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A relative trace formula proof of the Petersson trace formula

by

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1. Introduction. The Petersson trace formula relates spectral data coming from cusp forms to Kloosterman sums and Bessel functions. It was discovered in 1932 [Pe] long before Selberg's trace formula and can be regarded as the first type of trace formula for automorphic forms. It has proven to be an indispensable tool for estimating the size of the Fourier coefficients of modular forms in many situations. See for example [Se], [IK], and Section 5 of [Iw].

In this paper we will use the relative trace formula to prove a variant of the Petersson trace formula. The resulting generalized formula relates Hecke eigenvalues, Fourier coefficients and Petersson norms of cusp forms (on the spectral side) to Bessel functions and Kloosterman sums (on the geometric side). To state this result, let $S_{\mathtt{k}}(N,\omega')$ be the space of cusp forms of level N, weight $\mathtt{k}>2$, and Nebentypus ω' (see Section 3). For an integer $\mathtt{n}$ which is prime to N, let $\mathcal{F}$ be an orthogonal basis consisting of eigenfunctions for the Hecke operator $T_{\mathtt{n}}$. Then (see Theorem 3.9)

$$\frac{\psi(N)^{-1}(\mathtt{k}-2)!}{(4\pi\sqrt{\mathtt{n}m_1m_2})^{\mathtt{k}-1}}\sum_{h\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(h)a_{m_1}(h)\overline{a_{m_2}(h)}}{\|h\|^2} = T(m_1,m_2,\mathtt{n})\omega'(\sqrt{m_1\mathtt{n}/m_2})^{-1} + \frac{2\pi}{i^{\mathtt{k}}}\sum_{\substack{c>0\\ N|c}}\frac{1}{c}\,S_{\omega'}(m_2,m_1;\mathtt{n};c)J_{\mathtt{k}-1}\biggl(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\biggr),$$

where $a_m(h)$ is the mth Fourier coefficient of h, $\lambda_{\mathtt{n}}(h)$ is the eigenvalue of $T_{\mathtt{n}}$ relative to h, $J_{\mathtt{k}-1}$ is a Bessel function, $S_{\omega'}(m_2,m_1;\mathtt{n};c)$ is a generalized Kloosterman sum defined in (12), $\psi(N)=[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$, the Petersson norm $\|h\|$ is normalized in (3) below, and $T(a_1,a_2,a_3)\in\{0,1\}$ is nonzero if and only if a_ia_j/a_k is a perfect square for all distinct $i,j,k\in\{1,2,3\}$.

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In the special case $\mathtt{n} = 1$, we recover the classical Petersson trace formula (see Corollary 3.12 in the last section). The chief difference is that the above formula includes Hecke eigenvalues, whereas the classical version involves only the Fourier coefficients. Of course, for $\mathrm{GL}_2(\mathbb{Q})$ the two concepts are essentially the same. In the last section, we will briefly explain how the generalized formula can also be derived from the classical formula.

The modern theory of modular forms uses the viewpoint of representation theory. In this context, much has been done by the experts to develop various kinds of trace formulas for studying the spectral data attached to automorphic forms. Two noteworthy examples are Arthur's generalization of the Selberg trace formula to higher rank groups (cf. [Ar] and its references), and Jacquet's relative trace formulas obtained by integrating kernel functions over different subsets (refer to [Ja] and the bibliography of Lecture VIII in [Ge2]). These tools are very well suited for determining the nature of the functorial connections between the cuspidal representations of two different groups. However such abstract works can seem far removed from the realm of classical modular forms. The present proof of the Petersson trace formula illustrates a method which takes such work into an explicit form which is useful analytically in the classical sense.

This approach suggests itself for generalization to other groups, where:

- a reproduction of the classical argument (via Poincaré series) would be much more complicated,
- the relationship between Hecke eigenvalues and Fourier coefficients is not as transparent as it is for $\mathrm{GL}_2(\mathbb{Q})$.

We hope to pursue this idea in future work.

2. General setting. Let F be a number field, with adele ring $\mathbb{A}$. Let G be a reductive group over F. Let H be an abelian subgroup of $G \times G$. We assume that $H(F)\backslash H(\mathbb{A})$ is compact. Define a right action of H on G by $g(x,y) = x^{-1}gy$. For $g \in G$, let H_g be the stabilizer of g, i.e.

$$H_g = \{(x,y) \in H \mid x^{-1}gy = g\}.$$

For $\delta \in G(F)$, let $[\delta]$ be the $H(F)$-orbit of δ in $G(F)$, i.e.

$$[\delta] = \{x^{-1}\delta y \mid (x,y) \in H(F)\}.$$

Each element of $[\delta]$ can be expressed uniquely in the form $u^{-1}\delta v$ for some $(u,v) \in H_\delta(F)\backslash H(F)$.

Let f be a continuous function on $G(\mathbb{A})$, and let

$$K(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \qquad (x,y \in G(\mathbb{A})) \tag{1}$$

be the associated kernel function. We assume that the above sum is uniformly absolutely convergent on compact subsets of $H(\mathbb{A})$. In particular, $K(x,y)$ is a continuous function on the compact set $H(F)\backslash H(\mathbb{A})$. Let $\chi(x,y)$ be a character of $H(\mathbb{A})$, invariant under $H(F)$. Consider the expression

$$\int_{H(F)\backslash H(\mathbb{A})} K(x,y)\chi(x,y)\,d(x,y), \tag{2}$$

where $d(x,y)$ is an $H(\mathbb{A})$-invariant measure. A relative trace formula results from computing this integral using spectral and geometric expressions for $K(x,y)$.

Using the geometric expression (1), the integral (2) can be rewritten as

$$\begin{aligned}\int_{H(F)\backslash H(\mathbb{A})} &\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\chi(x,y)\,d(x,y)\\ &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{[\delta]}\sum_{\gamma\in[\delta]} f(x^{-1}\gamma y)\chi(x,y)\,d(x,y)\\ &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{[\delta]}\sum_{(u,v)\in H_\delta(F)\backslash H(F)} f(x^{-1}u^{-1}\delta v y)\chi(x,y)\,d(x,y)\\ &= \sum_{[\delta]}\int_{H_\delta(F)\backslash H(\mathbb{A})} f(x^{-1}\delta y)\chi(x,y)\,d(x,y).\end{aligned}$$

The last step follows because χ is $H(F)$-invariant. Let

$$I_\delta(f) = \int_{H_\delta(F)\backslash H(\mathbb{A})} f(x^{-1}\delta y)\chi(x,y)\,d(x,y)$$

so that (2) is equal to $\sum_{[\delta]} I_\delta$. An orbit $[\delta]$ is *relevant* if χ is trivial on $H_\delta(\mathbb{A})$.

