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On the number of primitive Pythagorean triangles

by

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1. Introduction. A *primitive Pythagorean triangle* is a triple (a,b,c) of natural numbers with

$$a^2+b^2=c^2, \quad a\le b, \quad \gcd(a,b,c)=1.$$

For a large real number N, let $P(N)$ denote the number of Pythagorean triangles with area less than N. Many authors studied the asymptotic behaviour of $P(N)$.

J. Lambek and L. Moser [3] proved that

$$P(N)=cN^{1/2}+O(N^{1/3}) \tag{1.1}$$

with $c=(2\pi^5)^{-1/2}\Gamma^2(1/4)$.

R. E. Wild [10] proved that

$$P(N)=cN^{1/2}-c'N^{1/3}+R(N) \tag{1.2}$$

with

$$c'=\frac{|\zeta(1/3)|(1+2^{-1/3})}{\zeta(4/3)(1+4^{-1/3})}, \quad R(N)=O(N^{1/4}\log N).$$

The best known bound for $R(N)$ depends on the estimation of

$$M(x):=\sum_{n\le x}\mu(n),$$

where $\mu(n)$ is the Möbius function. Using the bound $M(x)=O(xe^{-\delta\log^{1/2}x})$, J. Duttlinger and W. Schwarz [1] proved (1.2) with

$$R(N)=O(N^{1/4}e^{-\delta\log^{1/2}N}).$$

The same argument with the bound $M(x)=O(xe^{-\delta\log^{3/5}x\log\log^{-1/5}x})$ yields

$$R(N)=O(N^{1/4}e^{-\delta\log^{3/5}N\log\log^{-1/5}N}).$$

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Under the assumption of the Riemann Hypothesis (RH), J. Duttlinger and W. Schwarz [1] proved

$$R(N) = O(N^{5/22+\varepsilon}). \tag{1.3}$$

The exponent 5/22 can be replaced by 137/604, 127/560, 37/164, 269/1238, as proved respectively by Menzer [4], Müller–Nowak–Menzer [6], Müller and Nowak [5], Nowak [7].

In this paper, we shall prove the following

THEOREM. *If RH is true, then*

$$R(N) = O(N^{127/616} \log^{963/308} N). \tag{1.4}$$

Numerically, we have

$$5/22 = 0.22727\ldots, \quad 137/604 = 0.22681\ldots, \quad 127/560 = 0.22678\ldots,$$
$$37/164 = 0.2256\ldots, \quad 269/1238 = 0.21728\ldots, \quad 127/616 = 0.2061\ldots$$

Notations. $\mathbb{N}$ denotes the set of all natural numbers. Define

$$\mathbb{N}' = \sqrt{2}\,\mathbb{N}, \qquad \mathbb{N}'' = \sqrt{2}\,\mathbb{N} - \sqrt{2}/2.$$

$\{t\}$ denotes the fractional part of t,

$$\psi(t) = \{t\} - 1/2, \qquad \psi^*(t) = \{t\}^2/2 - \{t\}/2 + 1/12.$$

$d \sim D$ means $D < d \le 2D$.

2. Reduction of the problem. J. Lambek and L. Moser [3] reduced the problem of calculating $P(N)$ to that of counting the number $L(N)$ of lattice points inside the planar domain

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : xy(x^2 - y^2) < N,\ 0 < y < x\}, \qquad N \in \mathbb{R}^+.$$

They established the formula

$$P(N) = \sum_{k=0}^{\infty} (-1)^k L'\left(\frac{N}{4^k}\right), \tag{2.1}$$

where $L'(N)$ is the number of primitive lattice points in $\mathcal{D}$. Thus the problem is reduced to estimating $L'(N)$. By a usual device, we have

$$L'(N) = \sum_{d=1}^{\infty} \mu(d) L\left(\frac{N}{d^4}\right). \tag{2.2}$$

$L(N)$ can be written as

$$L(N) = kN^{1/2} - k'N^{1/3} + F(N), \tag{2.3}$$

where

$$k = \frac{\Gamma^2(1/4)}{4(2\pi)^{1/2}}, \qquad k' = |\zeta(1/3)|(1 + 2^{-1/3}).$$

Suppose $1 \le y < N^{1/4}$ is a parameter. We write

$$L'(N) = \sum_{d\le y} \mu(d) L\left(\frac{N}{d^4}\right) + \sum_{d>y} \mu(d) L\left(\frac{N}{d^4}\right) = S_1 + S_2. \tag{2.4}$$

By (2.3), we have

$$\begin{aligned} S_1 &= kN^{1/2} \sum_{d\le y} \frac{\mu(d)}{d^2} - k'N^{1/3} \sum_{d\le y} \frac{\mu(d)}{d^{4/3}} + \sum_{d\le y} \mu(d) F\left(\frac{N}{d^4}\right) \\ &= kN^{1/2} \sum_{d\le y} \frac{\mu(d)}{d^2} - k'N^{1/3} \sum_{d\le y} \frac{\mu(d)}{d^{4/3}} + S_1^*. \end{aligned} \tag{2.5}$$

Now the problem is reduced to estimating S_1^* and S_2.

