tau(p^{n+1}), \ldots, \tau(p^{n+l}))}$$

is $\mathrm{lcm}(1,2,\ldots,l)$-periodic.

Let E be an elliptic curve defined over $\mathbb{Q}$ and let $L(E,s)=\sum_{n=1}^{\infty} a_n/n^s$ be the L-function attached to E. Fix $l\in\mathbb{N}$. Then, for every prime p where E has good ordinary reduction, the arithmetic function

$$F_{(l,E,p)}(n)=\frac{|a_{p^n}a_{p^{n+1}}\cdots a_{p^{n+l}}|}{\operatorname{lcm}(a_{p^n},a_{p^{n+1}},\ldots,a_{p^{n+l}})},\qquad n\geq 0,$$

is $\operatorname{lcm}(1,2,\ldots,l)$-periodic.

EXAMPLE 2. Let E be the elliptic curve defined by $y^2=x^3+x$. Let p be a prime. Then E has good ordinary reduction at p if and only if $p\equiv 1 \pmod 4$. Fix $l\in\mathbb{N}$. Hence, for every prime $p\equiv 1 \pmod 4$, the arithmetic function

$$F_{(l,E,p)}(n)=\frac{|a_{p^n}a_{p^{n+1}}\cdots a_{p^{n+l}}|}{\operatorname{lcm}(a_{p^n},a_{p^{n+1}},\ldots,a_{p^{n+l}})}$$

is $\operatorname{lcm}(1,2,\ldots,l)$-periodic. Let $T_{(l,E,p)}$ be the least period of $F_{(l,E,p)}$. Then, by Proposition 3.2, $T_{(2,E,p)}=2$ for all prime $p\equiv 1 \pmod 4$.

4.4. Other sequences. Let a,b be integers such that $\gcd(a,b)=1$ and $|a|\neq|b|$. For each $n\geq 0$, let $L_n=(a^n-b^n)/(a-b)$. Then it is easy to verify that the sequence L_n is the Lucas sequence $L_{(P,Q)}(n)$ with parameters $P=a+b, Q=-ab$. In particular, if $b=1$, one obtains the sequence $L_n=\frac{a^n-1}{a-1}$, now the parameters are $P=a+1, Q=-a$. Finally, if also $a=2$, one gets $L_n=2^n-1$, and now the parameters are $P=3, Q=-2$.

Since $P^2+4Q=(a-b)^2>0$, by Theorem 1.3, the function $F_{(l,a+b,-ab)}$ is periodic and the least period $T_{(l,a+b,-ab)}$ depends only on l.

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