PROPOSITION 2.1. *If $[\delta]$ is not relevant, then $I_\delta = 0$.*

Proof. If $[\delta]$ is not relevant, there exists $(u,v)\in H_\delta(\mathbb{A})$ such that $\chi(u,v) \neq 1$. Because H is abelian and the measure is $H(\mathbb{A})$-invariant, we have

$$\begin{aligned} I_\delta(f) &= \int_{H_\delta(F)\backslash H(\mathbb{A})} f(x^{-1}u^{-1}\delta v y)\chi(ux,vy)\,d(ux,vy)\\ &= \chi(u,v)\int_{H_\delta(F)\backslash H(\mathbb{A})} f(x^{-1}\delta y)\chi(x,y)\,d(x,y) = \chi(u,v)I_\delta(f).\end{aligned}$$

It follows that $I_\delta(f)=0$. ■

3. The Petersson trace formula

3.1. *Background and notation.* In this section we recall various facts and notation from [KL]. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$, let $G=\mathrm{GL}(2)$, and let Z be the center of G. We write $\overline{G}$ for G/Z. Fix a weight $\mathtt{k}$ and a level N, and let ω' be a Dirichlet character with conductor dividing N. For $\gamma\in\Gamma_0(N)$, define

$$\omega'(\gamma) = \omega'(d), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $S_{\mathtt{k}}(N, \omega')$ be the space of cusp forms on $\Gamma_0(N)$ satisfying

$$h(\gamma z) = \omega'(\gamma)^{-1} j(\gamma, z)^{\mathtt{k}} h(z)$$

for all $\gamma \in \Gamma_0(N)$. To allow for the possibility of nonzero cusp forms, we assume

$$\omega'(-1) = (-1)^{\mathtt{k}}.$$

We normalize the Petersson inner product by taking

(3) $$\|h\|^2 = \frac{1}{\psi(N)} \iint_{\Gamma_0(N)\backslash \mathbf{H}} |h(z)|^2 y^{\mathtt{k}} \frac{dx\,dy}{y^2},$$

where $\mathbf{H}$ is the complex upper half-plane, and

$$\psi(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Using the decomposition $\mathbb{A}^* = \mathbb{Q}^*(\mathbb{R}_+^* \times \widehat{\mathbb{Z}}^*)$, we associate a Hecke character ω to ω':

$$\omega : \mathbb{A}^* \to \widehat{\mathbb{Z}}^* \to (\mathbb{Z}/N\mathbb{Z})^* \xrightarrow{\omega'} \mathbb{C}^*.$$

For an idele x, let x_N denote the idele which agrees with x at the places $p \,|\, N$, and which is 1 at all other places. Then for any integer d prime to N,

(4) $$\omega(d_N) = \omega'(d).$$

It is also straightforward to check that for $x \in \mathbb{R}$,

(5) $$\omega_\infty(x) = \mathrm{sgn}(x)^{\mathtt{k}}.$$

Let $L^2(\omega)$ be the Hilbert space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ with central character ω which are square integrable over $\overline{G}(\mathbb{Q})\backslash \overline{G}(\mathbb{A})$. The inner product depends on a choice of Haar measure, which we normalize so that the measure of $\overline{G}(\mathbb{Q})\backslash \overline{G}(\mathbb{A})$ is $\pi/3$ (see [KL] for details). Let $L_0^2(\omega)$ be the subspace of cuspidal functions. Let R denote the right regular representation of $G(\mathbb{A})$ on $L^2(\omega)$. Define

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid c \in N\widehat{\mathbb{Z}} \right\}.$$

Then by strong approximation, $G(\mathbb{A}) = G(\mathbb{Q})(G(\mathbb{R})^+ \times K_0(N))$.

With the Petersson inner product normalized as above, there is an isometric embedding $S_{\mathtt{k}}(N, \omega') \hookrightarrow L_0^2(\omega)$ given by

$$h \mapsto \varphi_h, \qquad \varphi_h(\gamma(g_\infty \times k)) = \omega(k) j(g_\infty, i)^{-\mathtt{k}} h(g_\infty(i)),$$

for $\gamma \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})^+$, and $k \in K_0(N)$. Here $\omega(k) = \omega(d_N)$ if $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(N)$.

Fix an integer $\mathtt{n} > 0$, prime to N. In [KL] we defined an operator $R(f)$ on $L^2(\omega)$ which factors through the orthogonal projection to $S_{\mathtt{k}}(N,\omega)$ and acts like the Hecke operator $T_{\mathtt{n}}$ on $S_{\mathtt{k}}(N,\omega)$. The function $f = f_\infty \times f_{\mathrm{fin}}$ is constructed as follows: Let

$$M(\mathtt{n},N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathbb{Z}}) \;\middle|\; \det(g) \in \mathtt{n}\widehat{\mathbb{Z}}^* \text{ and } c \equiv 0 \bmod N\widehat{\mathbb{Z}} \right\}.$$

Then

$$f_{\mathrm{fin}} = f^{\mathtt{n}} : G(\mathbb{A}_{\mathrm{fin}}) \to \mathbb{C}$$

is the Hecke operator supported on $Z(\mathbb{A}_{\mathrm{fin}})M(\mathtt{n},N)$ defined by

$$f^{\mathtt{n}}(zm) = \frac{\psi(N)}{\omega(z)\omega(m)},$$

where $\omega(m) = \omega(d_N)$ for $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(\mathtt{n},N)$, and $\omega(z) = \omega(1_\infty \times z_{\mathrm{fin}})$ for $z = \left(\begin{smallmatrix} z_{\mathrm{fin}} & \\ & z_{\mathrm{fin}} \end{smallmatrix}\right) \in Z(\mathbb{A}_{\mathrm{fin}})$. The following is easily established (cf. [KL]):

Lemma 3.1. *Suppose $g \in G(\mathbb{A}_{\mathrm{fin}})$, and $\det g \in \mathtt{n}\widehat{\mathbb{Z}}^*$. Then $f^{\mathtt{n}}(g) \neq 0$ if and only if $g \in M(\mathtt{n},N)$.*

Let $\pi_{\mathtt{k}}$ denote the weight $\mathtt{k}$ discrete series representation of $G(\mathbb{R})$, and let v_0 be a lowest weight unit vector in the space of $\pi_{\mathtt{k}}$. For $g \in G(\mathbb{R})$, let $f_{\mathtt{k}}(g) = \langle \pi_{\mathtt{k}}(g)v_0, v_0 \rangle$ be the matrix coefficient attached to v_0. Define

$$f_\infty = d_{\mathtt{k}} \overline{f}_{\mathtt{k}},$$

where $d_{\mathtt{k}}$ is the formal degree of $\pi_{\mathtt{k}}$. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$f_\infty(g) = \begin{cases} \dfrac{\mathtt{k}-1}{4\pi} \dfrac{\det(g)^{\mathtt{k}/2}(2i)^{\mathtt{k}}}{(-b+c+(a+d)i)^{\mathtt{k}}} & \text{if } \det(g) > 0, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

(see [KL]).