3. Properties of $F(N)$. In this section we study the error term $F(N)$, which is very important in our proof.

For a large real number N, let

$$Y = Y(N) = \sqrt{2}\sin(\pi/8) N^{1/4}.$$

For any $0 < y < Y$, let $\xi(y)$ denote the unique solution of the equation

$$x^3 y - y^3 x - N = 0.$$

By Cardano's formula,

$$\xi(y) = \left(\frac{N}{2y} + \sqrt{D}\right)^{1/3} + \left(\frac{N}{2y} - \sqrt{D}\right)^{1/3}, \quad D = \frac{1}{4}\left(\frac{N}{y}\right)^2 - \frac{1}{27} y^6.$$

In particular, $\xi(y) > y$ and $\xi(Y) = (\sqrt{2}+1)Y$. Let $\alpha = \sqrt{2} - 1$. Define

$$F_1(N) = \sum_{\substack{y<Y \\ y\in\mathbb{N}}} \psi(\xi(y)), \quad F_2(N) = \sum_{\substack{y<Y \\ y\in\mathbb{N}'}} \psi\left(\frac{\xi(y)}{\sqrt{2}}\right),$$

$$F_3(N) = \sum_{\substack{y<Y \\ y\in\mathbb{N}''}} \psi\left(\frac{\xi(y)}{\sqrt{2}} + \frac{1}{2}\right).$$

J. Duttlinger and W. Schwarz [1] proved that

$$F(N) = -F_1(N) - F_2(N) - F_3(N) + O(N^{2/15}). \tag{3.1}$$

We first prove the following

LEMMA 3.1. *We have*

$$F(N) = -F_1(N) - F_2(N) - F_3(N) + O(\log^2 N).$$

Proof. We begin with formula (2.2) of [1], which reads

$$L(N) = H_1 - H_2 - F_1 - F_2 - F_3 + G, \tag{3.2}$$

where

$$H_1 = \sum_{\substack{y<Y \\ y\in\mathbb{N}}} \xi(y) + \frac{1}{\sqrt{2}} \sum_{\substack{y<Y \\ y\in\mathbb{N}'}} \xi(y) + \frac{1}{\sqrt{2}} \sum_{\substack{y<Y \\ y\in\mathbb{N}''}} \xi(y),$$

$$H_2 = \sum_{\substack{y<Y \\ y\in\mathbb{N}}} \alpha^{-1}y + \frac{1}{\sqrt{2}} \sum_{\substack{y<Y \\ y\in\mathbb{N}'}} \alpha^{-1}y + \frac{1}{\sqrt{2}} \sum_{\substack{y<Y \\ y\in\mathbb{N}''}} \alpha^{-1}y,$$

$$G = \sum_{\substack{y<Y \\ y\in\mathbb{N}}} \psi(\alpha^{-1}y) + \sum_{\substack{y<Y \\ y\in\mathbb{N}'}} \psi\left(\frac{\alpha^{-1}y}{\sqrt{2}}\right) + \sum_{\substack{y<Y \\ y\in\mathbb{N}''}} \psi\left(\frac{\alpha^{-1}y}{\sqrt{2}} + \frac{1}{2}\right).$$

In order to estimate H_1, we use the following Euler–Maclaurin formula: suppose $f(u)$ is three times continuously differentiable on $[1, U]$; then

$$\sum_{1\le n\le U} f(n) = \int_1^U f(u)\,du - f(U)\psi(U) + f(1)/2 + \psi^*(U)f'(U) - \psi^*(1)f'(1) - \int_1^U \psi^*(u)f''(u)\,du. \tag{3.3}$$

This formula can be found in [8, Chapter 2].

We first evaluate the sum $\sum_{y<Y,\, y\in\mathbb{N}} \xi(y)$. By (3.3) we have

$$\sum_{\substack{y<Y \\ y\in\mathbb{N}}} \xi(y) = \int_1^Y \xi(u)\,du - \xi(Y)\psi(Y) + \xi(1)/2 + \psi^*(Y)\xi'(Y) - \psi^*(1)\xi'(1) - \int_1^Y \psi^*(u)\xi''(u)\,du. \tag{3.4}$$

Müller, Nowak and Menzer [6] proved that for $1 \le u \le Y$, the estimate

$$|\xi^{(r)}(u)| \asymp N^{1/3}u^{-1/3-r}$$

holds for $r = 1, 2, 3$. This implies that $\xi''(u)$ is monotone and $\xi''(u) \ll 1$ for $u \gg N^{1/7}$, $\xi'(Y) \ll 1$.

We write

$$\int_1^Y \psi^*(u)\xi''(u)\,du = \int_1^{N^{1/7}} \psi^*(u)\xi''(u)\,du + \int_{N^{1/7}}^Y \psi^*(u)\xi''(u)\,du. \tag{3.5}$$

By partial integration, we get

$$\int_{N^{1/7}}^Y \psi^*(u)\xi''(u)\,du \ll 1. \tag{3.6}$$

For $0 < u \le N^{1/7}$, it is easy to check that

$$\xi(u) = \frac{N^{1/3}}{u^{1/3}} + O\left(\frac{u^{7/3}}{N^{1/3}}\right), \tag{3.7}$$

$$\xi'(u) = -\frac{1}{3}\cdot\frac{N^{1/3}}{u^{4/3}} + O\left(\frac{u^{4/3}}{N^{1/3}}\right), \tag{3.8}$$

$$\xi''(u) = \frac{4}{9}\cdot\frac{N^{1/3}}{u^{7/3}} + O\left(\frac{u^{1/3}}{N^{1/3}}\right). \tag{3.9}$$

