On irregularities in the graph of generalized divisor functions

by

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1. Introduction. It is partly known [1], partly easy to prove that for the divisor function
\[ d(n) := \sum_{d|n} 1, \] (1)
it is true that for all \( \omega > 0 \) there is an \( n \in \mathbb{N} \) such that
\[ d(n) > \omega + \max(d(n - 1), d(n + 1)) \] (2)
and also there is an \( m \in \mathbb{N} \) such that
\[ d(m) + \omega < \min(d(m - 1), d(m + 1)). \] (3)
P. Erdős [1] proved (2) in the following stronger form: for all \( k \in \mathbb{N} \) there are infinitely many \( n \in \mathbb{N} \) such that
\[ d(n) > \prod_{i=1}^{k} d(n - i)d(n + i). \] (4)
We will extend these theorems to generalized divisor functions \( d(\mathcal{A}, n) \) defined for any set \( \mathcal{A} \subseteq \mathbb{N} \) as
\[ d(\mathcal{A}, n) := \sum_{a \in \mathcal{A}, a|n} 1. \] (5)
These functions were introduced by Erdős and Sárközy [2]. Among other results they proved that for any infinite \( \mathcal{A} \) the large values of \( d(\mathcal{A}, n) \) are much greater than its average:
\[ \limsup_{N \to \infty} \frac{\max_{n \leq N} d(\mathcal{A}, n)}{\sum_{a \in \mathcal{A}, a \leq N} 1/a} = \infty. \] (6)
A. Sárközy posed the following three related problems in [5] (Problems 25–27):

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Problem 1. Is it true that $|d(A, n+1) - d(A, n)|$ cannot be bounded for an infinite set $A \subseteq \mathbb{N}$?

Problem 2. Is it true that for any infinite set $A \subseteq \mathbb{N}$ there are infinitely many $n$ with

$$d(A, n) > \max(d(A, n+1), d(A, n-1))$$

Problem 3. What assumption is needed to ensure that

$$d(A, n) < \min(d(A, n-1), d(A, n+1))$$

for infinitely many $n$?

This article solves these problems and also generalizes Erdős’s theorem.

2. Notation and the lemma. Following [4], we will use the following notations: Let $B \subseteq \mathbb{N}$ be an arbitrary finite sequence, $X := |B|$. Let $P \subseteq \mathbb{N}$ be an arbitrary set of primes. Set

$$P(z) := \prod_{p \in P, p \leq z} p.$$  \hspace{1cm} (7)

$$S(B, P, z) := \{b : b \in B, (b, P(z)) = 1\}.$$ \hspace{1cm} (8)

Let $\omega$ be a multiplicative arithmetical function such that $\omega(n) = 0$ if $n$ is not squarefree and also if $n$ has a prime factor not in $P$, and $\omega(1) := 1$. Let $\gamma$ be Euler’s constant and $\Gamma$ be the well-known Gamma function, $\mu$ be the Möbius function, and $\nu(d)$ be the number of distinct prime divisors of $d$. We define

$$W(z) := \prod_{p \leq z} \left(1 - \frac{\omega(p)}{p}\right).$$ \hspace{1cm} (9)

$$\sigma_\kappa(u) := 2^{-\kappa} \frac{e^{-\gamma \kappa}}{\Gamma(\kappa + 1)} u^\kappa \quad \text{if } 0 \leq u \leq 2,$$ \hspace{1cm} (10)

$$(u^{-\kappa} \sigma_\kappa(u))' := -\kappa u^{-\kappa - 1} \sigma_\kappa(u - 2) \quad \text{if } u > 2,$$ \hspace{1cm} (11)

with $\sigma_\kappa$ required to be continuous at $u = 2$. We set

$$\eta_\kappa(u) := \kappa u^{-\kappa} \int_u^\infty \frac{1}{\sigma_\kappa(t - 1)} dt \quad (u > 1).$$ \hspace{1cm} (12)

$$R_d := |\{b \in B : d \mid b\}| \cdot \frac{\omega(d)}{d} X \quad \text{if } \mu(d) \neq 0.$$ \hspace{1cm} (13)

Let us now define four properties as in [4]:

$(\Omega_1)$: There exists $A_1$ such that $0 \leq \omega(p)/p \leq 1 - 1/A_1$ for all primes $p$.