Let $f = f_\infty f^{\mathtt{n}}$. Define an operator $R(f)$ on $L^2(\omega)$ by

$$R(f)\phi(x) = \int_{\overline{G}(\mathbb{A})} f(g)\phi(xg)\,dg.$$

Then as shown in [KL], when $\mathtt{k} > 2$ we have the following commutative diagram:

$$\begin{array}{ccc} L^2(\omega) & \xrightarrow{\;\mathtt{n}^{\mathtt{k}/2-1}R(f)\;} & L^2(\omega) \\ {\scriptstyle \text{orthog. proj.}}\downarrow & & \uparrow \\ S_{\mathtt{k}}(N,\omega') & \xrightarrow{\;T_{\mathtt{n}}\;} & S_{\mathtt{k}}(N,\omega') \end{array}$$

The kernel of the operator $R(f)$ is given by

$$K(x,y)=\sum_{\gamma\in\overline{G}(\mathbb{Q})} f(x^{-1}\gamma y).$$

This is the geometric expansion of $K(x,y)$. As shown in [KL], we also have a spectral expansion

$$K(x,y)=\sum_{h\in\mathcal{F}}\frac{R(f)\varphi_h(x)\,\overline{\varphi_h(y)}}{\|\varphi_h\|^2},$$

where $\mathcal{F}$ is any orthogonal basis for $S_{\mathtt{k}}(N,\omega')$. Suppose $\mathcal{F}$ consists of eigenvectors for the Hecke operator $T_{\mathtt{n}}$. The existence of such a basis is guaranteed by the fact that $\omega'(\mathtt{n})^{1/2}T_{\mathtt{n}}$ is self-adjoint relative to the Petersson inner product. In this case, $R(f)\varphi_h(x)=\mathtt{n}^{1-\mathtt{k}/2}\lambda_{\mathtt{n}}(h)\varphi_h(x)$, so

$$K(x,y)=\mathtt{n}^{1-\mathtt{k}/2}\sum_{h\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(h)\varphi_h(x)\overline{\varphi_h(y)}}{\|h\|^2}. \tag{7}$$

3.2. *The spectral side.* Define a unitary character $\theta:\mathbb{A}\to\mathbb{C}^*$ by

$$\theta_\infty(x)=e^{-2\pi ix},\quad x\in\mathbb{R},\quad\text{and}\quad \theta_p(x)=e^{2\pi ir(x)},\quad x\in\mathbb{Q}_p,$$

where $r(x)\in\mathbb{Q}$ is the principal part of x, a number with p-power denominator characterized (up to $\mathbb{Z}_p$) by $x\in r(x)+\mathbb{Z}_p$. Then θ is trivial on $\mathbb{Q}$ and $\theta_{\text{fin}}=\prod_p\theta_p$ is trivial precisely on $\widehat{\mathbb{Z}}$. In particular, for any $q\in\mathbb{Q}$, $\theta_{\text{fin}}(q)=\theta_\infty(q)^{-1}=e^{2\pi iq}$.

Let $N=\left\{\left(\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right)\right\}$ be the unipotent subgroup of G. The Petersson trace formula will result from applying the technique in Section 2 to the above kernel function, taking $H(\mathbb{A})=N(\mathbb{A})\times N(\mathbb{A})$. We use the usual Lebesgue measure on $\mathbb{R}$, and normalize Haar measure on $\mathbb{A}$ so that $\text{meas}(\mathbb{Q}\backslash\mathbb{A})=\text{meas}(N(\mathbb{Q})\backslash N(\mathbb{A}))=1$. In particular, this implies $\text{meas}(\widehat{\mathbb{Z}})=1$.

We need to fix a character on $N(\mathbb{Q})\backslash N(\mathbb{A})\times N(\mathbb{Q})\backslash N(\mathbb{A})$. This amounts to choosing two characters on $\mathbb{Q}\backslash\mathbb{A}$. Recall that every character on $\mathbb{Q}\backslash\mathbb{A}$ is of the form

$$\theta_m(x)=\theta(-mx)$$

for some $m\in\mathbb{Q}$. This identifies a character on $N(\mathbb{Q})\backslash N(\mathbb{A})$ in the obvious way. For two rational numbers m_1,m_2, we shall compute the integral

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})}\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} K(n_1,n_2)\,\overline{\theta_{m_1}(n_1)}\,\theta_{m_2}(n_2)\,dn_1\,dn_2. \tag{8}$$

We begin with the spectral side, where one immediately sees the motivation for integrating this kernel over $H(\mathbb{Q})\backslash H(\mathbb{A})$. Using the spectral expansion of the kernel (7), this equals

$$\mathtt{n}^{1-\mathtt{k}/2}\sum_{h\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(h)}{\|h\|^2}\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})}\varphi_h(n_1)\,\overline{\theta_{m_1}(n_1)}\,dn_1\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})}\overline{\varphi_h(n_2)}\,\theta_{m_2}(n_2)\,dn_2.$$

These integrals can be computed using the following proposition.

PROPOSITION 3.2. *Let $h\in S_{\mathtt{k}}(N,\omega')$, with Fourier expansion $h(z)=\sum_{n>0}a_nq^n$, where $q=e^{2\pi iz}$. Let φ_h be the associated function on $G(\mathbb{A})$. Then for $m\in\mathbb{Q}$,*

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}}\varphi_h\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\right)\theta(mt)\,dt=\begin{cases}e^{-2\pi m}a_m & \textit{if } m\in\mathbb{Z}^+,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. The $N=1$ case is given in [Ge1, p. 46]. For the general case, see [KL]. ∎

In light of this, (8) is nonzero only if $m_1,m_2\in\mathbb{Z}^+$. Under this assumption, we deduce that (8) equals

$$\mathtt{n}^{1-\mathtt{k}/2}e^{-2\pi(m_1+m_2)}\sum_{h\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(h)a_{m_1}(h)\,\overline{a_{m_2}(h)}}{\|h\|^2}. \tag{9}$$

3.3. *The geometric side.* Here we use the procedure in Section 2 to compute (8) using the geometric expansion of the kernel. The setting in Section 2 is slightly different from our present situation, since we are using a central character and integrating over $\overline{G}$. However, the same method goes through with the obvious minor adjustments.

The geometric side is a sum $\sum_{[\delta]}I_\delta$, where

$$I_\delta=I_\delta(f)=\int\limits_{H_\delta(\mathbb{Q})\backslash H(\mathbb{A})}f(n_1^{-1}\delta n_2)\,\overline{\theta_{m_1}(n_1)}\,\theta_{m_2}(n_2)\,dn_1\,dn_2$$

for $H(\mathbb{A})=N(\mathbb{A})\times N(\mathbb{A})$. As shown above, in order for the spectral side to be nonzero, we must have $m_1,m_2\in\mathbb{Z}^+$. We can also see this directly on the geometric side. Observe that

$$f^{\mathtt{n}}(g)=f^{\mathtt{n}}\left(g\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)=f^{\mathtt{n}}\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}g\right)$$

for any $t\in\widehat{\mathbb{Z}}$ and $g\in G(\mathbb{A}_{\mathrm{fin}})$. Thus

$$I_\delta(f)=\int\limits_{H_\delta(\mathbb{Q})\backslash H(\mathbb{A})}f\left(n_1^{-1}\delta n_2\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\right)\overline{\theta_{m_1}(n_1)}\,\theta_{m_2}(n_2)\,dn_1\,dn_2.$$

Replacing n_2 by $n_2\left(\begin{smallmatrix}1 & -t\\ & 1\end{smallmatrix}\right)$, we then have

$$I_\delta(f)=\theta_{\mathrm{fin}}(m_2t)I_\delta(f).$$

It follows that $I_\delta(f) \neq 0$ only if $m_2\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$, i.e. only if $m_2 \in \mathbb{Z}$. Similarly $m_1 \in \mathbb{Z}$. We will see below that in fact m_1, m_2 must be positive as well.