So we get

$$\begin{aligned}
&\int_1^{N^{1/7}} \psi^*(u)\xi''(u)\,du \\
&= \frac{4N^{1/3}}{9}\int_1^{N^{1/7}} \psi^*(u)u^{-7/3}\,du + O\Big(N^{-1/3}\int_1^{N^{1/7}} u^{1/3}\,du\Big) \\
&= \frac{4N^{1/3}}{9}\int_1^{\infty} \psi^*(u)u^{-7/3}\,du - \frac{4N^{1/3}}{9}\int_{N^{1/7}}^{\infty} \psi^*(u)u^{-7/3}\,du + O(1) \\
&= c_0N^{1/3} + O(1),
\end{aligned} \tag{3.10}$$

with

$$\begin{aligned}
c_0 &= \frac{4}{9}\int_1^{\infty} \psi^*(u)u^{-7/3}\,du = \zeta(1/3) + 1 + 1/36, \\
\xi(1)/2 &= N^{1/3}/2 + O(1), \qquad \psi^*(1)\xi'(1) = -N^{1/3}/36 + O(1).
\end{aligned} \tag{3.11}$$

From (3.4)–(3.11) we get

$$\sum_{\substack{y<Y\\ y\in\mathbb{N}}} \xi(y) = \int_1^Y \xi(u)\,du + c_1N^{1/3} - \xi(Y)\psi(Y) + O(1), \tag{3.12}$$

where $c_1 = 1/2 + 1/36 - c_0$.

Similarly we get

$$\begin{aligned}
\frac{1}{\sqrt{2}}\sum_{\substack{y<Y\\ y\in\mathbb{N}'}} \xi(y) &= \frac{1}{\sqrt{2}}\sum_{\substack{m<Y/\sqrt{2}\\ m\in\mathbb{N}}} \xi(\sqrt{2}\,m) \\
&= \frac{1}{\sqrt{2}}\int_1^{Y/\sqrt{2}} \xi(\sqrt{2}\,u)\,du + c_2'N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\left(\frac{Y}{\sqrt{2}}\right) + O(1) \\
&= \frac{1}{2}\int_{\sqrt{2}}^{Y} \xi(u)\,du + c_2'N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\left(\frac{Y}{\sqrt{2}}\right) + O(1)
\end{aligned} \tag{3.13}$$

$$= \frac{1}{2}\int_1^Y \xi(u)\,du - \frac{1}{2}\int_1^{\sqrt{2}} \xi(u)\,du + c_2' N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}}\biggr) + O(1)$$

$$= \frac{1}{2}\int_1^Y \xi(u)\,du + c_2 N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}}\biggr) + O(1),$$

$$\frac{1}{\sqrt{2}}\sum_{\substack{y<Y\\ y\in\mathbb{N}''}} \xi(y) = \frac{1}{\sqrt{2}}\sum_{\substack{m<Y/\sqrt{2}+1/2\\ m\in\mathbb{N}}} \xi\biggl(\sqrt{2}\,m - \frac{\sqrt{2}}{2}\biggr) \tag{3.14}$$

$$= \frac{1}{\sqrt{2}}\int_1^{Y/\sqrt{2}+1/2} \xi\biggl(\sqrt{2}\,u - \frac{\sqrt{2}}{2}\biggr)\,du + c_3' N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1)$$

$$= \frac{1}{2}\int_{\sqrt{2}/2}^{Y} \xi(u)\,du + c_3' N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1)$$

$$= \frac{1}{2}\int_1^{Y} \xi(u)\,du + \frac{1}{2}\int_{\sqrt{2}/2}^{1} \xi(u)\,du + c_3' N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1)$$

$$= \frac{1}{2}\int_1^{Y} \xi(u)\,du + c_3 N^{1/3} - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1),$$

where c_2, c_2', c_3, c_3' are constants.

From (3.12)–(3.14) we get

$$H_1 = 2\int_1^Y \xi(u)\,du - k' N^{1/3} - \xi(Y)\psi(Y) - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}}\biggr) - \frac{\xi(Y)}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1). \tag{3.15}$$

For $\int_1^Y \xi(u)\,du$, we have

$$\int_1^Y \xi(u)\,du = \frac{1}{4}\,N^{1/2}\,\frac{\Gamma^2(1/4)}{2(2\pi)^{1/2}} + \frac{\alpha^{-1}Y^2}{2}. \tag{3.16}$$

This is contained in [1].

It is easy to check that

$$H_2 = \alpha^{-1}Y^2 - \alpha^{-1}\xi(Y)Y - \alpha^{-1}\,\frac{Y}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}}\biggr) - \alpha^{-1}\,\frac{Y}{\sqrt{2}}\,\psi\biggl(\frac{Y}{\sqrt{2}} + \frac{1}{2}\biggr) + O(1) \tag{3.17}$$

$$= \alpha^{-1}Y^2 - \xi(Y)\psi(Y) - \frac{\xi(Y)}{\sqrt{2}}\,\psi\Big(\frac{Y}{\sqrt{2}}\Big) - \frac{\xi(Y)}{\sqrt{2}}\,\psi\Big(\frac{Y}{\sqrt{2}} + \frac{1}{2}\Big) + O(1),$$

if we notice $\xi(Y) = (\sqrt{2}+1)Y$.

For G, we have

$$G = O(\log^2 N). \tag{3.18}$$

This is formula (3.8) of [1]. Now Lemma 3.1 follows from (3.2), (3.15)–(3.18). ■

LEMMA 3.2. *Suppose $T \ge 10$. Then*

$$\int_1^T |F(u)|^2\,du \ll T^{5/4}\log^4 T.$$

Proof. By Lemma 3.1, it suffices to prove that

$$\int_T^{T+\varepsilon T} |F_i(u)|^2\,du \ll T^{5/4}\log^4 T \quad (i = 1, 2, 3) \tag{3.19}$$

for a small $\varepsilon > 0$. We only consider the case $i = 1$. The proofs for the other two cases are the same.

