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On various mean values of Dirichlet L-functions

by

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1. Introduction. Let χ be a Dirichlet character modulo $k \geq 2$ and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the Dirichlet L-function for $\Re s > 1$. The mean value

$$\sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi}) \tag{1.1}$$

has been studied when $m, n \geq 1$ are positive integers of the same parity. The aim of such studies is to express the mean value in terms of the values of the Riemann zeta-function $\zeta(\ell)$, the Euler function $\phi(k)$ and Jordan's totient functions

$$J_\ell(k) = k^\ell \prod_{p\mid k} (1 - p^{-\ell}), \tag{1.2}$$

where ℓ is an even positive integer. In the case $m = n = 1$ and $k = p$ is a prime, Walum [9] studied (1.1), and Alkan [3], Qi [8], Louboutin [6] and Zhang [10] independently gave the explicit formula

$$\sum_{\chi(-1)=-1} |L(1,\chi)|^2 = \frac{\pi^2 \phi(k)}{12k^2} (J_2(k) - 3\phi(k)) \tag{1.3}$$

for $k \geq 2$. Louboutin [7] proved that the mean values can be expressed as rational linear combinations of products of $\zeta(m+n)$, the Euler function and Jordan's totient functions when $m = n$; however, he did not determine the rational coefficients. Liu and Zhang [5] achieved the determination of the rational coefficients in the result of Louboutin [7], and gave the following

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explicit formula for positive integers $m, n \geq 1$ with the same parity [5, Theorem 1.1]:

$$(1.4)\qquad \sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi}) = \frac{(-1)^{(m-n)/2}(2\pi)^{m+n}\phi(k)}{4m!n!k^{m+n}} \times \left(\sum_{l=1}^{m+n} r_{m,n,l}J_l(k) - \frac{\epsilon_{m,n}}{4}\phi(k)\right),$$

where

$$r_{m,n,l} = B_{m+n-l} \sum_{\substack{0\leq a\leq m,\, 0\leq b\leq n\\ a+b\geq m+n-l}} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1},$$

B_m is the mth Bernoulli number and

$$\epsilon_{m,n} = \begin{cases} 1 & \text{if } m=n=1,\\ 0 & \text{otherwise.}\end{cases}$$

Also, in the case $\chi(-1) = (-1)^n$ and $k > 2$, Alkan [2] proved

$$(1.5)\qquad L(n,\chi) = (-1)^{n+1}\frac{i^n 2^{n-1}\pi^n}{kn!}\sum_{m=1}^{k-1}\chi(m)\sum_{q=0}^{2[n/2]}\binom{n}{q}\frac{B_q}{k^{n-q}}\sum_{j=1}^{n-q}\binom{n-q}{j}k^j \times \lim_{w\to 2\pi i m/k}\frac{d^{n-q-j}}{dw^{n-q-j}}\frac{1}{e^w-1},$$

and gave explicit formulas for $L(n,\chi)$ in terms of twisted trigonometric sums when $1 \leq n \leq 7$. Using (1.5), Alkan [4] gave the explicit formula

$$(1.6)\qquad \sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) = \frac{\phi(k)^2\pi^4}{4}\left(\frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4}\right)$$

for $k > 2$.

In this paper, we consider a generalization of (1.6). Our major purpose is to give an explicit formula for the following mean values (Theorem 1.1):

$$(1.7)\qquad \sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2}).$$

Using (1.5), we give an explicit formula for (1.7); however, it is complicated since we have to compute derivatives of $1/(e^w-1)$. We will apply the following result of Louboutin [7, Proposition 3(1)]:

Let $n \geq 1$ and $k > 2$ be positive integers. Let $\cot^{(n)} x$ denote the nth derivative of $\cot x$. If χ is a Dirichlet character modulo k and $\chi(-1) = (-1)^n$

then

$$L(n,\chi)=\frac{(-1)^{n-1}\pi^n}{2k^n(n-1)!}\sum_{j=1}^{k-1}\chi(j)\cot^{(n-1)}(\pi j/k). \tag{1.8}$$

In the following theorems, we use vectors and matrices. Let $(b_\alpha)_{\alpha=1}^m=(b_1,\dots,b_m)$ be a row vector and $(c_{ij})_{1\le i,j\le m}$ be an $m\times m$ matrix.