The orbits $[\delta]$ are in one-to-one correspondence with the double cosets

$$N(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})/N(\mathbb{Q}).$$

Let M be the group of invertible diagonal matrices. The Bruhat decomposition is the following disjoint union:

$$G(\mathbb{Q}) = N(\mathbb{Q})M(\mathbb{Q}) \cup N(\mathbb{Q})M(\mathbb{Q})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}N(\mathbb{Q}).$$

Thus

$$N(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})/N(\mathbb{Q}) = \left\{\left[\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\right]\,\middle|\,\gamma \in \mathbb{Q}^*\right\} \cup \left\{\left[\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix}\right]\,\middle|\,\mu \in \mathbb{Q}^*\right\}.$$

We need to determine which of these orbits are relevant.

First let $\delta = \left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)$. If $\left(\left(\begin{smallmatrix}1 & t_1\\ & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & t_2\\ & 1\end{smallmatrix}\right)\right) \in H_\delta(\mathbb{A})$, then

$$\begin{pmatrix}1 & -t_1\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix} = z\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}$$

for some $z \in Z(\mathbb{Q})$. A simple calculation shows that $z = 1$ and $t_1 = \gamma t_2$, so

$$H_\delta(\mathbb{A}) = \left\{\left(\begin{pmatrix}1 & \gamma t\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)\,\middle|\, t \in \mathbb{A}\right\}.$$

Thus δ is relevant if and only if

$$\theta((m_1\gamma - m_2)t) = 1$$

for all $t \in \mathbb{A}$, or equivalently, if and only if $m_1\gamma = m_2$.

On the other hand, if $\delta = \left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right) \in G(\mathbb{Q})$, one sees easily that

$$H_\delta(\mathbb{A}) = \{(e, e)\}$$

where e is the identity matrix, so all of these δ's are relevant.

3.3.1. *Computation of the first type of I_δ.* Here we take $m_1, m_2 \in \mathbb{Z}$, and $\delta = \left(\begin{smallmatrix}\gamma & \\ & 1\end{smallmatrix}\right)$ where $m_1\gamma = m_2$. Note that if $m_1 = 0$ or $m_2 = 0$, then they are both zero since $\gamma \neq 0$. Now

$$\begin{aligned} I_\delta(f) &= \int\limits_{\{(\gamma t, t)\in\mathbb{Q}^2\}\backslash\mathbb{A}\times\mathbb{A}} f\left(\begin{pmatrix}1 & -t_1\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix}\right)\theta(m_1t_1 - m_2t_2)\,dt_1\,dt_2 \\ &= \int\limits_{\{(\gamma t, t)\in\mathbb{Q}^2\}\backslash\mathbb{A}\times\mathbb{A}} f\left(\begin{pmatrix}\gamma & \gamma t_2 - t_1\\ 0 & 1\end{pmatrix}\right)\theta(m_1t_1 - m_2t_2)\,dt_1\,dt_2. \end{aligned}$$

Let $t_1' = \gamma t_2 - t_1$ and $t_2' = t_2$. Then because $m_1\gamma = m_2$,we have $m_1t_1 - m_2t_2 = -m_1t_1'$, so

$$I_\delta = \int\limits_{0\times\mathbb{Q}\backslash(\mathbb{A}\times\mathbb{A})} f\left(\begin{pmatrix}\gamma & t_1'\\ 0 & 1\end{pmatrix}\right)\theta(-m_1t_1')\,dt_1'\,dt_2'$$
$$= \mathrm{meas}(\mathbb{Q}\backslash\mathbb{A})\int\limits_{\mathbb{A}} f\left(\begin{pmatrix}\gamma & t\\ 0 & 1\end{pmatrix}\right)\theta(-m_1t)\,dt.$$

If $m_1 = m_2 = 0$, then

$$I_\delta = \int\limits_{\mathbb{A}} f\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\right)dt = 0.$$

This follows by a direct computation of the archimedean factor of I_δ using a contour integral in the spirit of Proposition 3.4 below. Full details are given in [KL].

We may therefore assume that m_1, m_2 are both nonzero integers. Then $\gamma = m_2/m_1$, and

$$I_\delta = \int\limits_{\mathbb{A}} f\left(\begin{pmatrix}m_2/m_1 & t\\ 0 & 1\end{pmatrix}\right)\theta(-m_1t)\,dt = \int\limits_{\mathbb{A}} f\left(\begin{pmatrix}m_2 & m_1t\\ & m_1\end{pmatrix}\right)\theta(-m_1t)\,dt$$
$$= \int\limits_{\mathbb{A}} f\left(\begin{pmatrix}m_2 & t\\ 0 & m_1\end{pmatrix}\right)\theta(-t)\,dt.$$

Here we used the fact that $f(zg) = f(g)$ for $z \in Z(\mathbb{Q})$.

We factorize the above integral into $(I_\delta)_\infty(I_\delta)_{\mathrm{fin}}$. First we compute

$$(I_\delta)_{\mathrm{fin}} = \int\limits_{\mathbb{A}_{\mathrm{fin}}} f^{\mathtt{n}}\left(\begin{pmatrix}m_2 & t\\ & m_1\end{pmatrix}\right)\theta_{\mathrm{fin}}(-t)\,dt.$$

Suppose $\left(\begin{smallmatrix}m_2 & t\\ 0 & m_1\end{smallmatrix}\right) \in \mathrm{Supp}\, f^{\mathtt{n}} = Z(\mathbb{A}_{\mathrm{fin}})M(\mathtt{n}, N)$. Then taking determinants we see that $m_1m_2 \in \mathtt{n}\mathbb{Q}_p^{*2}\mathbb{Z}_p^*$ for all p. Thus $\mathrm{ord}_p(m_1m_2/\mathtt{n})$ is even for all p. As a result, $m_1m_2 = \pm\mathtt{n}s^2$ for some $s \in \mathbb{Q}^*$. Here we can take $s > 0$. Under this condition, Lemma 3.1 shows that $\left(\begin{smallmatrix}m_2 & t\\ 0 & m_1\end{smallmatrix}\right) \in \mathrm{Supp}\, f^{\mathtt{n}}$ if and only if $\left(\begin{smallmatrix}m_2/s & t/s\\ 0 & m_1/s\end{smallmatrix}\right) \in M_2(\widehat{\mathbb{Z}})$, i.e. $m_1/s, m_2/s \in \mathbb{Z}$ and $t \in s\widehat{\mathbb{Z}}$. Assuming this, we have

$$(I_\delta)_{\mathrm{fin}} = \omega(1_\infty \times s_{\mathrm{fin}})^{-1}\int\limits_{s\widehat{\mathbb{Z}}} f^{\mathtt{n}}\left(\begin{pmatrix}m_2/s & t/s\\ 0 & m_1/s\end{pmatrix}\right)\theta_{\mathrm{fin}}(-t)\,dt.$$