We write

$$F_1(u) = \sum_{\substack{y<\sqrt{2}\sin(\pi/8)u^{1/4}\\ y\in\mathbb{N}}} \psi(\xi(y,u)) = \sum_{j=1}^{J} \sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^j\\ y\in\mathbb{N}}} \psi(\xi(y,u)) + O(T^{1/8}), \tag{3.20}$$

where $J = [\log T/(8\log 2)]$, and $\xi(y,u)$ is the unique solution of the equation

$$\xi^3 y - y^3\xi = u,$$

where

$$(u,y) \in \{(u,y) : T \le u \le T+\varepsilon T,\ 0 < y < \sqrt{2}\sin(\pi/8)u^{1/4}\}.$$

Thus

$$F_1(u) \ll \Big|\sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^{j_0}\\ y\in\mathbb{N}}} \psi(\xi(y,u))\Big|\log T + T^{1/8}$$

for some $1 \le j_0 \le J$.

We use the following expression (3.21) of $\psi(t)$ to transform the above sum into the exponential sum: for any $H_0 \geq 2$,

$$\psi(t) = \sum_{1\leq|h|\leq H_0} a(h)e(ht) + O\Big(\sum_{1\leq h\leq H_0} b(h)e(ht)\Big) + O(1/H_0) \tag{3.21}$$

with $a(h) \ll 1/|h|$ and $b(h) \ll 1/H_0$. This is due to Vaaler [9].

Take $H_0 = T^{1/4}2^{-j_0}$ in (3.21). We get

$$F_1(u) \ll \sum_{1\leq h\leq H_0} \frac{1}{h}\Big| \sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^{j_0}\\ y\in\mathbb{N}}} e(\xi(y,u))\Big| \log T + T^{1/8}.$$

By Cauchy's inequality,

$$\begin{aligned}
&|F_1(u)|^2\\
&\ll \Big(\sum_{1\leq h\leq H_0} \frac{1}{h^{1/2}}\cdot\frac{1}{h^{1/2}}\Big| \sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^{j_0}\\ y\in\mathbb{N}}} e(h\xi(y,u))\Big|\Big)^2 \log^2 T + T^{1/4}\\
&\ll \sum_{1\leq h\leq H_0} \frac{1}{h}\Big| \sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^{j_0}\\ y\in\mathbb{N}}} e(h\xi(y,u))\Big|^2 \log^3 T + T^{1/4}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\log^{-3} T \int_T^{T+\varepsilon T} |F_1(u)|^2\,du \qquad (3.22)\\
&\ll \sum_{1\leq h\leq H_0} \frac{1}{h}\int_T^{T+\varepsilon T} \Big| \sum_{\substack{y\sim\sqrt{2}\sin(\pi/8)u^{1/4}/2^{j_0}\\ y\in\mathbb{N}}} e(h\xi(y,u))\Big|^2 du + T^{5/4}\\
&= \sum_{1\leq h\leq H_0} \frac{1}{h}\int_T^{T+\varepsilon T} \sum_{y_1,y_2} e(h\xi(y_1,u)-h\xi(y_2,u))\,du + T^{5/4}\\
&\ll \varepsilon T N_0 \sum_{1\leq h\leq H_0}\frac{1}{h} + \sum_{1\leq h\leq H_0}\frac{1}{h}\Big|\int_T^{T+\varepsilon T}\sum_{y_1\neq y_2} e(h\xi(y_1,u)-h\xi(y_2,u))\,du\Big| + T^{5/4}\\
&\ll \sum_{1\leq h\leq H_0}\frac{1}{h}\sum_{\substack{y_1\neq y_2\\ y_i\sim N_0,\,y_i\in\mathbb{N}}}\Big|\int_{I(y_1,y_2)} e(h\xi(y_1,u)-h\xi(y_2,u))\,du\Big| + T^{5/4}\log T,
\end{aligned}$$

where $I(y_1,y_2)$ is a subinterval of $[T, T+\varepsilon T]$, $N_0 = \sqrt{2}\sin(\pi/8)T^{1/4}/2^{j_0}$.

In order to estimate the last integral in (3.22), we need the following well known result (see, for example, [8, Chapter 21]): suppose $f(t)$ is a realvalued function defined on $[a,b]$ such that $f'(t)$ is monotone on $[a,b]$ and $|f'(t)| \gg \delta > 0$; then

$$\int_a^b e(f(t))\,dt \ll \delta^{-1}. \tag{3.23}$$

Let $D(u) = h\xi(y_1,u) - h\xi(y_2,u)$. We need to compute D_u.