Theorem 1.1. *Let $m,n\ge 1$ and $k>2$ be positive integers. Then*

$$\begin{aligned}\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} & L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2})\\ &=\frac{-\phi(k)^2\pi^{2m+2n}}{8k^{2m+2n}(m-1)!(n-1)!(m+n-1)!}\\ &\quad\times(C_\alpha(m,n))_{\alpha=1}^{m+n}\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le m+n}\begin{pmatrix}J_2(k)\\ \vdots\\ J_{2m+2n}(k)\end{pmatrix},\end{aligned}$$

where B_m is the mth Bernoulli number,

$$a_{2i-1,2j}=\begin{cases}(-1)^i4^{i-1} & \text{if } j=1,\\ 2(j-1)(2j-1)a_{2i-3,2(j-1)}-4j^2a_{2i-3,2j} & \text{if } 1<j<i,\\ -(2i-1)! & \text{if } j=i,\\ 0 & \text{if } j>i,\end{cases}$$

and

$$C_\alpha(m,n)=\begin{cases}\displaystyle\sum\sum\sum_{\substack{j_1+j_2+j=\alpha\\ 1\le j_1\le[m/2]\\ 1\le j_2\le[n/2]\\ 1\le j\le[(m+n)/2]}} a_{\sigma(m),2j_1}a_{\sigma(n),2j_2}a_{\sigma(m+n),2j} & \text{if } m,n\in 2\mathbb{N},\\ \displaystyle\sum\sum\sum_{\substack{j_1+j_2+j=\alpha \text{ or } \alpha-1\\ 1\le j_1\le[m/2]\\ 1\le j_2\le[n/2]\\ 1\le j\le[(m+n)/2]}} f(m,n;j_1,j_2,j,\alpha)a_{\sigma(m),2j_1}a_{\sigma(n),2j_2}a_{\sigma(m+n),2j} & \text{otherwise.}\end{cases}$$

Here we set

$$\sigma(m)=\begin{cases}m-1 & \text{if } m\in 2\mathbb{N},\\ m-2 & \text{if } m\in 2\mathbb{N}+1,\end{cases}$$

and $f(m,n;j_1,j_2,j,\alpha)=(-1)^{j_1+j_2+j+\alpha+1}\rho(m;j_1)\rho(n;j_2)\rho(m+n;j)$, where *for positive integers a and b,*

$$\rho(a;b)=\begin{cases}-2b & \textit{if } a\in 2\mathbb{N}+1,\\ 1 & \textit{otherwise.}\end{cases}$$

Also in $C_\alpha(m,n)$, the sum corresponding to j_1 or j_2 does not appear when $m=1$ or $n=1$, that is,

$$C_\alpha(m,1)$$
$$=\begin{cases}\displaystyle\sum\sum_{\substack{j_1+j=\alpha \textit{ or } \alpha-1\\ 1\le j_1\le [m/2]\\ 1\le j\le [(m+1)/2]}}(-1)^{j_1+j+\alpha+1}\rho(m;j_1)\rho(m+1;j)a_{\sigma(m),2j_1}a_{\sigma(m+1),2j} & \textit{if } m>1,\\ (-1)^{\alpha+1} & \textit{if } m=1.\end{cases}$$

We will prove Theorem 1.1 in Section 2. The main idea of the proof is to express $J_2(k),\dots,J_{2n}(k)$ in terms of

$$\sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2}(\pi l/k),\ \dots,\ \sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2n}(\pi l/k)$$

using (1.8). This idea is similar to Alkan's [2], [4], but in our case we use the result of Louboutin. Hence we can obtain Theorem 1.1 which is a generalization of (1.6) without carrying out complicated computations. Also we stress that we can apply this method to various mean values of Dirichlet L-functions. In fact, we can apply it to (1.1) to obtain

Theorem 1.2. *Let $m,n\ge 1$ and $k>2$ be positive integers with $m\equiv n \bmod 2$. If $(m,n)=(1,1)$, then (1.3) holds, while if $(m,n)\neq(1,1)$, we have*

$$\sum_{\chi(-1)=(-1)^m}L(m,\chi)L(n,\overline{\chi})=\frac{\phi(k)\pi^{m+n}}{4k^{m+n}(m-1)!(n-1)!}(D_\alpha(m,n))_{\alpha=1}^{(m+n)/2}$$
$$\times\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le (m+n)/2}\begin{pmatrix}J_2(k)\\ \vdots\\ J_{m+n}(k)\end{pmatrix},$$