Note that in fact $\omega(1_\infty \times s_{\mathrm{fin}}) = \omega(s)\omega_\infty(s)^{-1} = 1$ since $s > 0$ (cf. (5)). Hence the above equals

$$\psi(N)\omega((m_1/s)_N)^{-1}\int\limits_{s\widehat{\mathbb{Z}}} \theta_{\mathrm{fin}}(-t)\,dt.$$

This is nonzero only if $s\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$, i.e. only if $s \in \mathbb{Z}$. This being the case, the integral is equal to $\mathrm{meas}(s\widehat{\mathbb{Z}}) = |s|_{\mathbb{A}_{\mathrm{fin}}} = 1/s$. In addition, m_1/s is relatively prime to N since it is a factor of $\mathtt{n}$. Thus by (4), $\omega((m_1/s)_N) = \omega'(m_1/s)$, and

$$(I_\delta)_{\mathrm{fin}} = \frac{1}{s}\,\psi(N)\omega'(m_1/s)^{-1}.$$

This proves the following.

Proposition 3.3. *Let $m_1, m_2 \in \mathbb{Z}$, and let $\delta = \left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right) \in G(\mathbb{Q})$. Then $(I_\delta)_{\mathrm{fin}}$ is nonzero if and only if*

(1) *$m_1, m_2 \neq 0$ and $\gamma = m_2/m_1$,*
(2) *$m_1 m_2 = \pm s^2\mathtt{n}$ for some positive integer $s \mid \gcd(m_1, m_2)$.*

If these conditions are satisfied, then

$$(I_\delta)_{\mathrm{fin}} = \frac{1}{s}\,\psi(N)\omega'(m_1/s)^{-1}.$$

For the infinite part, we have the following (recall $\mathtt{k} > 2$).

Proposition 3.4. *Let $\delta = \left(\begin{smallmatrix}m_2/m_1 & 0\\ 0 & 1\end{smallmatrix}\right) \in G(\mathbb{Q})$. Then $(I_\delta)_\infty$ is nonzero if and only if $m_1, m_2 > 0$. Under this assumption,*

$$(I_\delta)_\infty = \int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} m_2 & t\\ 0 & m_1\end{pmatrix}\right)\theta_\infty(-t)\,dt = \frac{(4\pi)^{\mathtt{k}-1}}{(\mathtt{k}-2)!}\,(m_1m_2)^{\mathtt{k}/2}e^{-2\pi(m_1+m_2)}.$$

Proof. Because f_∞ is supported on $G(\mathbb{R})^+$, the integrand is zero unless m_1 and m_2 have the same sign. Now by formula (6) for f_∞, we have

$$\begin{aligned}
&\int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} m_2 & t\\ 0 & m_1\end{pmatrix}\right)\theta_\infty(-t)\,dt\\
&\qquad = \frac{\mathtt{k}-1}{4\pi}\,(m_1m_2)^{\mathtt{k}/2}(2i)^{\mathtt{k}}\int_{\mathbb{R}} \frac{e^{2\pi i t}}{(-t+(m_1+m_2)i)^{\mathtt{k}}}\,dt\\
&\qquad = \frac{\mathtt{k}-1}{4\pi}\,(m_1m_2)^{\mathtt{k}/2}\left(\frac{2}{i}\right)^{\mathtt{k}}\int_{-\infty}^{\infty} \frac{e^{2\pi i t}}{(t-(m_1+m_2)i)^{\mathtt{k}}}\,dt.
\end{aligned}$$

The integrand has a pole at $(m_1+m_2)i$. Use a contour integral around a semicircle in the upper half-plane. If $m_1, m_2 < 0$, then there are no poles inside the contour, so the integral vanishes. If $m_1, m_2 > 0$, then the residue theorem gives

$$\begin{aligned}
(I_\delta)_\infty &= \frac{\mathtt{k}-1}{4\pi}\,\frac{(m_1m_2)^{\mathtt{k}/2}2^{\mathtt{k}}}{i^{\mathtt{k}}}\,\frac{2\pi i}{(\mathtt{k}-1)!}\,\frac{d^{\mathtt{k}-1}}{dt^{\mathtt{k}-1}}\,e^{2\pi i t}\Big|_{t=(m_1+m_2)i}\\
&= \frac{\mathtt{k}-1}{4\pi}\,\frac{(m_1m_2)^{\mathtt{k}/2}2^{\mathtt{k}}}{i^{\mathtt{k}}}\,\frac{2\pi i}{(\mathtt{k}-1)!}\,(2\pi i)^{\mathtt{k}-1}e^{-2\pi(m_1+m_2)}\\
&= \frac{(4\pi)^{\mathtt{k}-1}}{(\mathtt{k}-2)!}\,(m_1m_2)^{\mathtt{k}/2}e^{-2\pi(m_1+m_2)}. \quad\blacksquare
\end{aligned}$$

Multiplying $(I_\delta)_{\mathrm{fin}}$ and $(I_\delta)_\infty$, we have the following.

PROPOSITION 3.5. *If* $\delta = \left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right) \in G(\mathbb{Q})$, *then* $I_\delta(f)$ *is nonzero if and only if*

(1) $m_1, m_2 \in \mathbb{Z}^+$ *and* $\gamma = m_2/m_1$,
(2) $m_1 m_2 = s^2\mathtt{n}$ *for some positive integer* $s \mid \gcd(m_1, m_2)$.

Under these conditions,

$$I_\delta(f) = \frac{\psi(N)(4\pi\sqrt{m_1 m_2})^{\mathtt{k}-1}\sqrt{\mathtt{n}}}{(\mathtt{k}-2)!\, e^{2\pi(m_1+m_2)}\omega'(m_1/s)}.$$

3.3.2. *Computation of the second type of* I_δ. If $\delta = \left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right)$, then $H_\delta = \{(e,e)\}$, and

$$I_\delta(f) = \int\limits_{N(\mathbb{A})\times N(\mathbb{A})} f\left(n_1^{-1}\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix} n_2\right) \overline{\theta_{m_1}(n_1)}\, \theta_{m_2}(n_2)\, dn_1\, dn_2.$$

Once again we split the computation into the finite and infinite components. Let $n_i = \left(\begin{smallmatrix}1 & t_i\\ 0 & 1\end{smallmatrix}\right)$, $i = 1, 2$. Then

$$n_1^{-1}\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix} n_2 = \begin{pmatrix}-t_1 & \mu - t_1 t_2\\ 1 & t_2\end{pmatrix}. \tag{10}$$