$(\Omega_2(\kappa, A_2, A_3))$: There exist $\kappa \geq 0$ and $A_2, A_3 \geq 1$ such that
(14) \(- A_2 \leq \sum_{w \leq p < z \text{ prime}} \frac{\omega(p) \log p}{p} - \kappa \log \frac{z}{w} \leq A_3\) \(\text{if } 2 \leq w \leq z.\)

(R): \(|R_d| \leq \omega(d)\) if \(\mu(d) \neq 0\), and \((d, p) = 1\) for all \(p \notin \mathcal{P}\).

(R(\(\kappa, \alpha\))): There exist constants \(0 < \alpha < 1\) and \(A_4, A_5 \geq 1\) such that if \(X \geq 2\) then

\[
\sum_{d < X^\alpha/(\log X)^{A_4}} \mu^2(d) 3^{\nu(d)} |R_d| \leq A_5 \frac{X}{\log^{\kappa+1} X}.
\]

It is not difficult to see that (R(\(\kappa, \alpha\))) is less restrictive than (R) beside (\(\Omega_1\)) (see [4]). The strongest lower bound for \(S(B, \mathcal{P}, z)\) in [4] is the following:

**Lemma 1** (see [4, p. 219]). If (\(\Omega_1\)), (\(\Omega_2(\kappa, A_2, A_3)\)) and (R(\(\kappa, \alpha\))) hold and

\[
z^2 \leq X^\alpha/(\log X)^{A_4} \quad (X \geq 2),
\]

then

\[
S(B, \mathcal{P}, z) \geq X W(z) \left(1 - \eta_{\kappa} \left(\alpha \frac{\log X}{\log z}\right) - A_6 \frac{A_2 (\log \log 3X)^{3\kappa+2}}{\log X}\right)
\]

where \(A_6 \geq 1\) is a constant which depends only on \(\kappa, \alpha, A_1, A_2, A_3, A_4, A_5\).

**3. The results**

**Theorem 1.** Let \(\mathcal{A} = \{a_1 < a_2 < \ldots\} \subseteq \mathbb{N}\) and \(k \in \mathbb{N}\). Then there exist infinitely many \(n \in \mathbb{N}\) such that

\[
d(\mathcal{A}, n) > \prod_{i=1}^{k} d(\mathcal{A}, n - i)d(\mathcal{A}, n + i).
\]

**Proof.** We are going to prove that there exists a constant \(C = C(k) > 0\) such that there are infinitely many \(n\) for which

\[
\prod_{i=1}^{k} d(\mathcal{A}, n - i)d(\mathcal{A}, n + i) < C
\]

and \(d(\mathcal{A}, n)\) can be arbitrarily large for these \(n\)’s. Define

\[
X := \prod_{p \leq 2k+1 \text{ prime}} p^{1 + \left[\log_p k\right]} \prod_{j=1}^{N} a_j,
\]

\[
\mathcal{B} := \left\{\prod_{i=1}^{k} (jX - i)(jX + i) : j \in \{1, \ldots, X\}\right\},
\]

\[
\mathcal{P} := \{p : (p, X) = 1 \text{ prime}\},
\]

\[
\omega(p) := 2k \quad \text{if } p \in \mathcal{P},
\]
and extend $\omega$ multiplicatively to squarefree $d$’s for which $(d,p) = 1$ if $p \not\in \mathcal{P}$.
It is easy to see that $|\mathcal{B}| = X$. Now we should check the conditions we need
for the lemma:

(\Omega_1): Since $0 \leq \omega(p) \leq 2k$ and $p > 2k + 1$ if $\omega(p) \neq 0$, we have

$$0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{2k+1}. \quad (21)$$

(\Omega_2(\kappa, A_2, A_3)): This condition is trivial by the following well-known
statement:

$$\sum_{\text{prime } w \leq p < z} \frac{\log p}{p} = \log \left( \frac{z}{w} \right) + O(1) \quad \text{if } 2 \leq w \leq z \quad (22)$$

because $0 \leq \omega(p) \leq 2k$, and $\omega(p) = 2k$ if $p > 2k + 1$.