where

$$D_\alpha(m,n)=\begin{cases}\displaystyle\sum\sum_{\substack{j_1+j_2=\alpha\\ 1\le j_1\le [m/2]\\ 1\le j_2\le [n/2]}}a_{\sigma(m),2j_1}a_{\sigma(n),2j_2} & \textit{if } m,n\in 2\mathbb{N},\\ \displaystyle\sum\sum_{\substack{j_1+j_2=\alpha \textit{ or } \alpha-1\\ 1\le j_1\le [m/2]\\ 1\le j_2\le [n/2]}}f(m,n;j_1,j_2,\alpha)a_{\sigma(m),2j_1}a_{\sigma(n),2j_2} & \textit{otherwise,}\end{cases}$$

and $f(m,n;j_1,j_2,\alpha)=(-1)^{j_1+j_2+\alpha+1}\rho(m;j_1)\rho(n;j_2)$, *the other notations being the same as in Theorem* 1.1. *Also in* $D_\alpha(m,n)$, *the sum corresponding to* j_1 *or* j_2 *does not appear when* $m=1$ *or* $n=1$.

We will also prove Theorem 1.2 in Section 2. We note that Theorem 1.2 is a well-known result (see (1.3) and (1.4)), but our argument is different. Moreover, in Section 3, we will consider the limiting values for (1.1) and (1.7) by applying Theorems 1.1 and 1.2. We also give several evaluation formulas for (1.1) and (1.7) in Section 4.

Moreover, we can consider the mean value

$$\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} |L(m,n;\chi_1,\chi_2)+L(n,m;\chi_2,\chi_1)|^2, \tag{1.9}$$

where

$$L(s_1,s_2;\chi_1,\chi_2)=\sum_{l_1,l_2=1}^{\infty}\frac{\chi_1(l_1)\chi_2(l_1+l_2)}{l_1^{s_1}(l_1+l_2)^{s_2}}$$

is the double Dirichlet L-function corresponding to Dirichlet characters χ_1,χ_2 modulo $k>2$; this was defined by Akiyama and Ishikawa [1]. This function is absolutely convergent when $\Re s_1,\Re s_2>1$ (see [1]).

We remark that we have the harmonic product formula

$$\begin{aligned}L(m,n;\chi_1,\chi_2)&+L(n,m;\chi_2,\chi_1)\\ &=L(m,\chi_1)L(n,\chi_2)-L(m+n,\chi_1\chi_2).\end{aligned} \tag{1.10}$$

Substituting (1.10) into (1.9), we have

$$\begin{aligned}&\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} |L(m,n;\chi_1,\chi_2)+L(n,m;\chi_2,\chi_1)|^2\\ &=\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}}\big\{|L(m,\chi_1)L(n,\chi_2)|^2+|L(m+n,\chi_1\chi_2)|^2\\ &-L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2})-L(m,\overline{\chi_1})L(n,\overline{\chi_2})L(m+n,\chi_1\chi_2)\big\}.\end{aligned} \tag{1.11}$$

We can give an explicit formula for (1.11) by using Theorems 1.1 and 1.2. For the first term in (1.11), the sum can be written as a product of two sums,

$$\begin{aligned}&\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} |L(m,\chi_1)L(n,\chi_2)|^2\\ &\qquad=\Big(\sum_{\chi_1(-1)=(-1)^m}|L(m,\chi_1)|^2\Big)\Big(\sum_{\chi_2(-1)=(-1)^n}|L(n,\chi_2)|^2\Big),\end{aligned}$$

hence we can apply Theorem 1.2. For the second term in (1.11), we have to note that

$$\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} |L(m+n,\chi_1\chi_2)|^2 = \frac{\phi(k)}{2} \sum_{\chi(-1)=(-1)^{m+n}} |L(m+n,\chi)|^2,$$

since, for any Dirichlet character χ modulo k with $\chi(-1) = (-1)^{m+n}$, there exist $\phi(k)/2$ pairs (χ_1, χ_2) such that

$$\chi(m) = \chi_1(m)\chi_2(m),$$

where χ_1, χ_2 are Dirichlet characters modulo k with $\chi_1(-1) = (-1)^m$, $\chi_2(-1) = (-1)^n$. Therefore we can also apply Theorem 1.2 to the second term. For the third and fourth terms, we can clearly apply Theorem 1.1. Hence we can express the mean values (1.9) in terms of the values of the Riemann zeta-function, the Euler function and the Jordan's totient functions.

2. Proofs of Theorems 1.1 and 1.2. First we will give an evaluation formula for $\cot^{(2n-1)} x$. The notation $a_{2n-1,2j}$ is given in the statement of Theorem 1.1.