PROPOSITION 3.6. *For* $\mathtt{k} > 2$,

$$(I_\delta)_\infty = \iint\limits_{\mathbb{R}\times\mathbb{R}} f_\infty\left(\begin{pmatrix}-t_1 & \mu - t_1 t_2\\ 1 & t_2\end{pmatrix}\right)\theta_\infty(m_1 t_1 - m_2 t_2)\, dt_1\, dt_2 \tag{11}$$

is nonzero only if $m_1, m_2, -\mu$ *are all positive. Under these conditions*,

$$(I_\delta)_\infty = \frac{e^{-2\pi(m_1+m_2)}(4\pi i)^{\mathtt{k}}\sqrt{m_1 m_2}^{\,\mathtt{k}-1}}{2\cdot(\mathtt{k}-2)!}\,(-\mu)^{1/2} J_{\mathtt{k}-1}(4\pi\sqrt{-\mu m_1 m_2}),$$

where $J_{\mathtt{k}}$ *is the Bessel* J*-function.*

Proof. When $\mu > 0$, $\det\left(n_1^{-1}\left(\begin{smallmatrix} & \mu\\ 1 & \end{smallmatrix}\right)n_2\right) = -\mu < 0$, so f_∞ vanishes. Thus we can assume $\mu < 0$. Using the formula for f_∞, we have

$$\begin{aligned}(I_\delta)_\infty &= \frac{\mathtt{k}-1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(2i)^{\mathtt{k}}(-\mu)^{\mathtt{k}/2}e^{2\pi i(m_2t_2 - m_1t_1)}}{(t_2(t_1+i) - \mu - i(t_1+i))^{\mathtt{k}}}\, dt_1\, dt_2\\ &= \frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}(-\mu)^{\mathtt{k}/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{2\pi i(m_2t_2-m_1t_1)}}{(t_1+i)^{\mathtt{k}}\left(t_2 - \left(i + \frac{\mu}{t_1+i}\right)\right)^{\mathtt{k}}}\, dt_2\, dt_1.\end{aligned}$$

Note that $i + \frac{\mu}{t_1+i}$ is in the upper half-plane. Take the integral over t_2 along a semicircular contour. If $m_2 \le 0$, we can take a semicircle in the lower half-plane, and the integral vanishes. If $m_2 > 0$, we can use an upper half-plane contour, and by the residue theorem we have

$$\frac{\mathtt{k}-1}{4\pi}(2i)^{\mathtt{k}}(-\mu)^{\mathtt{k}/2}(2\pi i)\frac{(2\pi i m_2)^{\mathtt{k}-1}}{(\mathtt{k}-1)!}\int_{-\infty}^{\infty}\frac{e^{2\pi i(m_2(i+\frac{\mu}{t_1+i})-m_1t_1)}}{(t_1+i)^{\mathtt{k}}}\,dt_1$$
$$=\frac{(-1)^{\mathtt{k}}(4\pi m_2)^{\mathtt{k}-1}(-\mu)^{\mathtt{k}/2}}{(\mathtt{k}-2)!}\,e^{-2\pi(m_1+m_2)}\int_{-\infty}^{\infty}\frac{e^{2\pi i(m_2\frac{\mu}{t_1+i}-m_1(t_1+i))}}{(t_1+i)^{\mathtt{k}}}\,dt_1.$$

If $m_1 \le 0$, we can integrate over a contour along the real axis and a semicircle in the upper half-plane, and the integral vanishes.

For $m_1 > 0$, we can evaluate the above integral in terms of a Bessel function. The Bessel functions J_n may be defined using the generating function

$$e^{\frac{1}{2}\xi(\tau-1/\tau)}=\sum_{n=-\infty}^{\infty}\tau^n J_n(\xi),$$

see [Wa, Chapter 2.1]. Similarly, for any positively oriented simple closed curve C about the origin,

$$J_{n-1}(\xi)=\frac{1}{2\pi i}\int_C\frac{e^{\frac{1}{2}\xi(\tau-1/\tau)}}{\tau^n}\,d\tau.$$

To use this in our situation, solve

$$\frac{1}{2}\,\xi\tau=-2\pi i m_1(t_1+i),\qquad -\frac{1}{2}\,\frac{\xi}{\tau}=2\pi i m_2\,\frac{\mu}{t_1+i}.$$

We can take

$$\xi=4\pi\sqrt{-\mu m_1 m_2},\qquad \tau=\frac{-im_1}{\sqrt{-\mu m_1 m_2}}\,(t_1+i).$$

Thus if C is a clockwise semicircular contour along the real axis and enclosing $-i$ in the lower half-plane,

$$J_{\mathtt{k}-1}(4\pi\sqrt{-\mu m_1 m_2})=-\frac{1}{2\pi i}\int_C\frac{e^{\frac{1}{2}\xi(\tau-1/\tau)}}{\tau^{\mathtt{k}}}\,d\tau$$
$$=-\frac{1}{2\pi i}\int_C\frac{e^{2\pi i(m_2\frac{\mu}{t_1+i}-m_1(t_1+i))}}{\left(\frac{-im_1}{\sqrt{-\mu m_1 m_2}}\right)^{\mathtt{k}}(t_1+i)^{\mathtt{k}}}\left(\frac{-im_1}{\sqrt{-\mu m_1 m_2}}\right)dt_1.$$

As the radius of C goes to ∞, the contribution from the arc goes to 0. Therefore

$$\int_{-\infty}^{\infty}\frac{e^{2\pi i(m_2\frac{\mu}{t_1+i}-m_1(t_1+i))}}{(t_1+i)^{\mathtt{k}}}\,dt_1=(-2\pi i)\left(\frac{-im_1}{\sqrt{-\mu m_1 m_2}}\right)^{\mathtt{k}-1}J_{\mathtt{k}-1}(4\pi\sqrt{-\mu m_1 m_2}),$$

so we now see that

$$(I_\delta)_\infty = \frac{(-1)^{\mathtt{k}}(4\pi)^{\mathtt{k}-1}(-\mu)^{\mathtt{k}/2}m_2^{\mathtt{k}-1}}{(\mathtt{k}-2)!}\, e^{-2\pi(m_1+m_2)}(-2\pi i) \times \left(\frac{-im_1}{\sqrt{-\mu m_1 m_2}}\right)^{\mathtt{k}-1} J_{\mathtt{k}-1}(4\pi\sqrt{-\mu m_1 m_2})$$

$$= \frac{(4\pi i)^{\mathtt{k}}(-\mu)^{1/2}(m_1m_2)^{(\mathtt{k}-1)/2}}{2\cdot(\mathtt{k}-2)!}\, e^{-2\pi(m_1+m_2)} J_{\mathtt{k}-1}(4\pi\sqrt{-\mu m_1 m_2}). \quad ■$$

For a modulus $c \in N\mathbb{Z}$, the classical Kloosterman sum with character ω' is defined by

$$S_{\omega'}(n,m;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} \omega'(d)^{-1} e\left(\frac{nd+m\overline{d}}{c}\right),$$

where $d\overline{d} \equiv 1 \bmod c$ and $e(x) = e^{2\pi i x}$. We generalize this to the following sum for any integer a with $\gcd(a,N)=1$:

$$S_{\omega'}(n,m;a;c) = \sum_{\substack{d_1,d_2\in\mathbb{Z}/c\mathbb{Z}\\ d_1d_2=a}} \omega'(d_1)^{-1} e\left(\frac{nd_1+md_2}{c}\right). \tag{12}$$