(R(\kappa, \alpha)): It is enough to prove (R) because it is more restrictive beside
(\Omega_1). Suppose that $d = \prod_{r=1}^{l} p_r$ where $p_r \in \mathcal{P}$ are distinct primes. We can get $|\{b \in \mathcal{B} : d \mid b\}|$ by counting how many $j \in \{1, \ldots, X\}$ there exist such that $p_r \mid jX + i_r$ for fixed $i_r \in \{1, \ldots, k, -1, -2, \ldots, -k\}$ for all $1 \leq r \leq l$. Now $(X, d) = 1$ and this condition holds for $j$ if and only if it does for $j + d$,
so there are $[X/d]$ or $[X/d] + 1$ pieces of such $j$’s. Hence if we take it $X/d$
then the bias is at most 1. There are $(2k)^l = \omega(d)$ choices for the $i_r$’s and therefore $|R_d| \leq \omega(d)$.

Now we can use the lemma. Let $z = X^{1/c}$ and choose $c$ such that

$$z^2 \leq \frac{X}{\log X^{A_4}}, \quad (23)$$

$$\eta_\kappa \left( \frac{\alpha \log X}{\log z} \right) = \eta_\kappa(\alpha c) < 1 \quad (24)$$

for $X$ large enough. Such a $c$ exists because $\eta_\kappa$ is a decreasing function with
limit 0 at $+\infty$. Now we choose $N$ large enough and

$$N > \left( 2^{4kc} \prod_{p \leq k \text{ prime}} (2[k/p][\log_p k] + 1) \right)^{2k} \quad (25)$$

Then

$$1 - \eta_\kappa \left( \frac{\alpha \log X}{\log z} \right) - A_6 \frac{A_2(\log \log 3X)^{3\kappa+2}}{\log X} > 0. \quad (25)$$

So we can conclude from the lemma that $S(\mathcal{B}, \mathcal{P}, z) > 0$, which means
that there exists $b \in \mathcal{B}$ with $(b, p) = 1$ if $p \in \mathcal{P}$ and $p \leq z$, and $b = \prod_{i=1}^{k} (jX - i)(jX + i)$ for some $j \in \{1, \ldots, X\}$. In view of the lemma below, $n = jX$ is a good choice for the theorem.

**LEMMA 2.** We have

$$d(A, jX \pm i) \leq d(A, b) \leq d(b) \leq 2^{4kc} \prod_{p \leq k \text{ prime}} (2[k/p][\log_p k] + 1).$$
Proof. The first two inequalities are trivial. For the third one we use the formula 
\[ d(\prod_{i=1}^{m} p_{i}^{\alpha_{i}}) = \prod_{i=1}^{m} (\alpha_{i} + 1) \] :

1. If \( p \leq k \) then \( p^{1+[\log_{p} k]} | X \) so only \( 2[k/p] \) factors in 
   \[ b = \prod_{i=1}^{k} (jX - i)(jX + i) \]
   are divisible by \( p \) and all of them contain at most \( [\log_{p} k] \) factors \( p \) because \( p^{1+[\log_{p} k]} > k \).

2. If \( k < p \) and \( p | X \) then \((p, b) = 1\).

3. If \( k < p \) and \((p, X) = 1\) then \( p \in \mathcal{P} \). So if \( p \leq z \) then \((p, b) = 1\) else these primes give at most a multiplier of \( 2^{4kc} \) in \( d(b) \) because \( b < X^{4k} = z^{4kc} \leq p^{4kc} \).

Now the proof of the theorem can be completed: For \( n = jX \),

\[ d(A, n) \geq N > 2^{4kc} \prod_{p \leq k \text{ prime}} (2[k/p][\log_{p} k] + 1)^{2k} \]

\[ \geq \prod_{i=1}^{k} d(A, n - i)d(A, n + i). \]

From this theorem we know that the generalized divisor functions have isolated large values. One may ask: what about the isolated small values? The set \( \mathcal{A} = \{ a : a \in \mathbb{N}, 3 | a \} \) shows that it may occur that (27) never holds. The following two theorems answer the question by giving a necessary and sufficient condition on \( \mathcal{A} \).

**Theorem 2.** There are infinitely many \( n \in \mathbb{N} \) such that 
\[ d(A, n) < \min(d(A, n - 1), d(A, n + 1)) \] if and only if there exist \( a, b \in \mathcal{A} \) (not necessarily distinct) such that \( a, b > 1 \) and \((a, b) \leq 2\).

**Proof.** One direction is trivial because if there exists an \( n \in \mathbb{N} \) such that (27) holds then \( n - 1 \) and also \( n + 1 \) must have a divisor in \( \mathcal{A} \); the two divisors are greater than 1 and their greatest common divisor is at most 2.