LEMMA 2.1. *For any positive integer n,*

$$\cot^{(2n-1)} x = \sum_{j=1}^{n} a_{2n-1,2j} \sin^{-2j} x. \tag{2.1}$$

Proof. For $n = 1$, we have

$$\cot^{(1)} x = -\sin^{-2} x.$$

Hence $a_{1,2} = -1$. Next we assume the validity of (2.1) for $\cot^{(2n-1)} x$, and we will prove it for $\cot^{(2n+1)} x$. We have

$$\cot^{(2n+1)} x = \sum_{j=1}^{n} \{2j(2j+1)a_{2n-1,2j} \sin^{-2j-2} x - 4j^2 a_{2n-1,2j} \sin^{-2j} x\}.$$

Hence

$$\begin{aligned} a_{2n+1,2n+2} &= 2n(2n+1)a_{2n-1,2n} = \cdots = (2n+1)!a_{1,2} = -(2n+1)!,\\ a_{2n+1,2} &= -4a_{2n-1,2} = \cdots = (-4)^n a_{1,2} = (-1)^{n+1}4^n,\\ a_{2n+1,2j} &= 2(j-1)(2j-1)a_{2n-1,2(j-1)} - 4j^2 a_{2n-1,2j} \end{aligned}$$

for $2 \le j \le n+1$. ∎

Since $a_{2n-1,2j} = 0$ for $j > i$, equality (2.1) can be written as

$$\cot^{(2n-1)} x = \sum_{j=1}^{n+L} a_{2n-1,2j} \sin^{-2j} x$$

for any positive integer L.

Next, applying Lemma 2.1 to (1.8), we have

$$L(2n,\chi_0)=\frac{-\pi^{2n}}{2k^{2n}(2n-1)!}\sum_{j=1}^{n}a_{2n-1,2j}\sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2j}(\pi l/k),$$

where χ_0 is a principal Dirichlet character modulo k. On the other hand, from the Euler product expansion, we have

$$L(2n,\chi_0)=\zeta(2n)k^{-2n}J_{2n}(k).$$

Hence

$$J_{2n}(k)=\frac{-\pi^{2n}}{2\zeta(2n)(2n-1)!}(a_{2n-1,2}\ \cdots\ a_{2n-1,2n})\begin{pmatrix}\sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2}(\pi l/k)\\ \vdots\\ \sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2n}(\pi l/k)\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}J_2(k)\\ \vdots\\ J_{2n}(k)\end{pmatrix}=\left(\frac{-\pi^{2i}a_{2i-1,2j}}{2\zeta(2i)(2i-1)!}\right)_{i,j}\begin{pmatrix}\sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2}(\pi l/k)\\ \vdots\\ \sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2n}(\pi l/k)\end{pmatrix}. \tag{2.2}$$

We remark that all diagonal entries of the matrix $\left(\frac{-\pi^{2i}a_{2i-1,2j}}{2\zeta(2i)(2i-1)!}{}_{i,j}\right)$ are non-zero, and the matrix is lower triangular since $a_{2i-1,2j}=0$ whenever $j>i$. Hence it is an invertible matrix.

Now, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Using (1.8) and

$$\sum_{\chi(-1)=(-1)^n}\chi(j_1)\overline{\chi}(j_2)=\begin{cases}\phi(k)/2 & \text{if } j_1\equiv j_2 \bmod k,\ (j_1,k)=1,\\ (-1)^n\phi(k)/2 & \text{if } j_1\equiv -j_2 \bmod k,\ (j_1,k)=1,\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}&\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}}L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2})\\ &\quad=\frac{-\pi^{2m+2n}}{8k^{2m+2n}(m-1)!(n-1)!(m+n-1)!}\\ &\qquad\times\sum_{l_1,l_2,l=1}^{k-1}\cot^{(m-1)}(\pi l_1/k)\cot^{(n-1)}(\pi l_2/k)\cot^{(m+n-1)}(\pi l/k)\end{aligned}$$

$$\times \sum_{\chi_1(-1)=(-1)^m} \chi_1(l_1)\overline{\chi_1}(l) \sum_{\chi_2(-1)=(-1)^n} \chi_2(l_2)\overline{\chi_2}(l)$$
$$= \tfrac{1}{4}C\big(A_1 + (-1)^m A_{21} + (-1)^n A_{22} + (-1)^{m+n} A_3\big),$$