From the equation $\xi^3 y - y^3\xi = u$ it is easy to get

$$\xi_u = \frac{1}{3\xi^2 y - y^3}, \qquad \xi_y = \frac{3y^2\xi - \xi^3}{3\xi^2 y - y^3}.$$

Thus

$$D_u = h(\xi_u(y_1,u) - \xi_u(y_2,u)) = h\left(\frac{1}{3\xi(y_1,u)^2 y_1 - y_1^3} - \frac{1}{3\xi(y_2,u)^2 y_2 - y_2^3}\right).$$

For fixed $u \in [T, T+\varepsilon T]$, let $D_1(y) = 3\xi(y,u)^2 y - y^3$. By Lagrange's theorem, there exists a $y_0 \sim N_0$ such that

$$D_u = h\xi_{uy}(y_0,u)(y_1 - y_2).$$

It is easy to show that

$$\begin{aligned}\xi_{uy} &= -\frac{D_1'(y)}{D_1^2(y)} = -\frac{3}{D_1^2(y)}\,(2\xi y\xi_y + \xi^2 - y^2)\\ &= -\frac{3y(\xi^2+y^2)^2}{D_1^3(y)} = -\frac{3(\xi^2+y^2)^2}{y^2(3\xi^2-y^2)^3},\end{aligned}$$

which implies that (notice $\xi \asymp N^{1/3}y^{-1/3}$)

$$D_u \gg \frac{h|y_1-y_2|}{T^{2/3}N_0^{4/3}}.$$

Thus by (3.23) we get

$$\int_{I(y_1,y_2)} e(h\xi(y_1,u) - h\xi(y_2,u))\,du \ll \frac{T^{2/3}N_0^{4/3}}{h|y_1-y_2|},$$

which leads to

$$\begin{aligned}&\log^{-3} T \int_T^{T+\varepsilon T} |F_1(u)|^2\,du \\ &\quad \ll \sum_{1\le h\le H_0} \frac{1}{h} \sum_{\substack{y_1\ne y_2\\ y_i\sim N_0,\, y_i\in\mathbb{N}}} \frac{T^{2/3}N_0^{4/3}}{h|y_1-y_2|} + T^{5/4}\log T \ll T^{5/4}\log T,\end{aligned} \tag{3.24}$$

if we notice $N_0 \ll T^{1/4}$. Thus, (3.19) holds for $i=1$. In the same way we can prove the cases $i = 2, 3$. ∎

4. Estimation of S_1^*. In this section we shall estimate S_1^* by an argument similar to one of Huxley and Nowak [2].

Since $L(N)$ is increasing, we get $L(u) \geq L(v)$ for any $u \geq v \geq 1$, which implies that

$$F(u) - F(v) \geq k(v^{1/2} - u^{1/2}) - k'(v^{1/3} - u^{1/3}).$$

So, if V is some positive number and $t > 1$ is a value for which $|F(t)| \geq V$, there exists an interval I of length $\geq t^{1/2}V/100$ containing t such that

$$|F(u)| \geq V/3$$

for all $u \in I$.

For positive real parameters $D \ll N^{1/4}$ and $V \geq (N/D^4)^{1/8}$, let

$$\mathcal{M}(D,V) = \{d \in \mathbb{N} : D < d \leq 2D,\ V \leq |F(N/d^4)| \leq 2V\},$$
$$\mathcal{R}(D,V) = \#\mathcal{M}(D,V).$$

For any $d \in \mathcal{M}(D,V)$, by the above considerations, there exists an interval

$$J \subset \left[\frac{N}{(d+1/2)^4}, \frac{N}{(d-1/2)^4}\right]$$

of length $|J| \gg \min(N^{1/2}V/D^2, N/D^5)$ such that

$$|F(u)| \geq V/3$$

for all $u \in J$. By Lemma 3.2 we have

$$V^2\mathcal{R}(D,V)\min\left(\frac{N^{1/2}V}{D^2}, \frac{N}{D^5}\right) \ll \int_1^{N/(D-1/2)^4} |F(u)|^2\,du \ll \frac{N^{5/4}}{D^5}\log^4 N.$$

Thus we get

$$\mathcal{R}(D,V) \ll \left(\frac{N^{3/4}}{D^3V^3} + \frac{N^{1/4}}{V^2}\right)\log^4 N. \tag{4.1}$$

By a familiar device, we get

$$\begin{aligned}
&\sum_{D<d\leq 2D}\left|F\left(\frac{N}{d^4}\right)\right| \\
&\ll N^{1/8}D^{1/2} + \sum_{V=2^j \gg N^{1/8}D^{-1/2},\, j\in\mathbb{N}} V\min(D, \mathcal{R}(D,V)) \\
&\ll N^{1/8}D^{1/2} + \sum_{V=2^j \gg N^{1/8}D^{-1/2},\, j\in\mathbb{N}} \min\left(DV, \frac{N^{3/4}}{D^3V^2} + \frac{N^{1/4}}{V}\right)\log^4 N \\
&\ll N^{1/4}D^{-1/3}\log^4 N + N^{1/8}D^{1/2}\log^4 N.
\end{aligned} \tag{4.2}$$

For $F(u)$, we have the estimate (see [7, p. 176])

$$F(u) \ll u^{23/146} \log^{315/146} u, \tag{4.3}$$

which implies

$$\sum_{D<d\le 2D} \left| F\left(\frac{N}{d^4}\right)\right| \ll N^{23/146} D^{54/146} \log^{315/146} N. \tag{4.4}$$

Suppose $1 < y_1 < y$ is a parameter. From (4.2) we have

$$\sum_{y_1<d\le y} \left| F\left(\frac{N}{d^4}\right)\right| \ll N^{1/4} y_1^{-1/3} \log^4 N + N^{1/8} y^{1/2} \log^4 N. \tag{4.5}$$

From (4.4) we get

$$\sum_{d\le y_1} \left| F\left(\frac{N}{d^4}\right)\right| \ll N^{23/146} y_1^{54/146} \log^{315/146} N. \tag{4.6}$$