Here the summands are no longer necessarily invertible in $\mathbb{Z}/c\mathbb{Z}$, however they are invertible modulo N. Note that if $\gcd(a,c)=1$, then $d_2 = a\overline{d}_1$, so in this special case one has

$$S_{\omega'}(n,m;a;c) = S_{\omega'}(n,ma;c).$$

Proposition 3.7. *Assume $\mu < 0$, $m_1, m_2 \in \mathbb{Z}$, and $\delta = \left(\begin{smallmatrix} & \mu \\ 1 & \end{smallmatrix}\right)$. Then*

$$(I_\delta)_{\mathrm{fin}} = \iint\limits_{\mathbb{A}_{\mathrm{fin}}\times\mathbb{A}_{\mathrm{fin}}} f^{\mathtt{n}}\left(\begin{pmatrix} 1 & -t_1 \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & t_2 \\ & 1\end{pmatrix}\right)\theta_{\mathrm{fin}}(m_1t_1 - m_2t_2)\, dt_1\, dt_2$$

is nonzero only if $\mu = -\mathtt{n}/c^2$ for some positive integer $c \in N\mathbb{Z}$. Under this condition,

$$(I_\delta)_{\mathrm{fin}} = (-1)^{\mathtt{k}}\psi(N) S_{\omega'}(m_2,m_1;\mathtt{n};c).$$

Proof. Suppose

$$\begin{pmatrix} 1 & -t_1 \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & t_2 \\ & 1\end{pmatrix} = \begin{pmatrix} -t_1 & \mu - t_1t_2 \\ 1 & t_2\end{pmatrix} \in Z(\mathbb{A}_{\mathrm{fin}})M(\mathtt{n},N) = \operatorname{Supp} f^{\mathtt{n}}.$$

Then arguing as before, we have $\mu = -\mathtt{n}s^2$ for some $s \in \mathbb{Q}^+$. Under this condition,

$$\begin{pmatrix} -t_1 & \mu - t_1t_2 \\ 1 & t_2\end{pmatrix} \in \operatorname{Supp} f_{\mathrm{fin}} \;\Leftrightarrow\; \begin{pmatrix} -t_1/s & (\mu - t_1t_2)/s \\ 1/s & t_2/s\end{pmatrix} \in M(\mathtt{n},N)$$

by Lemma 3.1. Let $c = 1/s$, and let $t_1' = ct_1, t_2' = ct_2$. The above condition translates to

$$\begin{pmatrix} -t_1' & (-\mathtt{n} - t_1't_2')/c \\ c & t_2' \end{pmatrix} \in M(\mathtt{n}, N),$$

which means:

(1) $c \in N\mathbb{Z}$,
(2) $t_1', t_2' \in \widehat{\mathbb{Z}}$,
(3) $t_1't_2' \equiv -\mathtt{n} \bmod c\widehat{\mathbb{Z}}$.

Note that $dt_i' = |c|_{\mathbb{A}_{\mathrm{fin}}} dt_i = \frac{1}{c} dt_i$. Henceforth we will work with t_i', so we drop the $'$ from the notation. We have

$$(I_\delta)_{\mathrm{fin}}$$
$$= c^2 \iint_{\widehat{\mathbb{Z}}\times\widehat{\mathbb{Z}}} f^{\mathtt{n}}\left(\begin{pmatrix} c & \\ & c\end{pmatrix}^{-1} \begin{pmatrix} -t_1 & (-\mathtt{n} - t_1t_2)/c \\ c & t_2 \end{pmatrix}\right) \theta_{\mathrm{fin}}\left(\frac{m_1t_1 - m_2t_2}{c}\right) dt_1\, dt_2.$$

As before, because $c > 0$, $f^{\mathtt{n}}\left(\begin{pmatrix} c & \\ & c\end{pmatrix}^{-1} g\right) = f^{\mathtt{n}}(g)$. Hence the value of $f^{\mathtt{n}}$ in the integrand is $\psi(N)/\omega((t_2)_N)$, which depends only on the residue class of t_2 modulo $N\widehat{\mathbb{Z}}$. Now because θ_{fin} is trivial on $\widehat{\mathbb{Z}}$, the value $\theta_{\mathrm{fin}}((m_1t_1 - m_2t_2)/c)$ depends only on the cosets $t_1 + c\widehat{\mathbb{Z}}$ and $t_2 + c\widehat{\mathbb{Z}}$. This means that the entire integrand is constant on cosets of $c\widehat{\mathbb{Z}}$. Let $s_i \in \mathbb{Z}^+ \cap (t_i + c\widehat{\mathbb{Z}})$. Note that $\gcd(s_2, N) = 1$ since $s_1s_2 \equiv -\mathtt{n} \bmod N$, and consequently $\omega((t_2)_N) = \omega((s_2)_N) = \omega'(s_2)$. Therefore

$$(I_\delta)_{\mathrm{fin}} = \psi(N)c^2 \sum_{\substack{s_1,s_2 \in \mathbb{Z}/c\mathbb{Z} \\ s_1s_2 = -\mathtt{n}}} \mathrm{meas}(c\widehat{\mathbb{Z}})^2 \omega'(s_2)^{-1} e\left(\frac{m_1s_1 - m_2s_2}{c}\right)$$
$$= \psi(N) \sum_{\substack{s_1,s_2 \in \mathbb{Z}/c\mathbb{Z} \\ s_1s_2 = -\mathtt{n}}} \omega'(s_2)^{-1} e\left(\frac{m_1s_1 - m_2s_2}{c}\right).$$

Replacing s_2 by $-s_2$, this equals

$$\omega'(-1)^{-1}\psi(N) \sum_{\substack{s_1,s_2 \in \mathbb{Z}/c\mathbb{Z} \\ s_1s_2 = \mathtt{n}}} \omega'(s_2)^{-1} e\left(\frac{m_1s_1 + m_2s_2}{c}\right)$$
$$= (-1)^{\mathtt{k}}\psi(N) S_{\omega'}(m_2, m_1; \mathtt{n}; c). \quad \blacksquare$$

For the global integral, we now see the following.

Proposition 3.8. *Let* $\delta = \begin{pmatrix} & \mu \\ 1 & \end{pmatrix} \in G(\mathbb{Q})$. *Then* I_δ *is nonzero only if*

(1) $\mu = -\mathtt{n}/c^2$ *for some positive* $c \in N\mathbb{Z}$,
(2) $m_1, m_2 \in \mathbb{Z}^+$.