where

$$A_1 = \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \cot^{(m-1)}(\pi l/k)\cot^{(n-1)}(\pi l/k)\cot^{(m+n-1)}(\pi l/k),$$
$$A_{21} = \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \cot^{(m-1)}(\pi - \pi l/k)\cot^{(n-1)}(\pi l/k)\cot^{(m+n-1)}(\pi l/k),$$
$$A_{22} = \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \cot^{(m-1)}(\pi l/k)\cot^{(n-1)}(\pi - \pi l/k)\cot^{(m+n-1)}(\pi l/k),$$
$$A_3 = \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \cot^{(m-1)}(\pi - \pi l/k)\cot^{(n-1)}(\pi - \pi l/k)\cot^{(m+n-1)}(\pi l/k),$$
$$C = \frac{-\phi(k)^2\pi^{2m+2n}}{8k^{2m+2n}(m-1)!(n-1)!(m+n-1)!}.$$

Noting that $\cot^{(n-1)}(\pi - \pi l/k) = (-1)^n \cot^{(n-1)}(\pi l/k)$ for any positive integer n, we obtain

$$\sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2}) = CA_1.$$

In the case $m = 2p_1$ and $n = 2p_2$ with some positive integers p_1 and p_2, we have

$$A_1 = \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \Big(\sum_{j=1}^{p_1+p_2} a_{2p_1+2p_2-1,2j}\sin^{-2j}(\pi l/k)\Big)\prod_{i=1}^{2}\Big(\sum_{j_i=1}^{p_i} a_{2p_i-1,2j_i}\sin^{-2j_i}(\pi l/k)\Big)$$
$$= \left(\sum_{\substack{j_1+j_2+j=\alpha\\ 1\le j_1\le p_1,\,1\le j_2\le p_2,\,1\le j\le p_1+p_2}} a_{2p_1-1,2j_1}a_{2p_2-1,2j_2}a_{2p_1+2p_2-1,2j}\right)_{\alpha=1}^{2p_1+2p_2}$$
$$\times \left(\frac{-\pi^{2i}a_{2i-1,2j}}{2\zeta(2i)(2i-1)!}\right)^{-1}_{1\le i,j\le 2p_1+2p_2} \begin{pmatrix} J_2(k)\\ \vdots \\ J_{4p_1+4p_2}(k)\end{pmatrix},$$

where we have used (2.1) for the first equality and (2.2) for the second. Next we consider the case $m = 2p_1 + 1$ and $n = 2p_2 + 1$ with positive integers p_1 and p_2. Then, by (2.1),

$$\cot^{(2n)} x = -2\sum_{j=1}^{n} j a_{2n-1,2j} \sin^{-2j-1} x \cos x \tag{2.3}$$

for any positive integer n. Hence, by (2.1) and (2.3),

$$A_1 = \left(\sum_{\substack{j_1+j_2+j=\alpha \text{ or } \alpha-1 \\ 1\le j_1\le p_1,\, 1\le j_2\le p_2, \\ 1\le j\le p_1+p_2+1}} (-1)^{j_1+j_2+j+\alpha+1} 4 j_1 j_2 a_{2p_1-1,2j_1} a_{2p_2-1,2j_2} a_{2p_1+2p_2+1,2j} \right)_{\alpha=1}^{2p_1+2p_2+2} \times \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2p_1+2p_2+2} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{4p_1+4p_2+4}(k) \end{pmatrix}.$$

Similarly, for $m = 2p_1$ and $n = 2p_2 + 1$ with positive integers p_1 and p_2, we have

$$A_1 = \left(\sum_{\substack{j_1+j_2+j=\alpha \text{ or } \alpha-1 \\ 1\le j_1\le p_1,\, 1\le j_2\le p_2, \\ 1\le j\le p_1+p_2}} (-1)^{j_1+j_2+j+\alpha+1} 4 j_2 j a_{2p_1-1,2j_1} a_{2p_2-1,2j_2} a_{2p_1+2p_2-1,2j} \right)_{\alpha=1}^{2p_1+2p_2+1} \times \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2p_1+2p_2+1} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{4p_1+4p_2+2}(k) \end{pmatrix}.$$