Combining (4.5) and (4.6) we get the estimate

$$\begin{aligned} S_1^* = \sum_{d\le y} \mu(d) F\left(\frac{N}{d^4}\right) &\ll \sum_{d\le y_1} \left| F\left(\frac{N}{d^4}\right)\right| + \sum_{y_1<d\le y} \left| F\left(\frac{N}{d^4}\right)\right| \\ &\ll N^{23/146} y_1^{54/146} \log^{315/146} N \\ &\quad + N^{1/4} y_1^{-1/3} \log^4 N + N^{1/8} y^{1/2} \log^4 N \\ &\ll N^{127/616} \log^{963/308} N + N^{1/8} y^{1/2} \log^4 N \end{aligned} \tag{4.7}$$

on taking $y_1 = N^{81/616} \log^{807/308} N$. The estimate (4.7) is true for any $1 \le y \ll N^{1/4}$.

5. Estimation of S_2 and proof of the Theorem. In this section we shall estimate S_2 and give the proof of the Theorem. We first study the properties of

$$Z(s) = \sum_{n=1}^{\infty} r(n) n^{-s}, \qquad \Re s > 1/2,$$

where $r(n)$ is defined by

$$r(n) = \sum_{\substack{n=xy(x^2-y^2)\\ x,y\in\mathbb{N}}} 1.$$

LEMMA 5.1. *We have the following estimates*:

$$\sum_{n=1}^{\infty} \frac{r^2(n)}{n^\sigma} \ll 1, \qquad \sum_{n=1}^{\infty} \frac{r(n)}{n^\sigma} \ll 1, \qquad \sigma > 1/2;$$

$$\sum_{n\le x}\frac{r(n)}{n^{1/2}}\ll \log x,\qquad \sum_{n\le x}\frac{r(n)}{n^{\sigma}}\ll x^{1/2-\sigma},\qquad 0<\sigma<1/2.$$

Proof. These estimates follow from (2.3) by partial summation. ∎

LEMMA 5.2. *$Z(s)$ has the following properties*:

(1) *$Z(s)$ has an analytic continuation to $\sigma>1/8$ with two simple poles at $s=1/2$ and $s=1/3$.*

(2) *We have*

$$Z(\sigma+it)\ll \min\left(\log|t|,\frac{1}{\sigma-1/2}\right),\qquad \sigma\ge 1/2,\ |t|\ge 2.$$

(3) *The estimate*

$$Z(\sigma+it)\ll |t|^{8(1/2-\sigma)/3}\log|t|$$

holds uniformly for $1/8<\sigma_1\le\sigma\le 1/2, |t|\ge 2$.

(4) *For any real parameter $T\ge 10$, we have*

$$\int_T^{2T}|Z(24/73+it)|^2\,dt\ll T\log^7 T.$$

Proof. Suppose $X\ge 2$ is a parameter. For $\sigma>1/2$, by Stieltjes integration we get

$$\begin{aligned}(5.1)\qquad Z(s)&=\sum_{n\le X}\frac{r(n)}{n^s}+\int_X^{\infty}\omega^{-s}\,dL(\omega)\\ &=\sum_{n\le X}\frac{r(n)}{n^s}+\int_X^{\infty}\omega^{-s}\,d(k\omega^{1/2}-k'\omega^{1/3}+F(\omega))\\ &=\sum_{n\le X}\frac{r(n)}{n^s}+\frac{k}{2}\cdot\frac{X^{1/2-s}}{s-1/2}-\frac{k'}{3}\cdot\frac{X^{1/3-s}}{s-1/3}-X^{-s}F(X)+s\int_X^{\infty}\frac{F(\omega)}{\omega^{s+1}}\,d\omega.\end{aligned}$$

From Lemma 3.2 we get

$$(5.2)\qquad \int_1^M|F(u)|\,du\ll M^{9/8}\log^2 M.$$

This shows that the integration on the right side of (5.1) is absolutely convergent for $\sigma>1/8$. So the first assertion of Lemma 5.2 follows.

The second assertion follows from (5.1) and Lemma 5.1.

Suppose $1/8<\sigma_1<1/2$. Then from Lemma 5.1 and (5.1) we get

$$Z(\sigma_1+it)\ll X^{1/2-\sigma_1}+|t|X^{1/8-\sigma}\ll |t|^{8(1/2-\sigma)/3}$$

by choosing $X=|t|^{8/3}$. Now the third assertion of Lemma 5.2 follows from the Pragmén–Lindelöf argument.

Now we prove assertion (4). We always suppose $s=\sigma+it$, $T\le t\le 2T$, $24/73\le\sigma<1/2$, and $T\le X\le T^3$ is a parameter to be determined. From (5.1) we have

$$\int_T^{2T}|Z(\sigma+it)|^2\,dt \ll W_1+T^2W_2+T^{-1}X^{1-2\sigma}+T, \tag{5.3}$$

where

$$W_1=\int_T^{2T}\Big|\sum_{n\le X}\frac{r(n)}{n^{\sigma+it}}\Big|^2dt,\qquad W_2=\int_T^{2T}\Big|\int_X^{\infty}\frac{F(\omega)}{\omega^{\sigma+it+1}}\,d\omega\Big|^2\,dt.$$