For the other direction assume that \( a, b \in \mathcal{A} \) are such that \( a, b > 1 \) and \((a, b) \leq 2\). From the Chinese Remainder Theorem we know that there is a residue-class \( \mod[a, b] \) which is congruent to 1 (mod \( a \)) and \(-1 \) (mod \( b \)). From Dirichlet’s theorem we see that there are infinitely many prime numbers in this residue-class. If infinitely many of these primes do not belong to \( \mathcal{A} \) then we are done. If all but finitely many of these primes belong to \( \mathcal{A} \) then let \( p_{1} < p_{2} < p_{3} < p_{4} \) be such primes from the set \( \mathcal{A} \).
Applying again the Chinese Remainder Theorem and Dirichlet’s theorem we find that there are infinitely many primes $p$ such that $p \equiv 1 \pmod{p_1 p_2}$ and $p \equiv -1 \pmod{p_3 p_4}$ and for these primes $n = p$ satisfies (27).

**Theorem 3.** For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ such that

$$d(A, n) + \omega < \min(d(A, n - 1), d(A, n + 1))$$

if and only if for all $k \in \mathbb{N}$ there exist $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$ so that $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$, $([a_1, \ldots, a_k], [b_1, \ldots, b_k]) \leq 2$ and all $a_i, b_j > 1$.

**Proof.** One direction is trivial: if (28) holds for all $\omega$ with some $n \in \mathbb{N}$ then we choose $k = [\omega] + 1$, the numbers $n + 1$ and $n - 1$ have at least $k$ divisors ($> 1$) in $A$, and these $2k$ elements satisfy the condition.

To prove the other direction we use the Chinese Remainder Theorem and Dirichlet’s theorem to deduce that there are infinitely many prime numbers $p$ for which the following two relations hold for all $i, j \in \{1, \ldots, k\}$:

$$a_i \mid p - 1,$$

$$b_j \mid p + 1.$$ 

Now $n = p$ satisfies (28) with $\omega = k - 1$, and since $k$ was an arbitrary natural number, the proof is complete.

4. **Corollaries**

**Corollary 1** (Theorem of Erdős, see [1] and [3, p. 277]). For the divisor function $d(n)$, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with

$$d(n) > \prod_{i=1}^{k} d(n - i)d(n + i).$$

**Proof.** Choose $A = \mathbb{N}$ and apply Theorem 1.

**Corollary 2.** For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with

$$d(n) + \omega < \min(d(n - 1), d(n + 1)).$$

**Proof.** Choose $A = \mathbb{N}$ and apply Theorem 3.

**Corollary 3.** For the number $\nu(n)$ of distinct prime divisors, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with

$$\nu(n) > \prod_{i=1}^{k} \nu(n - i)\nu(n + i).$$

**Proof.** Choose $A = \{ p \in \mathbb{N} : \text{prime} \}$ and apply Theorem 1.

**Corollary 4.** For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with

$$\nu(n) + \omega < \min(\nu(n - 1), \nu(n + 1)).$$

**Proof.** Choose $A = \{ p \in \mathbb{N} : \text{prime} \}$ and apply Theorem 3.
Corollary 5. For the total number $\Omega(n)$ of prime divisors, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with

$$\Omega(n) > \prod_{i=1}^{k} \Omega(n - i) \Omega(n + i).$$

Proof. Choose $A = \{ q \in \mathbb{N} : \text{prime or power of a prime} \}$ and apply Theorem 1.

Corollary 6. For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with

$$\Omega(n) + \omega < \min(\Omega(n - 1), \Omega(n + 1)).$$

Proof. Choose $A = \{ q \in \mathbb{N} : \text{prime or power of a prime} \}$ and apply Theorem 3.

Corollary 7 (Problem of Sárközy, see [5, Problem 25]). For every infinite set $A \subseteq \mathbb{N}$, the sequence $|d(A, n + 1) - d(A, n)|$ cannot be bounded.

Proof. Apply Theorem 1 for the set $A \cup \{1\}$.

Corollary 8. For every infinite set $A \subseteq \mathbb{N}$ and any $\omega > 0$ there are infinitely many $n$ with

$$d(A, n) > \omega + \max(d(A, n - 1), d(A, n + 1)).$$

Proof. Apply Theorem 1 for the set $A \cup \{1\}$.

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References


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