Also, we consider the case $(m,n) = (2p_1, 1), (2p_1+1, 1), (1,1)$. If $(m,n) = (2p_1, 1)$, we have

$$A_1 = \left(\sum_{\substack{j_1+j=\alpha \text{ or } \alpha-1 \\ 1\le j_1\le p_1,\, 1\le j\le p_1}} -(-1)^{j_1+j+\alpha+1} 2 j a_{2p_1-1,2j_1} a_{2p_1-1,2j} \right)_{\alpha=1}^{2p_1+1} \times \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2p_1+1} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{4p_1+2}(k) \end{pmatrix},$$

if $(m,n) = (2p_1+1, 1)$ then

$$A_1 = \left(\sum_{\substack{j_1+j=\alpha \text{ or } \alpha-1 \\ 1\le j_1\le p_1,\, 1\le j\le p_1+1}} -(-1)^{j_1+j+\alpha+1} 2j_1 a_{2p_1-1,2j_1} a_{2p_1+1,2j} \right)_{\alpha=1}^{2p_1+2} \times \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2p_1+2} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{4p_1+4}(k) \end{pmatrix}$$

and if $(m,n)=(1,1)$ then

$$\begin{aligned} A_1 &= (-a_{12}\ a_{12}) \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2} \begin{pmatrix} J_2(k) \\ J_4(k) \end{pmatrix} \\ &= (1\ \ -1) \left(\frac{-\pi^{2i} a_{2i-1,2j}}{2\zeta(2i)(2i-1)!} \right)^{-1}_{1\le i,j\le 2} \begin{pmatrix} J_2(k) \\ J_4(k) \end{pmatrix}. \end{aligned}$$

Hence, using Euler's formula

$$\zeta(2i) = (-1)^{i-1} \frac{(2\pi)^{2i} B_{2i}}{2(2i)!}, \tag{2.4}$$

we obtain Theorem 1.1. ∎

Proof of Theorem 1.2. Similarly to the proof of Theorem 1.1, we have

$$\sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi}) = \frac{(-\pi)^{m+n}}{4k^{m+n}(m-1)!(n-1)!} \times \sum_{l_1,l_2=1}^{k-1} \cot^{(m-1)}(\pi l_1/k)\cot^{(n-1)}(\pi l_2/k) \sum_{\chi(-1)=(-1)^m} \chi(l_1)\overline{\chi}(l_2) = CA,$$

where

$$A = \sum_{\substack{l=1 \\ (l,k)=1}}^{k-1} \cot^{(m-1)}(\pi l/k)\cot^{(n-1)}(\pi l/k), \qquad C = \frac{\phi(k)(-\pi)^{m+n}}{4k^{m+n}(m-1)!(n-1)!}.$$

Also, if $(m,n)\ne(1,1)$, then

$$A = (D_\alpha(m,n))_{\alpha=1}^{(m+n)/2} \left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1} B_{2i}} \right)^{-1}_{1\le i,j\le (m+n)/2} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{m+n}(k) \end{pmatrix},$$

while for $(m,n)=(1,1)$ we obtain

$$A = \sum_{\substack{l=1 \\ (l,k)=1}}^{k-1} \cot(\pi l/k)\cot(\pi l/k) = \tfrac{1}{3} J_2(k) - \phi(k).$$

Hence Theorem 1.2 follows. ∎

3. Limiting values. Let p_r be the rth prime for a positive integer r and set $k = p_1 \cdots p_r$. Then we have the following corollary of Theorems 1.1 and 1.2.

Corollary 3.1. *Let $m, n \geq 1$. Then*

$$\lim_{\substack{k=p_1\cdots p_r \\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=(-1)^m \\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2}) = \frac{-C_{m+n}(m,n)}{(m-1)!(n-1)!(m+n-1)!} \tag{3.1}$$

and

$$\lim_{\substack{k=p_1\cdots p_r \\ r\to\infty}} \frac{2}{\phi(k)} \sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi}) = \frac{D_{(m+n)/2}(m,n)}{(m-1)!(n-1)!}, \tag{3.2}$$

where $C_{m+n}(m,n)$ and $D_{(m+n)/2}(m,n)$ are given in the statements of Theorems 1.1 *and* 1.2.