We first estimate W_1. Squaring, integrating and then using Lemma 5.1 we get

$$\begin{aligned} W_1&=\int_T^{2T}\sum_{m,n\le X}\frac{r(m)r(n)}{(mn)^\sigma}\Big(\frac{m}{n}\Big)^{it}dt \\ &\ll T\sum_{n\le X}\frac{r^2(n)}{n^{2\sigma}}+\sum_{m\ne n}\frac{r(m)r(n)}{(mn)^\sigma}\min\Big(T,\frac{1}{|\log\frac{m}{n}|}\Big)\\ &\ll T+\sum_{m<n\le X}\frac{r(m)r(n)}{(mn)^\sigma}\min\Big(T,\frac{1}{\log\frac{n}{m}}\Big)\\ &=T+\Sigma_1+\Sigma_2+\Sigma_3, \end{aligned}\tag{5.4}$$

where

$$\begin{aligned}\Sigma_1&=T\sum_{m\le X}\sum_{m<n\le e^{1/T}m}\frac{r(m)r(n)}{(mn)^\sigma},\\ \Sigma_2&=\sum_{m\le X}\sum_{e^{1/T}m<n\le 2m}\frac{r(m)r(n)}{(mn)^\sigma}\cdot\frac{1}{\log\frac{n}{m}},\\ \Sigma_3&=\sum_{m\le X}\sum_{n>2m}\frac{r(m)r(n)}{(mn)^\sigma}\cdot\frac{1}{\log\frac{n}{m}}.\end{aligned}$$

By Lemma 5.1 again we get

$$\Sigma_3\ll\Big(\sum_{m\le X}\frac{r(m)}{m^\sigma}\Big)^2\ll X^{1-2\sigma}. \tag{5.5}$$

For Σ_1, by Lemma 5.1 and (4.3) we get

$$\begin{aligned}\Sigma_1&\ll T\sum_{m\le X}\frac{r(m)}{m^{2\sigma}}\sum_{m<n\le e^{1/T}m}r(n)\\ &=T\sum_{m\le X}\frac{r(m)}{m^{2\sigma}}\,(L(e^{1/T}m)-L(m))\end{aligned}\tag{5.6}$$

$$= T\sum_{m\le X}\frac{r(m)}{m^{2\sigma}}\,(k(e^{1/(2T)}m^{1/2}-m^{1/2})-k'(e^{1/(3T)}m^{1/3}-m^{1/3}))$$
$$+T\sum_{m\le X}\frac{r(m)}{m^{2\sigma}}\,(F(e^{1/T}m)-F(m))$$
$$\ll X^{1-2\sigma}+T\log^4 T$$

if we notice $\sigma\ge 24/73$.

It remains to estimate Σ_2. Let $n=m+r$ and notice

$$\frac{1}{\log(n/m)}=\frac{1}{\log(1+r/m)}\ll m/r.$$

Hence we get

$$\Sigma_2\ll\sum_{m\le X}\frac{r(m)}{m^{2\sigma-1}}\Sigma_4\frac{r(m+r)}{r}, \tag{5.7}$$

where Σ_4 sums over $\max(1,m(e^{1/T}-1))\le r\le m$.

By a splitting argument and (4.3) we get

$$\begin{aligned}\Sigma_4 &\ll \log m\cdot\max_{a\ll m}\sum_{a<r\le 2a}\frac{r(m+r)}{r}\\ &\ll \log m\cdot\max_{a\ll m}\frac{1}{a}\sum_{a<r\le 2a}r(m+r)\\ &\ll \log m\cdot\max_{a\ll m}\frac{1}{a}\sum_{m+a<r\le m+2a}r(r)\\ &\ll \log m\cdot\max_{a\ll m}\frac{1}{a}\,((m+2a)^{1/2}-(m+a)^{1/2})\\ &\quad+\log m\cdot\max_{a\ll m}\frac{1}{a}\,((m+2a)^{1/3}-(m+a)^{1/3})\\ &\quad+\log m\cdot\max_{a\ll m}\frac{1}{a}\,m^{23/146}\log^3 m\\ &\ll m^{-1/2}\log m+Tm^{23/146-1}\log^4 m,\end{aligned} \tag{5.8}$$

if we notice $a\gg m(e^{1/T}-1)\gg m/T$.

Inserting (5.8) into (5.7) we get

$$\begin{aligned}\Sigma_2 &\ll \sum_{m\le X}\frac{r(m)}{m^{2\sigma-1}}\,(m^{-1/2}\log m+Tm^{23/146-1}\log^4 m)\\ &\ll X^{1-2\sigma}\log T+T\log^5 T.\end{aligned} \tag{5.9}$$