If these conditions hold, then

$$I_\delta = \psi(N) S_{\omega'}(m_2, m_1; \mathtt{n}; c)\, \frac{\sqrt{\mathtt{n}}(-4\pi i)^{\mathtt{k}} \sqrt{m_1 m_2}^{\,\mathtt{k}-1}}{2c \cdot (\mathtt{k}-2)! e^{2\pi(m_1+m_2)}}\, J_{\mathtt{k}-1}\left(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right).$$

3.4. *Final results.* Equating the geometric and spectral computations of the previous sections, we obtain the following upon multiplying both sides by

$$\frac{(\mathtt{k}-2)! e^{2\pi(m_1+m_2)}}{\psi(N)\sqrt{\mathtt{n}}\,(4\pi\sqrt{m_1 m_2})^{\mathtt{k}-1}}.$$

THEOREM 3.9. *Let* $\mathtt{k} > 2$, *and let* $\mathtt{n}, m_1, m_2 \in \mathbb{Z}^+$, *with* $\gcd(\mathtt{n}, N) = 1$. *Let* $\mathcal{F}$ *be an orthogonal basis for* $S_{\mathtt{k}}(N, \omega')$ *consisting of eigenfunctions for* $T_{\mathtt{n}}$. *Then*

$$\begin{aligned} \frac{\psi(N)^{-1}(\mathtt{k}-2)!}{(4\pi\sqrt{\mathtt{n} m_1 m_2})^{\mathtt{k}-1}} \sum_{h\in\mathcal{F}} \frac{\lambda_{\mathtt{n}}(h) a_{m_1}(h)\overline{a_{m_2}(h)}}{\|h\|^2} &= T(m_1, m_2, \mathtt{n})\omega'(\sqrt{m_1 \mathtt{n}/m_2})^{-1} \\ &\quad + \frac{2\pi}{i^{\mathtt{k}}} \sum_{c\in N\mathbb{Z}^+} \frac{1}{c} S_{\omega'}(m_2, m_1; \mathtt{n}; c) J_{\mathtt{k}-1}\left(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right), \end{aligned}$$

where

$$T(m_1, m_2, \mathtt{n}) = \begin{cases} 1 & \text{if } m_1 m_2 = s^2 \mathtt{n} \text{ for some integer } s \mid \gcd(m_1, m_2), \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $T(a_1, a_2, a_3) = 1$ if and only if $a_i a_j / a_k$ is a perfect square integer for all distinct $i, j, k \in \{1, 2, 3\}$.

By choosing an appropriate basis $\mathcal{F}$, the Hecke eigenvalues in the above formula can be replaced by Fourier coefficients, as we now explain.

LEMMA 3.10. *There exists an orthogonal basis* $\mathcal{F}$ *consisting of eigenfunctions of* $T_{\mathtt{n}}$, *each of which has* $a_1 \neq 0$.

Proof. We will see that in fact $\mathcal{F}$ can be taken to consist of Hecke eigenforms. We say that two Hecke eigenforms are *equivalent* if they have the same Hecke eigenvalues for every $T_{\mathtt{n}}, (\mathtt{n}, N) = 1$. Let $\mathcal{R}$ denote the set of equivalence classes of eigenforms in $S_{\mathtt{k}}(N, \omega')$. For $r \in \mathcal{R}$ let V_r denote the subspace spanned by the eigenforms in r. Then

$$S_{\mathtt{k}}(N, \omega') = \bigoplus_{r\in\mathcal{R}} V_r$$

is an orthogonal direct sum. Thus it suffices to show that each V_r has an orthogonal basis of forms with $a_1 \neq 0$ for each basis element.

As explained in [Ri, Theorem 1.2], there exists a new form $f_1 \in V_r$. It is well known that any new form has $a_1 \neq 0$ (see for example, [AL, Lemma

19]). Normalize f_1 so that $\|f_1\| = 1$, and extend $\{f_1\}$ to an orthonormal basis $\{f_1, \dots, f_s\}$ for V_r. Suppose $a_1(f_2) = 0$. Let

$$f_1' = \frac{f_1 + f_2}{\|f_1 + f_2\|}, \qquad f_2' = \frac{-f_1 + f_2}{\|f_1 + f_2\|},$$

and $f_i' = f_i$ for $i \geq 3$. Then $\{f_1', \dots, f_s'\}$ is still an orthonormal basis for V_r, and $a_1(f_1') = a_1(f_1)/\|f_1 + f_2\| \neq 0$, $a_1(f_2') = -a_1(f)/\|f_1 + f_2\| \neq 0$. Repeating this process if necessary, we can assume $a_1(f_i) \neq 0$ for all i. ■

Corollary 3.11. *Suppose $\mathcal{F}$ consists of eigenfunctions of $T_{\mathtt{n}}$ with $a_1 = 1$. Then*

$$\frac{\psi(N)^{-1}(\mathtt{k}-2)!}{(4\pi\sqrt{\mathtt{n}m_1m_2})^{\mathtt{k}-1}} \sum_{h\in\mathcal{F}} \frac{a_{\mathtt{n}}(h)a_{m_1}(h)\overline{a_{m_2}(h)}}{\|h\|^2}$$
$$= T(m_1, m_2, \mathtt{n})\omega'(\sqrt{m_1\mathtt{n}/m_2})^{-1}$$
$$+ \frac{2\pi}{i^{\mathtt{k}}} \sum_{\substack{c>0\\ N|c}} \frac{1}{c}\, S_{\omega'}(m_2, m_1; \mathtt{n}; c) J_{\mathtt{k}-1}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right).$$

Proof. When $a_1(h) = 1$ and h is an eigenfunction of $T_{\mathtt{n}}$, then $\lambda_{\mathtt{n}}(h) = a_{\mathtt{n}}(h)$. ■

If we take $\mathtt{n} = 1$ in the main theorem, then $T_{\mathtt{n}}$ is the identity map, so any orthogonal basis $\mathcal{F}$ will do, and $\lambda_1(h) = 1$ for all $h \in \mathcal{F}$. In this way we recover the classical Petersson trace formula (cf. [IK, Proposition 14.5]):

Corollary 3.12. *For any orthogonal basis $\mathcal{F}$ for $S_{\mathtt{k}}(N, \omega')$,*

$$\frac{\psi(N)^{-1}(\mathtt{k}-2)!}{(4\pi\sqrt{mn})^{\mathtt{k}-1}} \sum_{h\in\mathcal{F}} \frac{a_m(h)\overline{a_n(h)}}{\|h\|^2}$$
$$= \delta(m,n) + \frac{2\pi}{i^{\mathtt{k}}} \sum_{\substack{c>0\\ N|c}} \frac{1}{c}\, S_{\omega'}(n, m; c) J_{\mathtt{k}-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Remark. Although our main result involves both Hecke eigenvalues and Fourier coefficients, Corollary 3.11 shows that the formula can be rewritten in terms of three Fourier coefficients. As a result, the generalized formula holds nothing more than the Petersson trace formula. This is due to the fact that by the multiplicative relations among the Hecke operators ([Sh, Theorem 3.24]), for any eigenform h with $a_1(h) = 1$, we can express $a_{\mathtt{n}}(h)a_{m_1}(h)$ as a linear combination of other Fourier coefficients, with coefficients independent of h.

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