If $m, n \geq 2$, we have

$$\lim_{\substack{k=p_1\cdots p_r \\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=(-1)^m \\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2}) = 1 \tag{3.3}$$

and

$$\lim_{\substack{k=p_1\cdots p_r \\ r\to\infty}} \frac{2}{\phi(k)} \sum_{\chi(-1)=(-1)^m} L(m,\chi)L(n,\overline{\chi}) = 1. \tag{3.4}$$

Proof. We only show (3.1) and (3.3). Equalities (3.2) and (3.4) are proved similarly. By Theorem 1.1, we have

$$\begin{aligned} \frac{4}{\phi(k)^2} & \sum_{\substack{\chi_1(-1)=(-1)^m \\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2}) \\ &= \frac{-\pi^{2m+2n}}{2k^{2m+2n}(m-1)!(n-1)!(m+n-1)!}(C_\alpha(m,n))_{\alpha=1}^{m+n} \\ &\quad \times \left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\leq i,j\leq m+n} \begin{pmatrix} J_2(k) \\ \vdots \\ J_{2m+2n}(k) \end{pmatrix}. \end{aligned} \tag{3.5}$$

From (1.2) and $k = p_1 \cdots p_r$, we have

$$\lim_{\substack{k=p_1\cdots p_r \\ r\to\infty}} \frac{J_{2i}(k)}{k^{2i}} = \frac{1}{\zeta(2i)}$$

for a positive integer i. Hence

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{J_{2i}(k)}{k^{2m+2n}} = \begin{cases} 1/\zeta(2(m+n)) & \text{if } i=m+n,\\ 0 & \text{if } i<m+n.\end{cases}$$

Since the matrix on the right-hand side of (3.5) is lower triangular, we have

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2})$$
$$= \frac{(-1)^{m+n-1}\pi^{2m+2n}C_{m+n}(m,n)2^{2m+2n-1}B_{2m+2n}}{2(m-1)!(n-1)!(m+n-1)!(m+n)a_{2m+2n-1,2m+2n}\zeta(2m+2n)}.$$

From $a_{2m+2n-1,2m+2n} = -(2m+2n-1)!$ and (2.4), we obtain (3.1).

Next, let us prove (3.3). If $n\ge 2$ and χ is a Dirichlet character modulo $k=p_1\cdots p_r$, we have

$$L(n,\chi) = 1 + \sum_{l=p_{r+1}}^{\infty} \frac{\chi(l)}{l^n} = 1 + O(p_{r+1}^{1-n}).$$

Hence

$$\frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} L(m,\chi_1)L(n,\chi_2)L(m+n,\overline{\chi_1\chi_2})$$
$$= \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=(-1)^m\\ \chi_2(-1)=(-1)^n}} (1+O(p_{r+1}^{1-m}))(1+O(p_{r+1}^{1-n}))(1+O(p_{r+1}^{1-m-n}))$$
$$= 1 + O(p_{r+1}^{-1}). \ \blacksquare$$

4. Examples. We now give several evaluation formulas for (1.7) when $2\le m+n\le 4$:

In the case $(m,n)=(1,1)$, from

$$(C_\alpha(1,1))_{\alpha=1}^2 = (1\ \ -1)$$

and

$$\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)_{1\le i,j\le 2}^{-1} = \begin{pmatrix} 1/3 & 0\\ 2/9 & 1/45\end{pmatrix},$$

we have

$$\sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) = \frac{\phi(k)^2\pi^4}{8k^4}\left(\frac{1}{45}J_4(k) - \frac{1}{9}J_2(k)\right).$$

This coincides with Alkan's result. Also

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(1,\chi_2)L(2,\overline{\chi_1\chi_2}) = 1$$

by Corollary 3.1.

In the case $(m,n)=(1,2)$, from

$$(C_\alpha(1,2))_{\alpha=1}^3 = (0\ \ 2\ \ -2)$$

and

$$\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le 3} = \begin{pmatrix} 1/3 & 0 & 0 \\ 2/9 & 1/45 & 0 \\ 8/45 & 1/45 & 2/945 \end{pmatrix},$$

we have

$$\sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=1}} L(1,\chi_1)L(2,\chi_2)L(3,\overline{\chi_1\chi_2}) = \frac{\phi(k)^2\pi^6}{16k^6}\left(\frac{4}{945}J_6(k) - \frac{4}{45}J_2(k)\right)$$

and

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=1}} L(1,\chi_1)L(2,\chi_2)L(3,\overline{\chi_1\chi_2}) = 1.$$