From (5.4)–(5.9) we get

$$W_1\ll X^{1-2\sigma}\log T+T\log^5 T. \tag{5.10}$$

Now we estimate W_2. We have

$$
\begin{aligned}
(5.11)\quad W_2 &= \int_T^{2T} dt \int_X^{\infty}\int_X^{\infty} \frac{F(\omega_1)}{\omega_1^{\sigma+1+it}} \cdot \frac{F(\omega_2)}{\omega_2^{\sigma+1-it}}\, d\omega_1\, d\omega_2 \\
&= \int_X^{\infty}\int_X^{\infty} \frac{F(\omega_1)F(\omega_2)}{(\omega_1\omega_2)^{\sigma+1}}\, d\omega_1\, d\omega_2 \int_T^{2T} \left(\frac{\omega_2}{\omega_1}\right)^{it} dt \\
&\ll \int_X^{\infty}\int_X^{\infty} (\omega_1\omega_2)^{23/146-\sigma-1} (\log\omega_1\ \log\omega_2)^3 \min\left(T, \frac{1}{\left|\log\frac{\omega_2}{\omega_1}\right|}\right) d\omega_1\, d\omega_2 \\
&\ll \int_X^{\infty} \omega_1^{23/146-\sigma-1} \log^3\omega_1\, d\omega_1 \\
&\quad \times \int_X^{\omega_1} \omega_2^{23/146-\sigma-1} \log^3\omega_2 \min\left(T, \frac{1}{\log\frac{\omega_1}{\omega_2}}\right) d\omega_2 \\
&= \int_1 + \int_2 + \int_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_1 &= T \int_X^{\infty} \omega_1^{23/146-\sigma-1} \log^3\omega_1\, d\omega_1 \int_{e^{-1/T}\omega_1}^{\omega_1} \omega_2^{23/146-\sigma-1} \log^3\omega_2\, d\omega_2, \\
\int_2 &= \int_X^{\infty} \omega_1^{23/146-\sigma-1} \log^3\omega_1\, d\omega_1 \int_{e^{-1/3}\omega_1}^{e^{-1/T}\omega_1} \omega_2^{23/146-\sigma-1} \log^3\omega_2 \frac{1}{\log\frac{\omega_1}{\omega_2}}\, d\omega_2, \\
\int_3 &= \int_X^{\infty} \omega_1^{23/146-\sigma-1} \log^3\omega_1\, d\omega_1 \int_X^{e^{-1/3}\omega_1} \omega_2^{23/146-\sigma-1} \log^3\omega_2 \frac{1}{\log\frac{\omega_1}{\omega_2}}\, d\omega_2.
\end{aligned}
$$

Trivially we have

$$
(5.12)\qquad \int_3 \ll \left(\int_X^{\infty} \omega_1^{23/146-\sigma-1} \log^3\omega_1\, d\omega_1\right)^2 \ll X^{46/146-2\sigma} \log^6 X.
$$

For $\int_1$, we have

$$
\begin{aligned}
(5.13)\qquad \int_1 &\ll T \int_X^{\infty} \omega_1^{46/146-2\sigma-2} \log^6\omega_1\, d\omega_1 \int_{e^{-1/T}\omega_1}^{\omega_1} d\omega_2 \\
&\ll T(e^{1/T}-1) \int_X^{\infty} \omega_1^{46/146-2\sigma-1} \log^6\omega_1\, d\omega_1 \\
&\ll X^{46/146-2\sigma} \log^6 X.
\end{aligned}
$$

For $\int_2$, we have

$$(5.14)\qquad \int\limits_2 \ll \int\limits_X^\infty \omega_1^{46/146-2\sigma-2}\log^6\omega_1\,d\omega_1 \int\limits_{e^{-1/3}\omega_1}^{e^{-1/T}\omega_1} \frac{1}{\log\frac{\omega_1}{\omega_2}}\,d\omega_2$$

$$\ll \int\limits_X^\infty \omega_1^{46/146-2\sigma-2}\log^6\omega_1\,d\omega_1 \int\limits_{e^{-1/3}\omega_1}^{e^{-1/T}\omega_1} \frac{\omega_1}{\omega_1-\omega_2}\,d\omega_2$$

$$\ll \int\limits_X^\infty \omega_1^{46/146-2\sigma-1}\log^6\omega_1(-\log(\omega_1-e^{-1/T}\omega_1)+\log(\omega_1-e^{-1/3}\omega_1))\,d\omega_1$$

$$\ll X^{46/146-2\sigma}\log^7 X.$$

From (5.11)–(5.14) we get

$$(5.15)\qquad W_2 \ll X^{46/146-2\sigma}\log^7 X.$$

Combining (5.3), (5.10) and (5.15) we get

$$(5.16)\qquad \int\limits_T^{2T} |Z(\sigma+it)|^2\,dt \ll X^{1-2\sigma}\log T + T^2X^{46/146-2\sigma}\log^7 X + T\log^5 T.$$

Now the fourth assertion of Lemma 5.2 follows from (5.16) by taking $\sigma = 24/73$ and $X = T^{146/50}$. ∎

In order to estimate S_2, we also need the following lemma. Its proof is contained in [7].

LEMMA 5.3. *Suppose RH is true. If*

$$\int\limits_T^{2T} |Z(\sigma+it)|^2\,dt \ll T^{1+\varepsilon}$$

for some $\sigma \ge 1/4$, *then*

$$S_2 = kN^{1/2}\sum_{d>y}\frac{\mu(d)}{d^2} - k'N^{1/3}\sum_{d>y}\frac{\mu(d)}{d^{4/3}} + O\left(y^{1/2}\left(\frac{x}{y^4}\right)^{\sigma+\varepsilon}\right)$$

for $1 \le y < x^{1/4}$.

From Lemmas 5.2 and 5.3 we immediately get

PROPOSITION 5.1. *If RH is true, then*

$$S_2 = kN^{1/2}\sum_{d>y}\frac{\mu(d)}{d^2} - k'N^{1/3}\sum_{d>y}\frac{\mu(d)}{d^{4/3}} + O(N^{24/73+\varepsilon}y^{-119/146})$$

for $1 \le y < N^{1/4}$.

Now we give the proof of the Theorem. Take $y = N^{119/768}$. By (2.4), (2.5), (4.7) and Proposition 5.1 we get

$$L'(N) = kN^{1/2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - k'N^{1/3}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{4/3}} + O(N^{127/616}\log^{963/308} N), \tag{5.17}$$

which combined with (2.1) yields the Theorem.

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