In the case $(m,n)=(2,2)$, in view of

$$(C_\alpha(2,2))_{\alpha=1}^4 = (0\ \ 0\ \ 4\ \ -6)$$

and

$$\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le 4} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 2/9 & 1/45 & 0 & 0 \\ 8/45 & 1/45 & 2/945 & 0 \\ 16/105 & 14/675 & 8/2835 & 1/4725 \end{pmatrix},$$

we have

$$\sum_{\substack{\chi_1(-1)=1\\ \chi_2(-1)=1}} L(2,\chi_1)L(2,\chi_2)L(4,\overline{\chi_1\chi_2})$$
$$= \frac{\phi(k)^2\pi^8}{48k^8}\left(\frac{2}{1575}J_8(k) + \frac{8}{945}J_6(k) + \frac{8}{225}J_4(k) + \frac{64}{315}J_2(k)\right)$$

and

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=1\\ \chi_2(-1)=1}} L(2,\chi_1)L(2,\chi_2)L(4,\overline{\chi_1\chi_2}) = 1.$$

In the case $(m,n)=(1,3)$, similarly, we have

$$\sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(3,\chi_2)L(4,\overline{\chi_1\chi_2})$$
$$= \frac{\phi(k)^2\pi^8}{96k^8}\left(\frac{4}{1575}J_8(k)-\frac{8}{945}J_6(k)-\frac{4}{225}J_4(k)+\frac{16}{315}J_2(k)\right)$$

and

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{4}{\phi(k)^2} \sum_{\substack{\chi_1(-1)=-1\\ \chi_2(-1)=-1}} L(1,\chi_1)L(3,\chi_2)L(4,\overline{\chi_1\chi_2}) = 1.$$

Lastly, we give several evaluation formulas for (1.1) in the case $m+n=4$: In the case $(m,n)=(2,2)$, since

$$(D_\alpha(2,2))_{\alpha=1}^2 = (0\ \ 1)$$

and

$$\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le 2} = \begin{pmatrix} 1/3 & 0 \\ 2/9 & 1/45 \end{pmatrix},$$

we have

$$\sum_{\chi(-1)=1} |L(2,\chi)|^2 = \frac{\phi(k)\pi^4}{4k^4}\left(\frac{1}{45}J_4(k)+\frac{2}{9}J_2(k)\right)$$

and

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{2}{\phi(k)} \sum_{\chi(-1)=1} |L(2,\chi)|^2 = 1.$$

In the case $(m,n)=(1,3)$, as

$$(D_\alpha(1,3))_{\alpha=1}^2 = (-2\ \ 2)$$

and

$$\left(\frac{(-1)^i i a_{2i-1,2j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\le i,j\le 2} = \begin{pmatrix} 1/3 & 0 \\ 2/9 & 1/45 \end{pmatrix},$$

we have

$$\sum_{\chi(-1)=-1} L(1,\chi)L(3,\overline{\chi}) = \frac{\phi(k)\pi^4}{8k^4}\left(\frac{2}{45}J_4(k)-\frac{2}{9}J_2(k)\right)$$

and

$$\lim_{\substack{k=p_1\cdots p_r\\ r\to\infty}} \frac{2}{\phi(k)} \sum_{\chi(-1)=-1} L(1,\chi)L(3,\overline{\chi}) = 1.$$

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References

[1] S. Akiyama and H. Ishikawa, *On analytic continuation of multiple L-functions and related zeta-functions*, in: Analytic Number Theory, C. Jia and K. Matsumoto (eds.), Kluwer, 2002, 1–16.

[2] E. Alkan, *Values of Dirichlet L-functions, Gauss sums and trigonometric sums*, Ramanujan J. 26 (2011), 375–398.

[3] E. Alkan, *On the mean square average of special values of L-functions*, J. Number Theory 131 (2011), 1470–1485.

[4] E. Alkan, *Averages of values of L-series*, Proc. Amer. Math. Soc. 141 (2013), 1161–1175.

[5] H. Liu and W. Zhang, *On the mean value of* $L(m,\chi)L(n,\overline{\chi})$ *at positive integers* $m,n \geq 1$, Acta Arith. 122 (2006), 51–56.

[6] S. Louboutin, *Quelques formules exactes pour des moyennes de fonctions L de Dirichlet*, Canad. Math. Bull. 36 (1993), 190–196; Addendum, ibid. 37 (1994), 89.

[7] S. Louboutin, *The mean value of* $|L(k,\chi)|^2$ *at positive rational integers* $k \geq 1$, Colloq. Math. 90 (2001), 69–76.

[8] M. G. Qi, *A class of mean square formulas for L-functions*, J. Tsinghua Univ. 31 (1991), 34–41.

[9] H. Walum, *An exact formula for an average of L-series*, Illinois J. Math. 26 (1982), 1–3.

[10] W. Zhang, *On the mean values of Dedekind sums*, J. Théor. Nombres Bordeaux 8 (1996), 429–442.

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